Non-hyperbolic measures in partially hyperbolic diffeormorphisms

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joint work with L. J. Díaz, M. Rams, B. Santiago

Hyperbolicity – Definition

[Oseledets]: μ be ergodic f-invariant Borel probability measure:

- $\Gamma \subset M$ with $\mu(\Gamma) = 1$
- Df-invariant splitting $T_\Gamma M = E_1 \oplus \cdots \oplus E_k$
- Lyapunov exponents $\chi_1(\mu) \leq \ldots \leq \chi_k(\mu)$, $k \leq \dim M$, $x \in \Gamma$ for $v \in E^i_x \setminus \{0\}$, $i \in \{1, \ldots, k\}$

$$\chi_i(x) \stackrel{\text{def}}{=} \lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n_x(v)\| = \chi_i(\mu).$$

μ hyperbolic if $\chi_i(\mu) \neq 0$ $\forall i$.

$\Gamma \subset M$ is hyperbolic (of saddle type) if compact and f-invariant with Df-invariant splitting $T_\Gamma M = E^s \oplus E^u$ so that (after a change of metric)

$$\log \|Df|_{E^s}\| \leq \chi_s < 0 < \chi_u \leq \log \|Df|_{E^u}\|.$$

Γ basic if hyperbolic, transitive (dense orbit), isolated ($\Gamma = \bigcap_{k \in \mathbb{Z}} f^k(U)$, U open).

Γ horseshoe if basic and Cantor.

Hyperbolicity

... and some of its consequences in the space of ergodic measures $\mathcal{M}_{\mathrm{erg}}$

Γ hyperbolic $\Rightarrow$ $\mathcal{M}_{\mathrm{erg}}(\Gamma)$ has only hyperbolic measures

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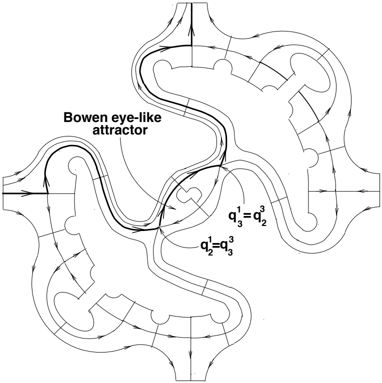

Bowen's eye-like construction:
only saddle-type hyperbolic measures

[Baladi-Bonatti-Schmitt'99]

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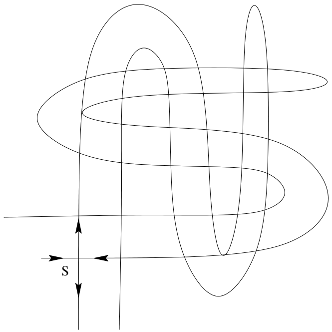

Hénon maps with internal tangencies: spectrum of Lyapunov exponents is

$$0 < \chi_{\min} \leq |\chi_i(\mu)|$$

[Cao-Luzzatto-Rios'06]

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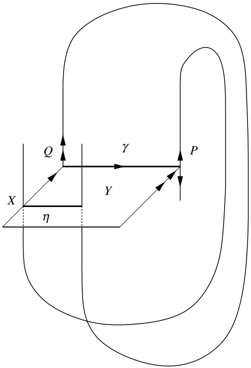

porcupine-like horseshoes:
saddles of different indices
spectrum of Lyapunov exponents for $\mu \neq \delta_P$ is

$$\chi^{\mathrm{c}}(\mu) \leq \chi_{\max} < 0 < \chi^{\mathrm{c}}(\delta_Q)$$

(saddle = hyperbolic periodic orbit)

[Díaz-Horita-Rios-Sambarino'09]

Hyperbolicity

... and some of its consequences in the space of ergodic measures $\mathcal{M}_{\mathrm{erg}}$

Assuming f $C^{\mathbf{2}}$ or $C^{\mathbf{1}}$+dominated splitting:

μ hyperbolic ergodic $\Rightarrow$ exist plenty of periodic orbits $\Rightarrow$ exist horseshoes

[Katok'80, Katok-Mendoza'95]

$$\mathcal{M}(\Gamma) = \overline{\mathrm{conv}} \mathcal{M}_{\mathrm{erg}}(\Gamma)$$

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[Katok'80, Katok-Mendoza'95]

