Ruelle resonances for pseudo-Anosov maps

Sébastien Gouëzel

CNRS and LMJL, Université de Nantes

June 18, 2018

Notations

- X a compact space
- $T : X \to X$ a continuous map
- μ a probability measure on X, preserved by T

Notations

- X a compact space
- $T : X \to X$ a continuous map
- μ a probability measure on X, preserved by T

T is mixing if $\int f \cdot g \circ T^n \,\mathrm{d}\mu \to \left(\int f \,\mathrm{d}\mu\right)\left(\int g \,\mathrm{d}\mu\right)$.

Notations

- X a compact space
- $T : X \to X$ a continuous map
- μ a probability measure on X, preserved by T

T is mixing if $\int f \cdot g \circ T^n \, \mathrm{d}\mu \to \left(\int f \, \mathrm{d}\mu\right)\left(\int g \, \mathrm{d}\mu\right)$.

Goal: speed of convergence? Asymptotic expansion?

The doubling map

$T(x) = 2x \mod 1$ on the circle $\mathbb{S}^1$
μ: Lebesgue measure

Theorem

Assume $f, g \in C^\infty(\mathbb{S}^1)$. Then, for any $\epsilon > 0$,

$$\int f \cdot g \circ T^n \, \mathrm{d}\mu = \left(\int f \, \mathrm{d}\mu\right)\left(\int g \, \mathrm{d}\mu\right) + O(\epsilon^n).$$

The doubling map

$T(x) = 2x \mod 1$ on the circle $\mathbb{S}^1$
μ: Lebesgue measure

Theorem

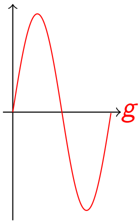
Assume $f, g \in C^\infty(\mathbb{S}^1)$. Then, for any $\epsilon > 0$,

$$\int f \cdot g \circ T^n \, \mathrm{d}\mu = \left(\int f \, \mathrm{d}\mu\right)\left(\int g \, \mathrm{d}\mu\right) + O(\epsilon^n).$$

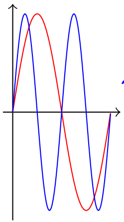
Proof.

$g(x) = \sum \hat{g}_k e^{2\mathbf{i}\pi kx}$.
$g(T^n x) = \sum \hat{g}_k e^{2\mathbf{i}\pi k 2^n x}$.
$\int f \cdot g \circ T^n \, \mathrm{d}\mu = \hat{f}_0 \overline{\hat{g}_0} + \sum_{k \neq 0} \hat{f}_{2^n k} \overline{\hat{g}_k}$.
The Fourier coefficients of f decay faster than any polynomial. $\square$

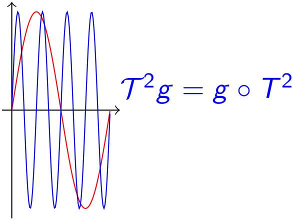
Iterates of the composition operator $\mathcal{T}g = g \circ T$.



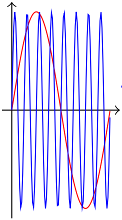
Iterates of the composition operator $\mathcal{T}g = g \circ T$.



Iterates of the composition operator $\mathcal{T}g = g \circ T$.



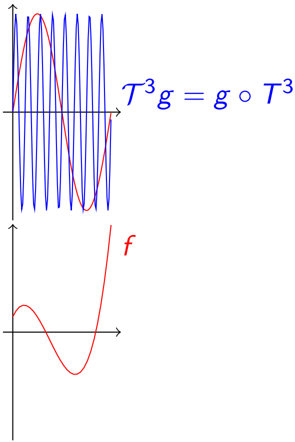
Iterates of the composition operator
$\mathcal{T} g = g \circ T$.



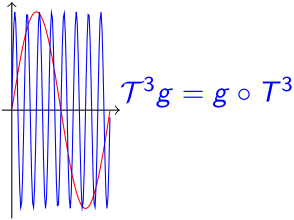
$\mathcal{T}^3 g = g \circ T^3$

Iterates of the composition operator
$\mathcal{T}g = g \circ T$.

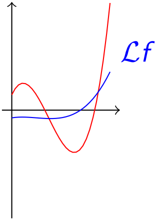
Transfer operator $\mathcal{L} = \mathcal{T}^*$:
$\int f \cdot g \circ T \,\mathrm{d}\mu = \int \mathcal{L}f \cdot g \,\mathrm{d}\mu$
$\mathcal{L}f(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$.



Iterates of the composition operator
$\mathcal{T}g = g \circ T$.

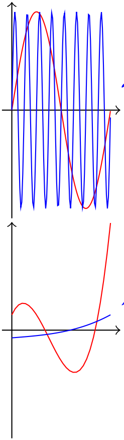


Transfer operator $\mathcal{L} = \mathcal{T}^*$:
$\int f \cdot g \circ T \, \mathrm{d}\mu = \int \mathcal{L}f \cdot g \, \mathrm{d}\mu$
$\mathcal{L}f(x) = \frac{1}{2}\big(f\big(\frac{x}{2}\big) + f\big(\frac{x+1}{2}\big)\big)$.



