Symbolic dynamics for non-uniformly hyperbolic systems with singularities

Yuri Lima

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June, 2018
Part 1: Introduction
Symbolic model for maps

- \( G = (V, E) \) oriented graph with \( V \) countable
- \( \Sigma = \{ \text{Z-indexed paths on } G \} \)
- \( \sigma : \Sigma \rightarrow \Sigma \) left shift
- \( (\Sigma, \sigma) \) topological Markov shift (TMS)
- \( \pi : \Sigma \rightarrow M \) coding
- \( (\Sigma, \sigma, \pi) \) symbolic model

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Symbolic models for uniformly hyperbolic systems

- Adler-Weiss 1967: 2–dim hyperbolic toral automorphisms
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Examples

○ Hyperbolic toral automorphisms:
○ Geodesic flows on manifolds with negative sectional curvature:
○ Dispersing billiards:
Examples

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![Diagram of hyperbolic toral automorphisms]
Examples

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Part 2: Non-Uniform Hyperbolicity
Framework

Non-uniform hyperbolicity parameters
Framework

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- \( f : M \to M \) surface diffeomorphism
Framework
Non-uniform hyperbolicity parameters

- $f : M \rightarrow M$ surface diffeomorphism
- $\chi > 0$
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- Angle parameter: $\alpha(x) = \angle(e^s_x, e^u_x)$
- $s$–parameter:

$$s(x) = \sqrt{2} \left( \sum_{n \geq 0} e^{2\chi^n} \| df^n e_x^s \|^2 \right)^{1/2}$$
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Non-uniformly hyperbolic set $\text{NUH}_\chi$

Asymptotic hyperbolicity
Non-uniformly hyperbolic set $\text{NUH}_\chi$

Asymptotic hyperbolicity

\[ \text{NUH} = \text{NUH}_\chi = \{ x \in M : s(x), u(x) < \infty \} \]
Symbolic dynamics for NUH surface diffeomorphisms

Katok 1980: surface diffeo with $h > 0$ has horseshoes of large topological entropy.

Theorem (Sarig 2013)

Let $f$ be a $C^1 + \beta$ surface diffeomorphism. For all $\chi > 0$ there is $(\Sigma, \sigma)$ TMS and $\pi : \Sigma \to M$ Hölder continuous s.t.:

1. $\pi \circ \sigma = f \circ \pi$.
2. $\pi[\Sigma]$ contains NUH.
3. Every $x \in NUH$ has finitely many pre-images in $\Sigma$.

NUH = recurrent subset of NUH

$\Sigma$ = recurrent subset of $\Sigma$.

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\[ \Psi_{x}^{p_s,p_u} \rightarrow \Psi_{y}^{q_s,q_u} \text{ if} \]
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- **Overlap:**
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\[ \psi_{x}^{p^{s}, p^{u}} \rightarrow \psi_{y}^{q^{s}, q^{u}} \text{ if} \]

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\[ \psi_{f(x)}^{q^{s} \wedge q^{u}} \approx \psi_{y}^{q^{s} \wedge q^{u}} \]
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(2) **\(\varepsilon\)-double charts** \(\Psi_{x}^{p_{s},p_{u}} = (\Psi_{x} \uparrow \left[ -p_{s}, p_{s} \right], \Psi_{x} \uparrow \left[ -p_{u}, p_{u} \right])\)

(3) **Transition rule:** \(\Psi_{x}^{p_{s},p_{u}} \rightarrow \Psi_{y}^{q_{s},q_{u}} \text{ if}\)

- **Overlap:** \(\Psi_{f(x)}^{q_{s} \wedge q_{u}} \approx \Psi_{y}^{q_{s} \wedge q_{u}} \text{ and } \Psi_{x}^{p_{s} \wedge p_{u}} \approx \Psi_{f^{-1}(y)}^{p_{s} \wedge p_{u}}\)
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- **Overlap:** \(\Psi^{q_s,q_u}_f(x) \approx \Psi^{q_s,q_u}_y\) and \(\Psi^{p_s,p_u}_x \approx \Psi^{p_s,p_u}_{f^{-1}(y)}\)
- **Maximality of parameters:**
Main ingredients

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- **Maximality of parameters:**

\[
p^s = \min\{ e^\varepsilon q^s, Q(x) \}
\]
\[
q^u = \min\{ e^\varepsilon p^u, Q(y) \}
\]
Transition between $\varepsilon$–double charts
Transition between $\varepsilon$–double charts

