Symbolic dynamics for non-uniformly hyperbolic systems with singularities

Yuri Lima

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June, 2018

Where?

Where?

SANTA FE DE BOGOTÁ
COLOMBIA
PARAMARIBO
Cayenne
SURINAM
FRENCH GUYANA
GUYANA
Boa Vista
RORAIMA
AMAPÁ
Macapá
EQUATOR
Belém
São Luis
Manaus
Santarém
PARÁ
Iquitos
AMAZONAS
MARANHÃO
Teresina
Fortaleza
CEARÁ
RIO GRANDE DO NORTE
Natal
João Pessoa
PARAÍBA
Recife
PIAUÍ
PERNAMBUCO
ACRE
Rio Branco
Pôrto Velho
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ALAGOAS
Maceió
Aracaju
SERGIPE
Palmas
TOCANTINS
MATO GROSSO
BAHIA
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Goiânia
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MATO GROSSO DO SUL
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Part 1: Introduction

Symbolic model for maps

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- (Σ, σ, π) symbolic model

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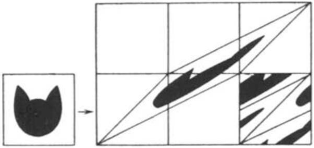

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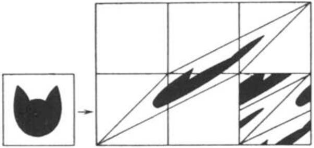

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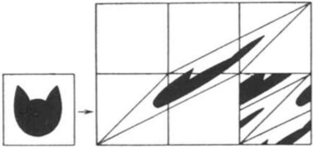

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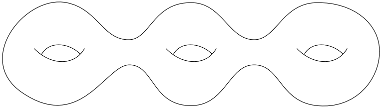

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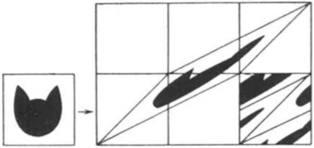

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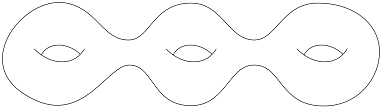

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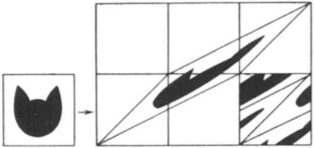

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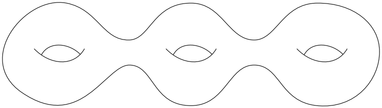

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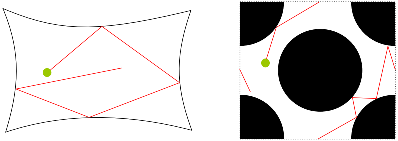

Part 2: Non-Uniform Hyperbolicity

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Non-uniformly hyperbolic set NUH_χ

Asymptotic hyperbolicity

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$$\mathrm{NUH} = \mathrm{NUH}_\chi = \{x \in M : s(x), u(x) < \infty\}$$

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$$p^s = \min\{e^\varepsilon q^s, Q(x)\}$$
$$q^u = \min\{e^\varepsilon p^u, Q(y)\}$$

Transition between ε–double charts

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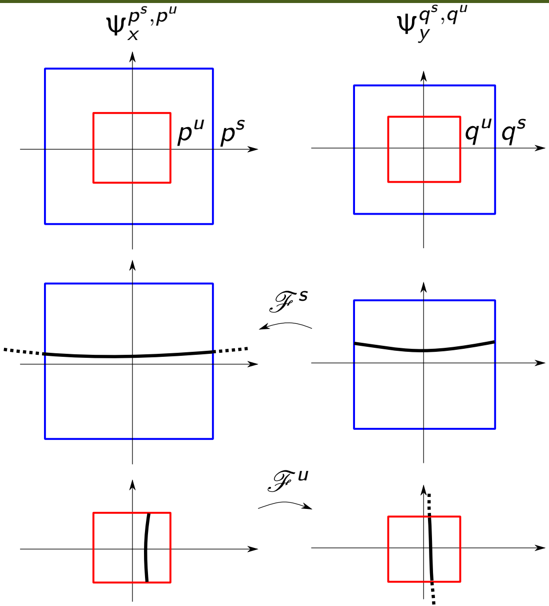