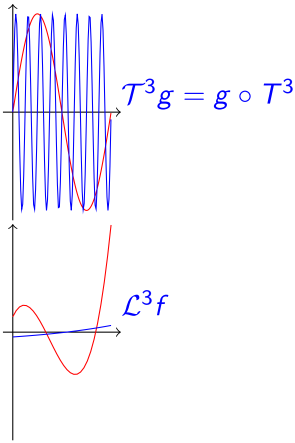
Iterates of the composition operator $\mathcal{T}g = g \circ T$.

Transfer operator $\mathcal{L} = \mathcal{T}^*$:
$\int f \cdot g \circ T \, \mathrm{d}\mu = \int \mathcal{L}f \cdot g \, \mathrm{d}\mu$
$\mathcal{L}f(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$.

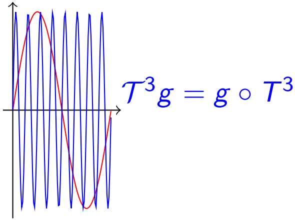


Iterates of the composition operator
$\mathcal{T}g = g \circ T$.

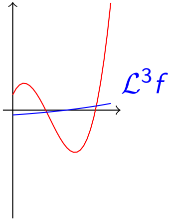
Transfer operator $\mathcal{L} = \mathcal{T}^*$:
$\int f \cdot g \circ T \,\mathrm{d}\mu = \int \mathcal{L}f \cdot g \,\mathrm{d}\mu$
$\mathcal{L}f(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$.



Iterates of the composition operator
$\mathcal{T}g = g \circ T$.

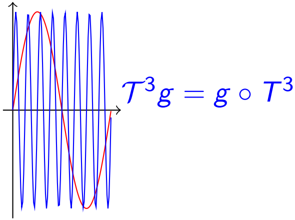


Transfer operator $\mathcal{L} = \mathcal{T}^*$:
$\int f \cdot g \circ T \,\mathrm{d}\mu = \int \mathcal{L}f \cdot g \,\mathrm{d}\mu$
$\mathcal{L}f(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$.

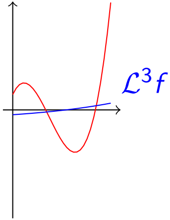


If f is C^∞ and $\int f \,\mathrm{d}\mu = 0$, then $\mathcal{L}^n f$ tends to 0 superexponentially fast in C^∞.

Iterates of the composition operator
$\mathcal{T}g = g \circ T$.



Transfer operator $\mathcal{L} = \mathcal{T}^*$:
$\int f \cdot g \circ T \,\mathrm{d}\mu = \int \mathcal{L}f \cdot g \,\mathrm{d}\mu$
$\mathcal{L}f(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right)$.



If f is C^∞ and $\int f \,\mathrm{d}\mu = 0$, then $\mathcal{L}^n f$ tends to 0 superexponentially fast in C^∞.
If $\int g \,\mathrm{d}\mu = 0$, then $\mathcal{T}^n g$ tends to 0 superexponentially fast as a distribution.

Uniformly expanding maps of the circle

$T : \mathbb{S}^1 \to \mathbb{S}^1$ a C^∞ map, with $T' \geqslant \beta > 1$.
μ Lebesgue measure. Assume T preserves μ.

Theorem

Let $f, g \in C^\infty$. For all $\epsilon > 0$, there is an expansion

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n),$$

where

- λ_i *is a sequence tending to* 0 *(the Ruelle resonances).*
- $c_i(f, g)$ *coefficients depending on* f *and* g*.*

Uniformly expanding maps of the circle

$T : \mathbb{S}^1 \to \mathbb{S}^1$ a C^∞ map, with $T' \geqslant \beta > 1$.
μ Lebesgue measure. Assume T preserves μ.

Theorem

Let $f, g \in C^\infty$. For all $\epsilon > 0$, there is an expansion

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n),$$

where

- *λ_i is a sequence tending to 0 (the Ruelle resonances).*
- *$c_i(f, g)$ coefficients depending on f and g.*

$\lambda_0 = 1$ dominating term, with $c_0(f, g) = \left(\int f \,\mathrm{d}\mu\right)\left(\int g \,\mathrm{d}\mu\right)$.

Uniformly expanding maps of the circle

$T : \mathbb{S}^1 \to \mathbb{S}^1$ a C^∞ map, with $T' \geqslant \beta > 1$.
μ Lebesgue measure. Assume T preserves μ.

Theorem

Let $f, g \in C^\infty$. For all $\epsilon > 0$, there is an expansion

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n),$$

where

- *λ_i is a sequence tending to 0 (the Ruelle resonances).*
- *$c_i(f, g)$ coefficients depending on f and g.*

$\lambda_0 = 1$ dominating term, with $c_0(f, g) = \left(\int f \,\mathrm{d}\mu\right)\left(\int g \,\mathrm{d}\mu\right)$.
In general, c_i are distributions.