$\Psi_{x}^{p_{s}, p_{u}}$  

$\Psi_{y}^{q_{s}, q_{u}}$

$\mathcal{F}^{s}$  

$\mathcal{F}^{u}$
Part 3: Billiards
NUH billiards

Sinai’s billiard table

Bunimovich billiard tables
Framework

Invertible map with singularities
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○ $T = \text{billiard table}$
Framework

Invertible map with singularities

- $T =$ billiard table
- Phase space $= \partial T \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$
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- $T = $ billiard table
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- Singular set $\mathcal{S} =$
Framework

Invertible map with singularities

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- Phase space $= \partial T \times [-\frac{\pi}{2}, \frac{\pi}{2}]$
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- Behavior near $\mathcal{S}$:
Invertible map with singularities

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- Singular set $\mathcal{S} =$ break points $+$ glancing orbits
- Behavior near $\mathcal{S}$:

$$\text{dist}(x, \mathcal{S})^a \leq \| df_x \| \leq \text{dist}(x, \mathcal{S})^{-a}$$
Non-uniformly hyperbolic set NUH

Asymptotic hyperbolicity
Old NUH
Old NUH

\[ \lim_{n \to \pm \infty} \frac{1}{n} \log \text{dist}(f^n(x), \mathcal{S}) = 0 \]
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Theorem (L-Matheus)

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Theorem (L-Matheus)

Let \( f \) be the derived map of a piecewise \( C^3 \) table. For all \( \chi > 0 \) there is \((\Sigma, \sigma)\) TMS and \( \pi : \Sigma \rightarrow M \) Hölder continuous s.t.:

1. \( \pi \circ \sigma = f \circ \pi \).
2. \( \pi[\Sigma^#] \) contains \( NUH^# \).
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1. $\pi \circ \sigma = f \circ \pi$.
2. $\pi[\Sigma^\#]$ contains NUH$^\#$.
3. Every $x \in \text{NUH}^\#$ has finitely many pre-images in $\Sigma^\#$. 
Bowen’s 17th problem
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15. Renewal thm. for dependent r.v.’s
   a. derive via motivation of A, A flows mixingness
   b. how fast is the mixing for A, A flows

16. Brownian motion or diffusion quinoa-flow

17. Symbolic dynamics for billiards

18. Interpret log f(x) as a potential fcn? Kohn, idea on surfaces req. curv.

19. Can you construct some Banach space so that H is an eigenvalue of some coin. operator

20. Oth. systems in stat. mech. — top dyn. formulation?
Proof

Size of Pesin chart:

\[ \text{New } Q = \text{Old } Q \times \rho \times \rho \]

where \( \rho \times \rho \) = dist\(\{f^{-1}(x), x, f(x)\}, S\).

Finer local estimates:

estimates from Sarig are uniform, while ours are not ↔ bounded distortion.

Exponential beats polynomial and sub-exponential:

behavior of \( df \) beats explosion of derivative and proximity to \( S \).
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(2) **Finer local estimates:**
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2. **Finer local estimates:** estimates from Sarig are uniform, while ours are not \(\leftrightarrow\) bounded distortion.

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Part 4: Interval Maps
Non-uniformly expanding maps
Non-invertible map with singularities

- Singular set \( S \) = critical points + discontinuities
- Behavior near \( S \):
  \[
  \text{dist}(x, S) \leq \text{df}_x \leq \text{dist}(x, S) - a
  \]
- Size of inverse branches:
  for \( x \in [0, 1] \) and \( g = \text{inverse branch of } f \) s.t.
  \( g(f(x)) = x \), \( \exists r(x) > \text{dist}\{x, f(x)\} \) s.t.
  \( f^{-1}[x - r(x), x + r(x)] \) and \( g^{-1}[f(x) - r(x), f(x) + r(x)] \) are diffeos onto their images.
Framework

Non-invertible map with singularities

- $f : [0, 1] \rightarrow [0, 1]$ of class $C^{1+\beta}$
**Framework**

Non-invertible map with singularities

- $f : [0, 1] \rightarrow [0, 1]$ of class $C^{1+\beta}$
- Singular set $\mathcal{S} =$
Framework

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Non-invertible map with singularities

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$$\text{dist}(x, \mathcal{I})^a \leq \|df_x\| \leq \text{dist}(x, \mathcal{I})^{-a}$$
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Looking at the past

Natural extension
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Natural extension