Part 3: Billiards

NUH billiards

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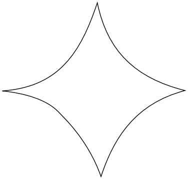
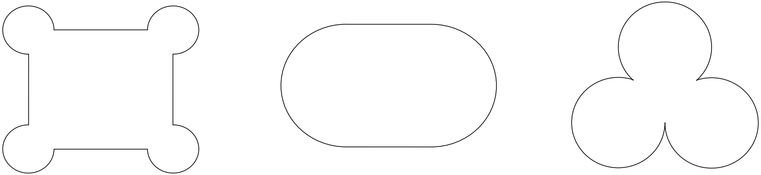


Sinai's billiard table



Bunimovich billiard tables

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- Behavior near $\mathscr{S}$:

$$\text{dist}(x, \mathscr{S})^a \leq \|df_x\| \leq \text{dist}(x, \mathscr{S})^{-a}$$

Non-uniformly hyperbolic set NUH

Asymptotic hyperbolicity

Old NUH

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$$+$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \operatorname{dist}(f^n(x), \mathscr{S}) = 0$$

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Bowen's 17th problem

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15. Renewal thm. for dependent r.v.'s
    a. derive via motivation of Ax. A flows mixingness
    b. how fast is the mixing for Ax. A flows

16. Brownian motion or diffusion given a flow

17. Symbolic dynamics for billiards

18. Interpret log $\lambda^u(x)$ as a potential fcn?)— Kolm. idea on surfaces neg. curv.

19. Can you construct some Banach space so that h_μ is an eigenvalue? of some can. operator

20. Cl. systems in stat. mech. — top. dyn. formulation?

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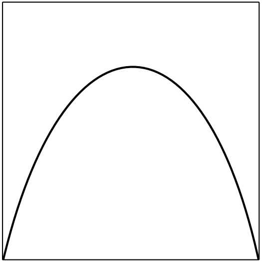
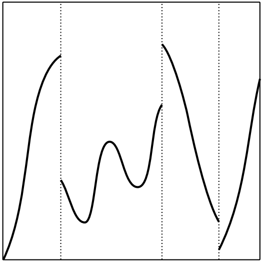
Part 4: Interval Maps

Non-uniformly expanding maps

Non-uniformly expanding maps





Framework

Non-invertible map with singularities

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Looking at the past

Natural extension

- $\widehat{M} = \{\widehat{x} = (x_n)_{n\in\mathbb{Z}} : f(x_n) = x_{n+1}, \forall\, n \in \mathbb{Z}\}$.

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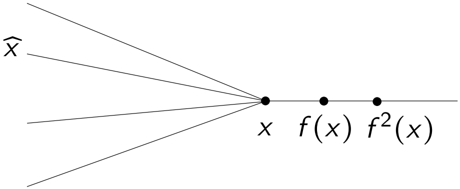

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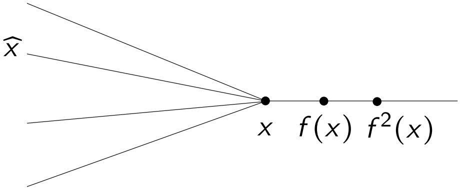

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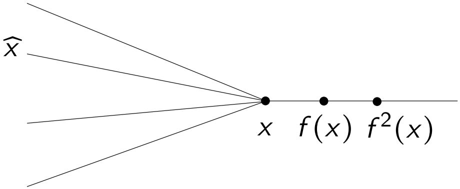

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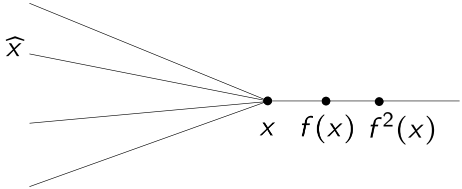

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    - $p = \min\{e^{\varepsilon} q, Q(\widehat{x})\}$.

Hofbauer tower × previous theorem

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Fiber cardinality	Infinite-to-one	Finite-to-one

Thanks!