Uniformly expanding maps of the circle

$T : \mathbb{S}^1 \to \mathbb{S}^1$ a C^∞ map, with $T' \geqslant \beta > 1$.
μ Lebesgue measure. Assume T preserves μ.

Theorem

Let $f, g \in C^\infty$. For all $\epsilon > 0$, there is an expansion

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} P_i(n) \lambda_i^n c_i(f, g) + o(\epsilon^n),$$

where

- *λ_i is a sequence tending to 0 (the Ruelle resonances).*
- *$c_i(f, g)$ coefficients depending on f and g.*

$\lambda_0 = 1$ dominating term, with $c_0(f, g) = \left(\int f \,\mathrm{d}\mu\right)\left(\int g \,\mathrm{d}\mu\right)$.
In general, c_i are distributions.
Just like for powers of matrices, there could be polynomial terms.
We ignore them for simplicity.

Sketch of proof, using the transfer operator $\mathcal{L} = \mathcal{T}^*$

Let $C \geqslant 0$. Decompose a function f as

$$f = \sum_{|k| \leqslant C} \hat{f}_k e^{2\mathbf{i}\pi kx} + \sum_{|k| > C} \hat{f}_k e^{2\mathbf{i}\pi kx} = f_{\leqslant C} + f_{>C}.$$

Then

$$\mathcal{L}f = \mathcal{L}f_{\leqslant C} + \mathcal{L}f_{>C} = \mathcal{L}_{\leqslant C} f + \mathcal{L}_{>C} f.$$

Sketch of proof, using the transfer operator $\mathcal{L} = \mathcal{T}^*$

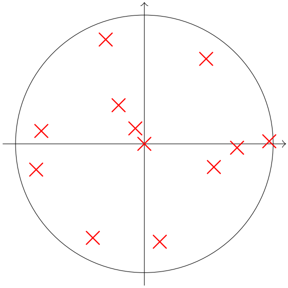
Let $C \geqslant 0$. Decompose a function f as

$$f = \sum_{|k| \leqslant C} \hat{f}_k e^{2\mathbf{i}\pi kx} + \sum_{|k| > C} \hat{f}_k e^{2\mathbf{i}\pi kx} = f_{\leqslant C} + f_{>C}.$$

Then

$$\mathcal{L}f = \mathcal{L}f_{\leqslant C} + \mathcal{L}f_{>C} = \mathcal{L}_{\leqslant C}f + \mathcal{L}_{>C}f.$$

$\mathcal{L}_{\leqslant C}$ has finite rank. Spectrum: finite set of eigenvalues, including 0.



Sketch of proof, using the transfer operator $\mathcal{L} = \mathcal{T}^*$

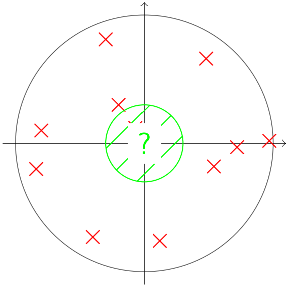
Let $C \geqslant 0$. Decompose a function f as

$$f = \sum_{|k| \leqslant C} \hat{f}_k e^{2\mathbf{i}\pi kx} + \sum_{|k| > C} \hat{f}_k e^{2\mathbf{i}\pi kx} = f_{\leqslant C} + f_{>C}.$$

Then

$$\mathcal{L}f = \mathcal{L}f_{\leqslant C} + \mathcal{L}f_{>C} = \mathcal{L}_{\leqslant C} f + \mathcal{L}_{>C} f.$$

$\mathcal{L}_{\leqslant C}$ has finite rank. Spectrum: finite set of eigenvalues, including 0.
$\mathcal{L}_{>C}$ is like for $x \mapsto 2x$: its norm on C^r is $\leqslant \beta^{-r}$.



Sketch of proof, using the transfer operator $\mathcal{L} = \mathcal{T}^*$

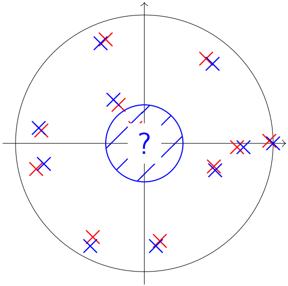
Let $C \geqslant 0$. Decompose a function f as

$$f = \sum_{|k| \leqslant C} \hat{f}_k e^{2\mathbf{i}\pi kx} + \sum_{|k|>C} \hat{f}_k e^{2\mathbf{i}\pi kx} = f_{\leqslant C} + f_{>C}.$$

Then

$$\mathcal{L}f = \mathcal{L}f_{\leqslant C} + \mathcal{L}f_{>C} = \mathcal{L}_{\leqslant C} f + \mathcal{L}_{>C} f.$$