\(\hat{M} = \{(\hat{x}_n)_{n \in \mathbb{Z}} : f(\hat{x}_n) = \hat{x}_{n+1}, \forall n \in \mathbb{Z}\}.\)
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- $\hat{M} = \{ \hat{x} = (x_n)_{n \in \mathbb{Z}} : f(x_n) = x_{n+1}, \forall n \in \mathbb{Z} \}$.
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\(f\)-invariant \(\mu\)
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- $f$–invariant $\mu \leftrightarrow \hat{f}$–invariant $\hat{\mu}$. 
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- \( f \)-invariant \( \mu \) ↔ \( \widehat{f} \)-invariant \( \widehat{\mu} \).
- Unstable manifold of \( x \):

\[ u(\widehat{x}) = \sum_{n \geq 0} e^{2n} \chi_{2n} \widehat{df}(x-n) \frac{1}{2} \]
Looking at the past

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- $f$–invariant $\mu \leftrightarrow \widehat{f}$–invariant $\widehat{\mu}$.
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- Non-invertible cocycle $df$
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- Non-invertible cocycle $df \leftrightarrow$ Invertible cocycle $\hat{df}$.

$$u(\hat{x}) = \left( \sum_{n \geq 0} e^{2n\chi} |\hat{df}|^{-n} \right)^{1/2}.$$
Non-uniformly expanding set NUE

Asymptotic expansion

\( \text{NUE} = \{ \hat{x} \in \hat{M} : u(\hat{x}) < \infty \text{ and } \lim_{n \to \pm\infty} \frac{1}{n} \log \text{dist}(x_n, S) = 0 \} \).
Non-uniformly expanding set $\text{NUE}$

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Non-uniformly expanding set $\overline{\text{NUE}}$

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$$\overline{\text{NUE}} = \{ \hat{x} \in \hat{M} : u(\hat{x}) < \infty \}$$
Non-uniformly expanding set $\emph{NUE}$

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$$\emph{NUE} = \{ \hat{x} \in \hat{M} : u(\hat{x}) < \infty \text{ and } \lim_{n \to \pm \infty} \frac{1}{n} \log \text{dist}(x_n, \mathcal{S}) = 0 \}.$$
Symbolic dynamics for interval maps

Theorem (L)

Let $\hat{f}$ be as above. For all $\chi > 0$ there is $(\hat{\Sigma},\hat{\sigma})$ a TMS and $\hat{\pi} : \hat{\Sigma} \to \mathbb{R}$ such that:

1. $\hat{\pi} \circ \hat{\sigma} = \hat{f} \circ \hat{\pi}$.
2. $\hat{\pi} [\hat{\Sigma}^\#]$ contains $\mathbb{N} U E ^\#$.
3. Every $\hat{x} \in \mathbb{N} U E ^\#$ has finitely many pre-images in $\hat{\Sigma}^\#$. 

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Let \( \hat{f} \) be as above. For all \( \chi > 0 \) there is \((\hat{\Sigma}, \hat{\sigma})\) TMS and \( \hat{\pi} : \hat{\Sigma} \rightarrow [0, 1] \) Hölder continuous s.t.:

1. \( \hat{\pi} \circ \hat{\sigma} = \hat{f} \circ \hat{\pi} \).
2. \( \hat{\pi}(\hat{\Sigma}^\#) \) contains \( \mathbb{NUE}^\# \).
3. Every \( \hat{\pi}(\hat{x}) \in \mathbb{NUE}^\# \) has finitely many pre-images in \( \hat{\Sigma}^\# \).
Symbolic dynamics for interval maps

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1. $\hat{\pi} \circ \hat{\sigma} = \hat{f} \circ \hat{\pi}$.
2. $\hat{\pi}[\hat{\Sigma}^\#]$ contains $\overline{\text{NUE}^\#}$.
Theorem (L)

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3. Every \( \hat{x} \in \overline{\text{NUE}^\#} \) has finitely many pre-images in \( \hat{\Sigma}^\# \).
Pesin theory in $\hat{M}$

$\hat{M} = u(\hat{x}) - \rho(\hat{x})$, where

$\rho(x) = \text{dist}\{x - 1, x_0, x_1\}.$

$\Psi_{p\hat{x}}, 0 < p \le Q(\hat{x})$.

Transition rule:

1. Overlap: $\Psi_{q\hat{f}^{-1}(\hat{x})} \approx \Psi_{q\hat{y}}$.

2. Control of parameters:
   - $\text{dist}(y_1, x_0) < q$.
   - $u(\hat{f}(\hat{y})) u(\hat{x}) = e^{\pm q}$.
   - $p = \min\{e^{\epsilon q}, Q(\hat{x})\}$. 

