Workshop on Applied Matrix Positivity

International Centre for Mathematical Sciences

19th – 23rd July 2021

Timetable

All times are in BST (GMT +1).

	0900	1000	1100	1600	1700	1800
Monday	Berg	Bhatia	Khare	Charina	Schilling	Skalski
Tuesday	Köstler	Franz	Fagnola	Apanasovich	Emery	Cuevas
Wednesday	Jain	Choudhury	Vishwakarma	Pascoe	Pereira	-
Thursday	Sharma	Vashisht	Mishra	-	Stöckler	Knese

Titles and abstracts

1. New classes of multivariate covariance functions

   Tatiyana Apanasovich (George Washington University, USA)

   The class which is refereed to as the Cauchy family allows for the simultaneous modeling of the long memory dependence and correlation at short and intermediate lags. We introduce a valid parametric family of cross-covariance functions for multivariate spatial random fields where each component has a covariance function from a Cauchy family. We present the conditions on the parameter space that result in valid models with varying degrees of complexity. Practical implementations, including reparameterizations to reflect the conditions on the parameter space will be discussed. We show results of various Monte Carlo simulation experiments to explore the performances of our approach in terms of estimation and cokriging. The application of the proposed multivariate Cauchy model is illustrated on a dataset from the field of Satellite Oceanography.

2. A unified view of covariance functions through Gelfand pairs

   Christian Berg (University of Copenhagen)

   In Geostatistics one examines measurements depending on the location on the earth and on time. This leads to Random Fields of stochastic variables $Z(\xi, u)$ indexed by (ξ, u) belonging to $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ – the 2-dimensional unit sphere – is a model for the earth, and $\mathbb{R}$ is a model for time.

   If the variables are real-valued one considers a basic probability space $(\Omega, \mathcal{F}, P)$, where all the random variables $Z(\xi, u)$ are defined as measurable mappings from Ω to $\mathbb{R}$.

   One is interested in isotropic and stationary random fields $Z(\xi, u)$, $(\xi, u) \in \mathbb{S}^2 \times \mathbb{R}$, i.e., the situation where there exists a continuous function $f : [-1, 1] \times \mathbb{R} \to \mathbb{R}$ such that the covariance kernel is given as

$$\operatorname{cov}(Z(\xi, u), Z(\eta, v)) = f(\xi \cdot \eta, v - u), \quad \xi, \eta \in \mathbb{S}^2,\ u, v \in \mathbb{R}.$$

   Here $\xi \cdot \eta = \cos(\theta(\xi, \eta))$ is the scalar product equal to cosine of the length of the geodesic arc (=angle) between ξ and η.

   We require with other words that the covariance kernel only depends on the geodesic distance between the points on the sphere and on the time difference.

   Porcu and the speaker (2017) gave a characterization of such kernels by having uniformly convergent expansions

$$f(x, u) = \sum_{n=0}^{\infty} b_n(u) P_n(x), \quad \sum_{n=0}^{\infty} b_n(0) < \infty,$$

   where (b_n) is a sequence of real-valued characteristic (= continuous positive definite) functions on $\mathbb{R}$ and P_n are the Legendre polynomials on $[-1, 1]$ normalized as $P_n(1) = 1$. The result can be generalized to spheres $\mathbb{S}^d$ of any dimension d and $\mathbb{R}$ can be replaced by an arbitrary locally compact group.

   In work of Peron, Porcu and the speaker (2018) it was pointed out that the spheres can be replaced by compact homogeneous spaces G/K, where (G, K) is a Gelfand pair.

   We shall explain the theory of Gelfand pairs and also show how recent work of several people can be extended to this framework.

3. Metrics and means on positive definite matrices

   Rajendra Bhatia (Ashoka University, India)

   Two metrics on the manifold of positive definite matrices, known as the affine-invariant (Cartan) metric and the Bures–Wasserstein distance, have been of much interest in various applications. We will describe the main ideas in their study with special emphasis on the barycentres associated with them.

4. Tight wavelet frames and matrix positivity

   Maria Charina (University of Vienna, Austria)

   Joint work with M. Putinar, C. Scheiderer and J. Stöckler.

   After a short introduction to the world of tight wavelet frames – loosely speaking to the world of redundant bases-like function families – we discover matrix positivity at a junction of

   - constructions of tight wavelet frames

   - matrix extension problems in the context of the unitary extension principle

   - sum of squares problems for non-negative trigonometric polynomials and

   - semi-definite programming.