$\mathcal{L}_{\leqslant C}$ has finite rank. Spectrum: finite set of eigenvalues, including 0.
$\mathcal{L}_{>C}$ is like for $x \mapsto 2x$: its norm on C^r is $\leqslant \beta^{-r}$.
The spectrum of $\mathcal{L}$ is a discrete set of eigenvalues in $\{z : |z| > \beta^{-r}\}$. $\square$



Sketch of proof, using the transfer operator $\mathcal{L} = \mathcal{T}^*$

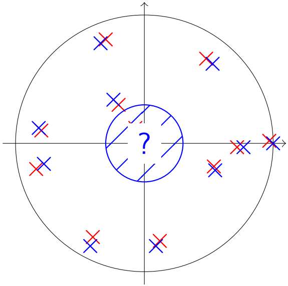
Let $C \geqslant 0$. Decompose a function f as

$$f = \sum_{|k| \leqslant C} \hat{f}_k e^{2\mathbf{i}\pi kx} + \sum_{|k| > C} \hat{f}_k e^{2\mathbf{i}\pi kx} = f_{\leqslant C} + f_{>C}.$$

Then

$$\mathcal{L}f = \mathcal{L}f_{\leqslant C} + \mathcal{L}f_{>C} = \mathcal{L}_{\leqslant C} f + \mathcal{L}_{>C} f.$$

$\mathcal{L}_{\leqslant C}$ has finite rank. Spectrum: finite set of eigenvalues, including 0.
$\mathcal{L}_{>C}$ is like for $x \mapsto 2x$: its norm on C^r is $\leqslant \beta^{-r}$.
The spectrum of $\mathcal{L}$ is a discrete set of eigenvalues in $\{z : |z| > \beta^{-r}\}$. $\square$



One could also do the same with $\mathcal{T}$ acting on $(C^r)^*$.

Ruelle resonances: general setting

Definition

A probability-preserving system (X, T, μ) has a Ruelle spectrum $\{\lambda_i\}$ if, for all C^∞ functions f and g, for all $\epsilon > 0$,

$$\int f \cdot g \circ T^n \, \mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n).$$

Ruelle resonances: general setting

Definition

A probability-preserving system (X, T, μ) has a Ruelle spectrum $\{\lambda_i\}$ if, for all C^∞ functions f and g, for all $\epsilon > 0$,

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n).$$

Theorem

C^∞ uniformly expanding maps have a Ruelle spectrum, for any Gibbs measure μ with smooth potential.

Ruelle resonances: general setting

Definition

A probability-preserving system (X, T, μ) has a Ruelle spectrum $\{\lambda_i\}$ if, for all C^∞ functions f and g, for all $\epsilon > 0$,

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n).$$

Theorem

C^∞ uniformly expanding/hyperbolic maps have a Ruelle spectrum, for any Gibbs measure μ with smooth potential.

Ruelle resonances: general setting

Definition

A probability-preserving system (X, T, μ) has a Ruelle spectrum $\{\lambda_i\}$ if, for all C^∞ functions f and g, for all $\epsilon > 0$,

$$\int f \cdot g \circ T^n \,\mathrm{d}\mu = \sum_{|\lambda_i| \geqslant \epsilon} \lambda_i^n c_i(f, g) + o(\epsilon^n).$$

Theorem

C^∞ uniformly expanding/hyperbolic maps have a Ruelle spectrum, for any Gibbs measure μ with smooth potential.

Further questions:

- Is the Ruelle spectrum trivial (like for $x \mapsto 2x \mod 1$)?
- Asymptotics of the number of eigenvalues with $|\lambda_i| \geqslant \epsilon$
- Speed of decay of correlations (i.e., gap between $\lambda_0 = 1$ and $|\lambda_1|$)
- Compute the Ruelle spectrum in some examples

Ruelle resonances for linear pseudo-Anosov maps

X: compact connected surface, genus g
T: linear pseudo-Anosov map, preserving orientations
μ: Lebesgue measure

Ruelle resonances for linear pseudo-Anosov maps

X: compact connected surface, genus g
T: linear pseudo-Anosov map, preserving orientations
μ: Lebesgue measure
$\lambda > 1$: expansion factor of the pseudo-Anosov map
Eigenvalues of T_* on H^1: $\{\lambda, \lambda^{-1}, \mu_1, \dots, \mu_{2g-2}\}$, with $|\mu_i| \in (\lambda^{-1}, \lambda)$.

Ruelle resonances for linear pseudo-Anosov maps

X: compact connected surface, genus g
T: linear pseudo-Anosov map, preserving orientations
μ: Lebesgue measure
$\lambda > 1$: expansion factor of the pseudo-Anosov map
Eigenvalues of T_* on H^1: $\{\lambda, \lambda^{-1}, \mu_1, \dots, \mu_{2g-2}\}$, with $|\mu_i| \in (\lambda^{-1}, \lambda)$.