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Pesin theory in $\hat{M}$

- $Q(\hat{x}) = \ldots$
Pesin theory in $\hat{M}$

- $Q(\hat{x}) = u(\hat{x})^{-\text{Large power}}$
Pesin theory in $\hat{M}$

- \( Q(\hat{x}) = u(\hat{x})^{-\text{Large power}} \times \rho(\hat{x})^{\text{Large power}} \),
\( Q(\hat{x}) = u(\hat{x})^{-\text{Large power}} \times \rho(\hat{x})^{\text{Large power}}, \) where
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\rho(x) = \text{dist}(\{x_{-1}, x_0, x_1\}, S).
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Pesin theory in $\hat{M}$

- $Q(\hat{x}) = u(\hat{x})^{-\text{Large power}} \times \rho(\hat{x})^{\text{Large power}}$, where
  $\rho(x) = \text{dist}(\{ x_{-1}, x_0, x_1 \}, \mathcal{L})$.

- **Pesin chart:**
○ $Q(\hat{x}) = u(\hat{x})^{-\text{Large power}} \times \rho(\hat{x})^{\text{Large power}}$, where
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○ **Pesin chart:** $\Psi^P_{\hat{x}}$, $0 < p \leq Q(\hat{x})$.  

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- **Pesin chart**: $\Psi^p_\hat{x}$, $0 < p \leq Q(\hat{x})$.

- **Transition rule**: $\Psi^q_\hat{y} \leftarrow \Psi^p_\hat{x}$ if
  - Overlap: $\Psi^q_\hat{f} - 1(\hat{x}) \approx \Psi^q_\hat{y}$.
  - Control of parameters:
    - $\text{dist}(y_1, x_0) < q$.
    - $u(\hat{f}(\hat{y})) - u(\hat{x}) = e^{\pm q}$.
    - $p = \min\{e^{\epsilon q}, Q(\hat{x})\}$.
Pesin theory in $\hat{M}$

- $Q(\hat{x}) = u(\hat{x})^{-}\text{Large power} \times \rho(\hat{x})^{\text{Large power}}$, where $\rho(x) = \text{dist}(\{x_{-1}, x_{0}, x_{1}\}, \mathcal{L})$.

- **Pesin chart**: $\psi_{\hat{x}}^{p}$, $0 < p \leq Q(\hat{x})$.

- **Transition rule**: $\psi_{\hat{y}}^{q} \leftarrow \psi_{\hat{x}}^{p}$ if
  
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○ \( Q(\hat{x}) = u(\hat{x})^{\text{Large power}} \times \rho(\hat{x})^{\text{Large power}} \), where \\
\( \rho(x) = \text{dist}(\{x_{-1}, x_0, x_1\}, \mathcal{S}) \).

○ **Pesin chart:** \( \Psi_x^p \), \( 0 < p \leq Q(\hat{x}) \).

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  1. **Overlap:** \( \Psi_{\hat{f}^{-1}(\hat{x})}^q \approx \Psi_{\hat{y}}^q \).

  2. **Control of parameters:**
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○ **Pesin chart:** $\Psi^p_{\hat{x}}$, $0 < p \leq Q(\hat{x})$.

○ **Transition rule:** $\Psi^q_{\hat{y}} \leftarrow \Psi^p_{\hat{x}}$ if
  (1) **Overlap:** $\Psi^q_{\hat{f}^{-1}(\hat{x})} \approx \Psi^q_{\hat{y}}$.
  (2) **Control of parameters:**
  - $\text{dist}(y_1, x_0) < q$.
  - $\frac{u(\hat{f}(\hat{y}))}{u(\hat{x})} = e^{\pm q}$. 
Pesin theory in $\hat{M}$

- $Q(\hat{x}) = u(\hat{x})^{-}\text{Large power} \times \rho(\hat{x})^{\text{Large power}}$, where $\rho(x) = \text{dist} (\{x_{-1}, x_{0}, x_{1}\}, \mathcal{L})$.

- **Pesin chart**: $\Psi_{\hat{x}}^{p}$, $0 < p \leq Q(\hat{x})$.

- **Transition rule**: $\Psi_{\hat{y}}^{q} \leftarrow \Psi_{\hat{x}}^{p}$ if
  1. **Overlap**: $\Psi_{\hat{f}^{-1}(\hat{x})}^{q} \approx \Psi_{\hat{y}}^{q}$.
  2. **Control of parameters**:
     - $\text{dist}(y_{1}, x_{0}) < q$.
     - $\frac{u(\hat{f}(\hat{y}))}{u(\hat{x})} = e^{\pm q}$.
     - $p = \min \{e^{\epsilon} q, Q(\hat{x})\}$.
Hofbauer tower \times previous theorem
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Thanks!