   The above mentioned problems belong to harmonic analysis, real algebraic geometry and convex optimization whose successful interaction yields new classes of tight wavelet frames.

5. Total positivity: characterizations and connections

   Projesh Nath Choudhury (Indian Institute of Science, Bengaluru, India)

   We will survey some foundational results on totally positive (TP) and totally non-negative (TN) matrices, including by Fekete (1912, following Laguerre 1883), Schoenberg (1930), Gantmacher–Krein (1950), and Brown–Johnstone–MacGibbon (1981). The last involves the well-known characterization of TP and TN matrices A via variation diminution: the number of sign changes of Ax is at most that for x, for all input vectors x. We then present a recent result (2021), which reduces this test set to a single vector for each square submatrix of A.

   We next provide an alternate characterization of TP and TN matrices (2021, joint with Kannan and Khare): via the sign non-reversal property, and again involving individual test vectors for each square submatrix. Finally, we present two applications. (a) Given an interval hull of matrices, we identify two matrices in it, whose total positivity implies the same for the entire hull. (b) We characterize total positivity via the Linear Complementarity Problem (LCP), and in fact strengthen this to checking the solution set of LCP at a single vector for each submatrix.

6. The F-family of covariance functions: a Matérn model on the sphere

   Francisco Cuevas (Federico Santa María Technical University, Chile)
   The Matérn family of isotropic covariance functions has been central to the theoretical development and application of statistical models for geospatial data. For global data defined over the whole sphere representing planet Earth, the natural distance between any two locations is the great circle distance. In this setting, the Matérn family of covariance functions has a restriction on the smoothness parameter, making it an unappealing choice to model smooth data. Finding a suitable analogue for modelling data on the sphere is still an open problem. This work proposes a new family of isotropic covariance functions for random fields defined over the sphere. The proposed family has four parameters, one of which indexes the mean square differentiability of the corresponding Gaussian field, and also allows for any admissible range of fractal dimension. We apply the proposed model to a dataset of precipitable water content over a large portion of the Earth, and show that the model gives more precise predictions of the underlying process at unsampled locations than does the Matérn model using chordal distances.

7. Flexible validity conditions for the multivariate Matérn covariance in any space dimension and for any number of components

   Xavier Emery (University of Chile, Chile)

   Joint work with Emilio Porcu and Philip White.

   Flexible multivariate covariance models for spatial data are in demand. This work addresses the problem of parametric constraints ensuring positive semidefiniteness of the multivariate Matérn model. Much attention has been given to the bivariate case, while highly multivariate cases have been explored to a limited extent only. The existing conditions often imply severe restrictions on the upper bounds for the collocated correlation coefficients, which makes the multivariate Matérn model appealing for the case of weak spatial crossdependence only. We provide a collection of validity conditions for the multivariate Matérn covariance model that allows for more flexible parameterizations than those currently available and prove that, under these new conditions, much higher upper bounds can be obtained for the collocated correlation coefficients.

8. Gaussian quantum Markov semigroups on Fock space: irreducibility and normal invariant states

   Franco Fagnola (Politecnico di Milano, Italy)

   We consider the most general Gaussian quantum Markov semigroup on the algebra of all bounded operators on a Fock space, discuss its construction from the generalized GKSL representation of the generator. We illustrate the known explicit formula on Weyl operators, irreducibility and its equivalence to a Hormander type condition on commutators and establish necessary and sufficient conditions for existence and uniqueness of normal invariant states. We will see that many properties can be deduced from those of matrices on the one-particle space of the Fock space.

9. The discrete Weyl relations and completely positive maps

   Uwe Franz (University of Bourgogne Franche-Comté, France)

   We use representations of the discrete Weyl relations to define kernels that parametrize linear maps on the matrix algebra M_n and show how positivity properties of these maps are reflected by their kernels. This allows us to reprove the Lindblad-Gorini-Kossakowski-Sudarshan classification of generators of completely positive semigroups. We also discuss extensions of this work to representations of groups of central type.