Theorem (Faure-Gouëzel-Lanneau)

T has Ruelle resonances. They are exactly 1 and $\lambda^{-n}\mu_i$ for $i \in \{1, \dots, 2g-2\}$ and $n \geqslant 1$.

Ruelle resonances for linear pseudo-Anosov maps

X: compact connected surface, genus g
T: linear pseudo-Anosov map, preserving orientations
μ: Lebesgue measure
$\lambda > 1$: expansion factor of the pseudo-Anosov map
Eigenvalues of T_* on H^1: $\{\lambda, \lambda^{-1}, \mu_1, \dots, \mu_{2g-2}\}$, with $|\mu_i| \in (\lambda^{-1}, \lambda)$.

Theorem (Faure-Gouëzel-Lanneau)

T has Ruelle resonances. They are exactly 1 (with multiplicity 1) and $\lambda^{-n}\mu_i$ for $i \in \{1, \dots, 2g-2\}$ and $n \geqslant 1$ (with multiplicity n).

Ruelle resonances for linear pseudo-Anosov maps

X: compact connected surface, genus g
T: linear pseudo-Anosov map, preserving orientations
μ: Lebesgue measure
$\lambda > 1$: expansion factor of the pseudo-Anosov map
Eigenvalues of T_* on H^1: $\{\lambda, \lambda^{-1}, \mu_1, \dots, \mu_{2g-2}\}$, with $|\mu_i| \in (\lambda^{-1}, \lambda)$.

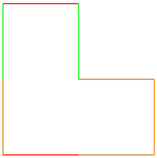
Theorem (Faure-Gouëzel-Lanneau)

T has Ruelle resonances. They are exactly 1 (with multiplicity 1) and $\lambda^{-n}\mu_i$ for $i \in \{1, \dots, 2g-2\}$ and $n \geqslant 1$ (with multiplicity n).

Example

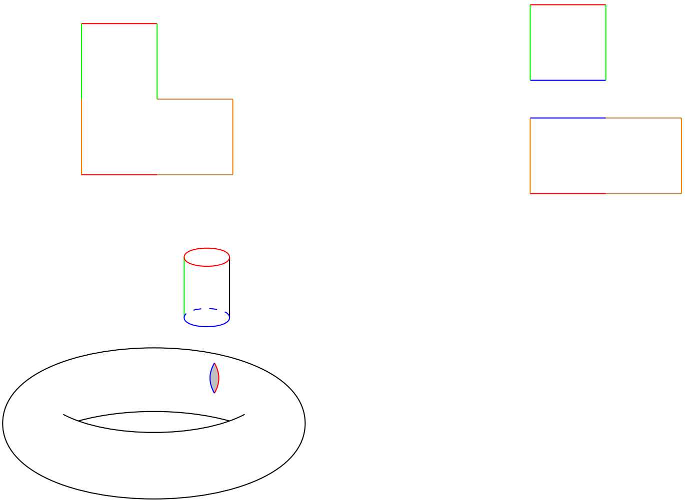
$T = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ the cat map, acting on $\mathbb{T}^2$.
Its Ruelle spectrum is $\{1\}$.

Constructing a translation surface

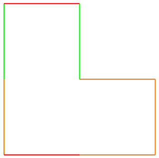


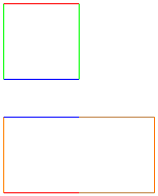
Constructing a translation surface

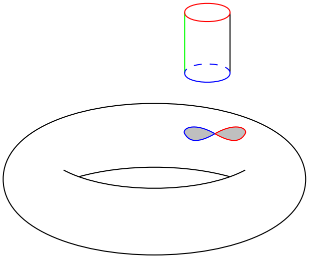
Constructing a translation surface



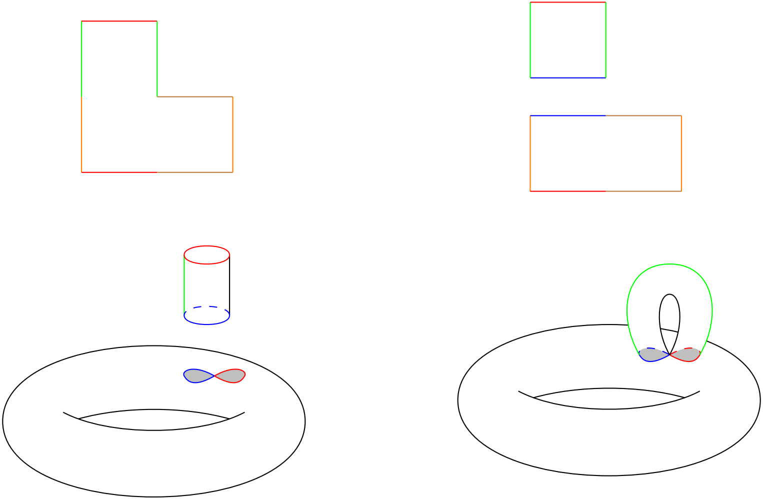
Constructing a translation surface



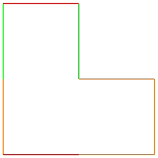




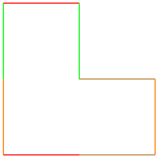
Constructing a translation surface



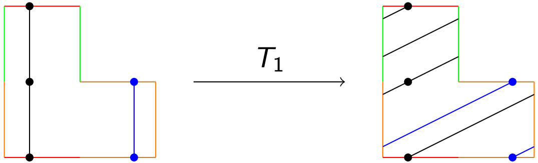
Constructing a pseudo-Anosov map



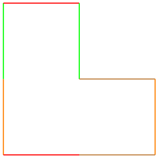
Constructing a pseudo-Anosov map



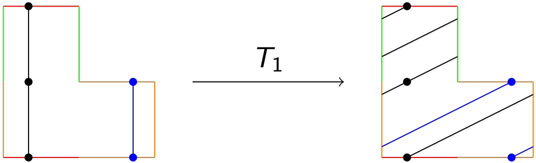
$T_1 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ acts on X:



Constructing a pseudo-Anosov map

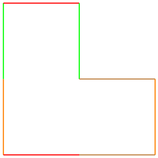


$T_1 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ acts on X:

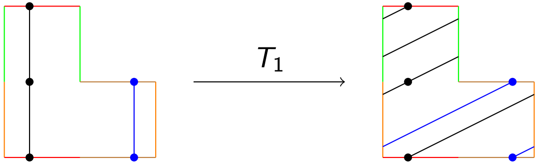


$T_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ also acts on X.

Constructing a pseudo-Anosov map



$T_1 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ acts on X:



$T_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ also acts on X.

Compose the above two: one gets a map $T : X \to X$ locally given by the matrix $\left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. This is a linear pseudo-Anosov map.

T linear pseudo-Anosov map, with expansion factor $\lambda > 1$ and spectrum on H^1 : $\{\lambda, \lambda^{-1}, \mu_1, \dots, \mu_{2g-2}\}$.

Theorem (Faure-Gouëzel-Lanneau)

T has Ruelle resonances. They are exactly 1 and $\lambda^{-n}\mu_i$ for $i \in \{1, \dots, 2g-2\}$ and $n \geqslant 1$.

T linear pseudo-Anosov map, with expansion factor $\lambda > 1$ and spectrum on $H^1 : \{\lambda, \lambda^{-1}, \mu_1, \ldots, \mu_{2g-2}\}$.

Theorem (Faure-Gouëzel-Lanneau)

T has Ruelle resonances. They are exactly 1 and $\lambda^{-n}\mu_i$ for $i \in \{1, \ldots, 2g-2\}$ and $n \geqslant 1$.

Steps of the proof:

1. Ruelle resonances make sense
2. Cohomology gives rise to Ruelle resonances
3. All Ruelle resonances come from cohomology

Ruelle resonances make sense

Goal: construct a Banach space on which $f \mapsto f \circ T$ has a small essential spectral radius.

Ruelle resonances make sense

Goal: construct a Banach space on which $f \mapsto f \circ T$ has a small essential spectral radius.
For expanding maps: distributions
For contracting maps: smooth functions

Ruelle resonances make sense

Goal: construct a Banach space on which $f \mapsto f \circ T$ has a small essential spectral radius.
For expanding maps: distributions
For contracting maps: smooth functions
In hyperbolic cases, anisotropic space

- smooth in the contracting (vertical) direction
- dual of smooth in the expanding (horizontal) direction

Ruelle resonances make sense

Goal: construct a Banach space on which $f \mapsto f \circ T$ has a small essential spectral radius.
For expanding maps: distributions
For contracting maps: smooth functions
In hyperbolic cases, anisotropic space

- smooth in the contracting (vertical) direction
- dual of smooth in the expanding (horizontal) direction

Smoothness indices k_h and k_v. Let L_v be the vertical derivative.

$$\|f\| = \sup_{i \leqslant k_v} \sup_{\substack{I \text{ horizontal interval} \\ |I|=1, I \cap \Sigma = \emptyset}} \sup_{\phi \in C_c^{k_h}(I), \|\phi\|_{C^{k_h}} \leqslant 1} \int_I \phi \cdot L_v^i f \, \mathrm{d}x$$

$\mathcal{B}$ = completion of $C_c^\infty(M - \Sigma)$ for $\|\cdot\|$. $\square$

Cohomology gives rise to Ruelle resonances

$h = [\omega] \neq 0$ a cohomology class with $T^* h = \mu h$ and $|\mu| \in (\lambda^{-1}, \lambda)$.
Goal: construct $f \in \mathcal{B}$ with $\mathcal{T} f = \lambda^{-1} \mu f$, where $\mathcal{T} f = f \circ T$.