10. Symplectic eigenvalues of positive definite matrices

    Tanvi Jain (Indian Statistical Institute, Delhi Centre, India)

    For every real positive definite matrix A of order $2n$ there exists a real symplectic matrix M such that $M^T AM = \mathrm{diag}\{D, D\}$, where D is a positive diagonal matrix. The diagonal entries $d_1, \ldots, d_n$ of D are called the symplectic eigenvalues of the matrix A, and are the complete invariants of A under the action of the real symplectic group. Symplectic eigenvalues are relevant in different areas of physics and mathematics. We address some fundamental questions on these and discuss a few results.

11. A survey of positivity preservers

    Apoorva Khare (Indian Institute of Science and Analysis & Probability Research Group, Bangalore, India)

    Partly based on joint works with Alexander Belton, Dominique Guillot, Mihai Putinar, Bala Rajaratnam, and Terence Tao.

    I will survey some of the developments in the theory of transformations that preserve positivity. The talk is divided into four parts:

    (1) Classical (old and new) characterizations of positivity preservers, from Fourier and harmonic analysis.

    (2) The journey from Euclidean metric embeddings to positivity and its preservers.

    (3) Motivations from modern-day applications.

    (4) Connections to total positivity via power-preservers, and to combinatorics via a novel graph invariant.

12. Local theory of stable polynomials and bounded rational functions

    Greg Knese (Washington University in St. Louis, USA)

    We present a detailed local analysis of polynomials with no zeros in the bi-upper half plane at a distinguished boundary zero in three ways: via homogeneous expansions, Puiseux series, and transfer function realizations. This theory is then applied to study non-tangential regularity and distinguished boundary behavior of bounded rational functions. We also investigate the question of characterizing numerators of bounded rational functions with prescribed denominator. This is joint work with Bickel, Pascoe, and Sola.

13. Markovianity and the Thompson monoid F^+

    Claus Köstler (University College Cork, Ireland)

    Markovianity is a probabilistic phenomenon which does not care about the past - the future is already determined through the presence. Surprisingly this phenomenon is connected to the representation theory of the Thompson monoid F^+. This connection manifests itself by a new distributional invariance principle called 'partial spreadability'. My talk will introduce into these recent developments in noncommutative probability. In particular I will present some new de Finetti type results for classical Markov sequences and address how these results relate to the de Finetti theorem of David Freedman and Persi Diaconis. Returning to the operator algebraic framework, possibly I will address how our approach relates to recent work of Vaughan Jones on representations of the Thompson group F. My talk is based in wide parts on joint work with Gwion Evans, Rolf Gohm, Arundhathi Krishnan and Stephen Wills.

    References

    - Claus Köstler, Arundhathi Krishnan, Stephen Wills. Markovianity and the Thompson monoid F^+. eprint arXiv:2009.14811.

    - D. Gwion Evans, Rolf Gohm, Claus Köstler. Semi-cosimplicial objects and spreadability. Rocky Mountain J. Math., 47(6), 1839–1873.

14. Derivatives of symplectic eigenvalues and a Lidskii type theorem

    Hemant Mishra (Indian Statistical Institute, Delhi Centre, India)

    Associated with every $2n \times 2n$ real positive definite matrix A, there exist n positive numbers called the symplectic eigenvalues of A, and a basis of $\mathbb{R}^{2n}$ called the symplectic eigenbasis of A corresponding to these numbers. In this talk, we discuss the differentiability (analyticity) of the symplectic eigenvalues and corresponding symplectic eigenbasis for differentiable (analytic) map $t \mapsto A(t)$, and compute their derivatives. We then derive an analogue of Lidskii's theorem for symplectic eigenvalues as an application.

15. Abstract preservers

    James Pascoe (University of Florida, USA)

    We develop the theory of (completely) invariant structure preserving functions on a general structured topological space. We use our theory to conclude various results essentially for free - for example, a function f on the real line such that if C is a closed convex cone in n-space such that C is invariant under coordinate-wise multiplication by some point in n-space $(x_1, \ldots, x_n)$, then it is invariant under multiplication by $(f(x_1), \ldots, f(x_n))$, then f is analytic, entire and has a power series at 0 with nonnegative coefficients. Perhaps surprisingly, the same result holds when we change the domain of f to be the integers.