Cohomology gives rise to Ruelle resonances

$h = [\omega] \neq 0$ a cohomology class with $T^*h = \mu h$ and $|\mu| \in (\lambda^{-1}, \lambda)$.
Goal: construct $f \in \mathcal{B}$ with $\mathcal{T} f = \lambda^{-1}\mu f$, where $\mathcal{T} f = f \circ T$.
Decompose $\omega = \omega_x \, \mathrm{d}x + \omega_y \, \mathrm{d}y$. Then

$$\mu^n \int_\gamma \omega = \int_\gamma (T^*)^n \omega = \int_\gamma \mathcal{T}^n \omega_x \cdot \lambda^n \, \mathrm{d}x + \int_\gamma \mathcal{T}^n \omega_y \cdot \lambda^{-n} \, \mathrm{d}y.$$

Therefore,

$$\int_\gamma (\mathcal{T}^n \omega_x) \, \mathrm{d}x = (\lambda^{-1}\mu)^n \int_\gamma \omega + O(\lambda^{-2n}).$$

Cohomology gives rise to Ruelle resonances

$h = [\omega] \neq 0$ a cohomology class with $T^* h = \mu h$ and $|\mu| \in (\lambda^{-1}, \lambda)$.
Goal: construct $f \in \mathcal{B}$ with $\mathcal{T} f = \lambda^{-1}\mu f$, where $\mathcal{T} f = f \circ T$.
Decompose $\omega = \omega_x \, \mathrm{d}x + \omega_y \, \mathrm{d}y$. Then

$$\mu^n \int_\gamma \omega = \int_\gamma (T^*)^n \omega = \int_\gamma \mathcal{T}^n \omega_x \cdot \lambda^n \, \mathrm{d}x + \int_\gamma \mathcal{T}^n \omega_y \cdot \lambda^{-n} \, \mathrm{d}y.$$

Therefore,

$$\int_\gamma (\mathcal{T}^n \omega_x) \, \mathrm{d}x = (\lambda^{-1}\mu)^n \int_\gamma \omega + O(\lambda^{-2n}).$$

Spectral expansion: $\mathcal{T}^n \omega_x = \sum \lambda_i^n f_i + o(\epsilon^n)$.
One of the λ_i should coincide with $\lambda^{-1}\mu$. $\square$

Cohomology gives rise to Ruelle resonances

$h = [\omega] \neq 0$ a cohomology class with $T^* h = \mu h$ and $|\mu| \in (\lambda^{-1}, \lambda)$.
Goal: construct $f \in \mathcal{B}$ with $\mathcal{T} f = \lambda^{-1}\mu f$, where $\mathcal{T} f = f \circ T$.
Decompose $\omega = \omega_x \, \mathrm{d}x + \omega_y \, \mathrm{d}y$. Then

$$\mu^n \int_\gamma \omega = \int_\gamma (T^*)^n \omega = \int_\gamma \mathcal{T}^n \omega_x \cdot \lambda^n \, \mathrm{d}x + \int_\gamma \mathcal{T}^n \omega_y \cdot \lambda^{-n} \, \mathrm{d}y.$$

Therefore,

$$\int_\gamma (\mathcal{T}^n \omega_x) \, \mathrm{d}x = (\lambda^{-1}\mu)^n \int_\gamma \omega + O(\lambda^{-2n}).$$

Spectral expansion: $\mathcal{T}^n \omega_x = \sum \lambda_i^n f_i + o(\epsilon^n)$.
One of the λ_i should coincide with $\lambda^{-1}\mu$. $\square$

To show that $\lambda^{-n}\mu$, $n > 1$, is also a Ruelle resonance, use $L_h^{n-1} f$ where f eigenfunction for $\lambda^{-1}\mu$.

Cohomological interpretation of elements of $\mathcal{B}$

To $f \in \mathcal{B}$, associate the current $f\,\mathrm{d}x$.

Cohomological interpretation of elements of $\mathcal{B}$

To $f \in \mathcal{B}$, associate the current $f\,\mathrm{d}x$.
It is closed if $\mathrm{d}(f\,\mathrm{d}x) = 0$, i.e., $L_v f = 0$.
Cohomology class $[f\,\mathrm{d}x]$ (or simply $[f]$) for $f \in \mathcal{B} \cap \ker L_v$.

Cohomological interpretation of elements of $\mathcal{B}$

To $f \in \mathcal{B}$, associate the current $f\,\mathrm{d}x$.
It is closed if $\mathrm{d}(f\,\mathrm{d}x) = 0$, i.e., $L_v f = 0$.
Cohomology class $[f\,\mathrm{d}x]$ (or simply $[f]$) for $f \in \mathcal{B} \cap \ker L_v$.

The current $f\,\mathrm{d}x$ is exact if, additionally, $[f] = 0$. Then one can write $f\,\mathrm{d}x = \mathrm{d}g$. Automatically, $L_v g = 0$ and $L_h g = f$

Cohomological interpretation of elements of $\mathcal{B}$

To $f \in \mathcal{B}$, associate the current $f\,\mathrm{d}x$.
It is closed if $\mathrm{d}(f\,\mathrm{d}x) = 0$, i.e., $L_v f = 0$.
Cohomology class $[f\,\mathrm{d}x]$ (or simply $[f]$) for $f \in \mathcal{B} \cap \ker L_v$.

The current $f\,\mathrm{d}x$ is exact if, additionally, $[f] = 0$. Then one can write $f\,\mathrm{d}x = \mathrm{d}g$. Automatically, $L_v g = 0$ and $L_h g = f$ and moreover $g \in \mathcal{B}$.

Cohomological interpretation of elements of $\mathcal{B}$

To $f \in \mathcal{B}$, associate the current $f\,\mathrm{d}x$.
It is closed if $\mathrm{d}(f\,\mathrm{d}x) = 0$, i.e., $L_v f = 0$.
Cohomology class $[f\,\mathrm{d}x]$ (or simply $[f]$) for $f \in \mathcal{B} \cap \ker L_v$.

The current $f\,\mathrm{d}x$ is exact if, additionally, $[f] = 0$. Then one can write $f\,\mathrm{d}x = \mathrm{d}g$. Automatically, $L_v g = 0$ and $L_h g = f$ and moreover $g \in \mathcal{B}$.

Let E_α be the eigenspace of $\mathcal{T}$ for the eigenvalue α.

$$\begin{tikzcd} \dots \arrow[r, bend left, "L_v"] & E_{\lambda^{-2}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^{-1}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_\alpha \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^2\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & \dots \arrow[l, bend left, "L_h"] \end{tikzcd}$$

All Ruelle resonances come from cohomology

$$
\begin{tikzcd}
\cdots \arrow[r, bend left, "L_v"] & E_{\lambda^{-2}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^{-1}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^2\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & \cdots \arrow[l, bend left, "L_h"]
\end{tikzcd}
$$

Consider α with $E_\alpha \neq 0$. We want to show that $\alpha = 1$, or α is of the form $\lambda^{-n}\mu_i$.

- Start from $f \in E_\alpha - \{0\}$.

All Ruelle resonances come from cohomology

$$
\begin{tikzcd}
\cdots \arrow[r, bend left, "L_v"] & E_{\lambda^{-2}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^{-1}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^2\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & \cdots \arrow[l, bend left, "L_h"]
\end{tikzcd}
$$

Consider α with $E_\alpha \neq 0$. We want to show that $\alpha = 1$, or α is of the form $\lambda^{-n}\mu_i$.

- Start from $f \in E_\alpha - \{0\}$.
- k maximal with $L_v^k f \neq 0$. Then $g = L_v^k f \in E_{\lambda^k\alpha}$, and $L_v g = 0$.

All Ruelle resonances come from cohomology

$$\begin{tikzcd} \cdots \arrow[r, bend left, "L_v"] & E_{\lambda^{-2}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^{-1}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^2\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & \cdots \arrow[l, bend left, "L_h"] \end{tikzcd}$$

Consider α with $E_\alpha \neq 0$. We want to show that $\alpha = 1$, or α is of the form $\lambda^{-n}\mu_i$.

- Start from $f \in E_\alpha - \{0\}$.
- k maximal with $L_v^k f \neq 0$. Then $g = L_v^k f \in E_{\lambda^k\alpha}$, and $L_v g = 0$.
- If $[g] \neq 0$, then $T^* : H^1(M) \to H^1(M)$ has an eigenvalue at $\lambda \cdot \lambda^k \alpha$. We are done (modulo the problem of the eigenvalue λ^{-1}, that we have to exclude – harder, ignored in this sketch).

All Ruelle resonances come from cohomology

$$\begin{tikzcd} \cdots \arrow[r, bend left, "L_v"] & E_{\lambda^{-2}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^{-1}\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & E_{\lambda^2\alpha} \arrow[l, bend left, "L_h"] \arrow[r, bend left, "L_v"] & \cdots \arrow[l, bend left, "L_h"] \end{tikzcd}$$

Consider α with $E_\alpha \neq 0$. We want to show that $\alpha = 1$, or α is of the form $\lambda^{-n}\mu_i$.

- Start from $f \in E_\alpha - \{0\}$.
- k maximal with $L_v^k f \neq 0$. Then $g = L_v^k f \in E_{\lambda^k\alpha}$, and $L_v g = 0$.
- If $[g] \neq 0$, then $T^* : H^1(M) \to H^1(M)$ has an eigenvalue at $\lambda \cdot \lambda^k \alpha$. We are done (modulo the problem of the eigenvalue λ^{-1}, that we have to exclude – harder, ignored in this sketch).
- If $[g] = 0$, then $g = L_h g_1$, with $g_1 \in E_{\lambda^{k+1}\alpha} \cap \ker L_v$. Repeat, until $[g_\ell] \neq 0$. Then $\lambda^{k+\ell+1}$ is an eigenvalue of T^*. $\square$