16. To be announced

    Nicolai Pastoors (TU Dortmund, Germany)

17. Eigenvalues of n by n doubly stochastic matrices and the Perfect–Mirsky conjecture

    Rajesh Pereira (University of Guelph, Canada)

    We will look at what is known about which complex numbers can be an eigenvalue of an n by n doubly stochastic matrix as well as certain quantum generalizations. We will look at known cases of the Perfect–Mirsky conjecture on the eigenvalues of doubly stochastic matrices and explain its connection to both the majorization order and to group representation theory. We use the connection between doubly stochastic matrices and majorization to derive the spectrum of different majorization relations (multivariate majorization, directional majorization etc...) which we explore. Several open questions will also be discussed.

18. The zoo of Bernstein functions

    René Schilling (TU Dresden, Germany)

    This is a survey on the class of Bernstein and related functions which appear in matrix positivity.

19. On some inequalities related to positive linear maps

    Rajesh Sharma (Himachal Pradesh University, Shimla, India)

    In this talk we focus on the non-commutative versions of some inequalities related to the the Cauchy-Schwarz inequality in matrix algebra. We discuss some inequalities involving positive linear maps on matrices and show how positive linear maps can be used to obtain bounds for the spread of matrices. We explain that the special cases of these inequalities involving positive linear functional provide inequalities involving moments of discrete and continuous random variable. These special cases relate some statistical parameters of interest in stochastic processes.

20. Positive matrices in noncommutative mathematics inspired by quantum theory

    Adam Skalski (Institute of Mathematics of the Polish Academy of Sciences, Poland)

    The talk will have a very elementary and introductory character. I will sketch the role played in the theory of C*-algebras, itself inspired by the quantum theory, by positive matrices and maps which preserve positivity on all matrix levels. I will attempt to indicate the delicate balance between the richness and ubiquity of such maps and relative rigidity of the class. Time permitting, I will also sketch recently developed related research directions inspired by the quantum information theory.

21. Matrix polynomials and rational functions for constructing multivariate tight wavelet frames

    Joachim Stöckler (TU Dortmund, Germany)

    We study polynomial and rational masks of wavelet frames from the point of view of linear system theory. An important step in the construction is the definition of matrix rational functions $A(z)$ in the polydisk from known positive definite $B = A^*A$ on the torus $\mathbb{T}^d$. The construction for the bidisk and for inner functions $A(z)$ was completed in [1] by A. Kummert, see also the review of Kummert's approach in [2]. In our talk, we give a short exposition of the connection with wavelet frames and present first ideas of generalizations to non-inner functions on the bidisk, which are current joint work with Nicolai Pastoors.

    [1] A. Kummert, Synthesis of two-dimensional lossless m-prts with prescried scattering matrix, Circuits Systems Signal Process. 8 (1989), 97-119.

    [2] G. Knese, Kummert's approach to realization on the bidisk, arXiv:1907.13191v2.

22. Frames of nonuniform wavelet systems

    Lalit Vashisht (University of Delhi, India)

    Frames are redundant building blocks which provide series representation, not necessarily unique, of each vector (signal) in the underlying Hilbert space. I will focus on frames of non-uniform wavelet systems, where the translation set is not necessarily a group but a spectrum which is based on the theory of spectral pairs. In this talk, First, I will present the construction of Parseval frames of non-uniform wavelet systems in the space $L^2(\mathbb{R})$ by the Unitary Extension Principle and its generalized version (also known as oblique extension principle). Secondly, I will discuss frames of non-uniform wavelets in discrete signal spaces. This is joint work with H.K. Malhotra.

23. Positivity preservers forbidden to operate on diagonal blocks

Prateek Kumar Vishwakarma (Indian Institute of Science, Bangalore, India)

The question of which functions acting entrywise preserve positive semidefiniteness has a long history, beginning with the Schur product theorem [Crelle 1911], which implies that absolutely monotonic functions (i.e., power series with nonnegative coefficients) preserve positivity on matrices of all dimensions. A famous result of Schoenberg and of Rudin [Duke Math. J. 1942, 1959] shows the converse: there are no other such functions.

Motivated by modern applications, Guillot and Rajaratnam [Trans. Amer. Math. Soc. 2015] classified the entrywise positivity preservers in all dimensions, which act only on the off-diagonal entries. These two results are at "opposite ends", and in both cases the preservers have to be absolutely monotonic.

We complete the classification of positivity preservers that act entrywise except on specified "diagonal/principal blocks", in every case other than the two above. (In fact we achieve this in a more general framework.) This yields the first examples of dimension-free entrywise positivity preservers – with certain forbidden principal blocks – that are not absolutely monotonic.