Γ basic $\Rightarrow$

$$\mathcal{M}(\Gamma) = \overline{\rm conv}\mathcal{M}_{\rm erg}(\Gamma) = \overline{\mathcal{M}_{\rm per}(\Gamma)}$$

is Poulsen simplex (dense extremes)

[Sigmund'70s]

Nonhyperbolic dynamics

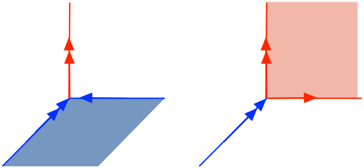
... seen in the space of ergodic measures $\mathcal{M}_{\rm erg}$

"To what extend is a (generic) dynamical system hyperbolic?"

[Gorodetski-Ilyashenko-Kleptsyn-Nalski'05]

How does nonhyperbolic behavior occur?

- critical behavior (tangencies)
- parabolic (topologically hyperbolic) behavior
- coexistence of hyperbolic periodic orbits of different indices



How can different types of (non-)hyperbolicity be distinguished?

To what extend ergodic theory can detect hyperbolic dynamics?

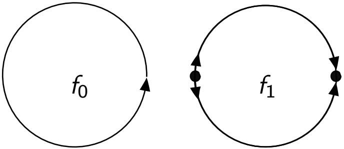
Coexistence of hyperbolicity

... very simple model

Consider step skew-product model with circle fiber maps

$$f\colon \Sigma_2 \times \mathbb{S}^1 \to \Sigma_2 \times \mathbb{S}^1, \quad (\xi, x) \mapsto f(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)).$$

where $\sigma\colon \Sigma_2 = \{0,1\}^{\mathbb{Z}} \to \Sigma_2$ models horseshoe map in the base.



- f_0 irrational rotation.
- f_1 Morse-Smale

Motivated by: [Gorodetskii-Ilyashenko-Kleptsyn-Nalskii'05]

Hypotheses (H): C^1 diffeomorphisms $f\colon M \to M$

- transitive
- partially hyperbolic $TM = E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$ but nonhyperbolic
- $\dim E^{\mathrm{c}} = 1$, a closed curve tangent to E^{c}

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- blender-horseshoes (special basic sets)

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$\Rightarrow$ open and dense in former [Bonatti-Díaz'12,Bonatti-Díaz-Ures'02,RodriguezHertz^2-Ures'07]

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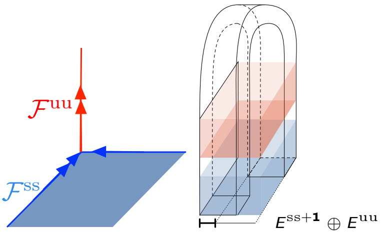
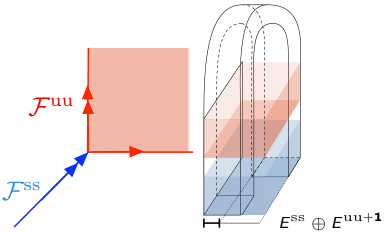




nonhyperbolic measures with positive entropy [Bochi-Bonatti-Díaz'16]

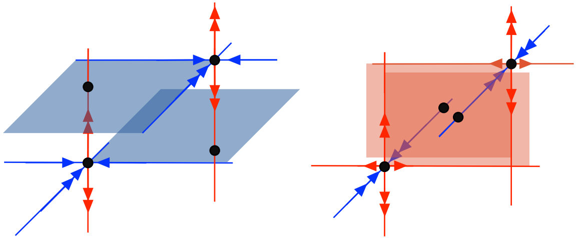
(Non-)Hyperbolicity

... is intermingled

- coexisting saddles with splitting

$$E^{\mathrm{ss}+1} \oplus E^{\mathrm{uu}} \quad \text{and} \quad E^{\mathrm{ss}} \oplus E^{\mathrm{uu}+1}$$

with rich homoclinic relations (manifolds of orbits intersect cyclically transversally)



- saturation by horseshoes of types $E^{\mathrm{ss}+1} \oplus E^{\mathrm{uu}}$ and $E^{\mathrm{ss}} \oplus E^{\mathrm{uu}+1}$
- nonhyperbolic ergodic measures with splitting

$$E^{\mathrm{ss}} \oplus E^{0} \oplus E^{\mathrm{uu}}$$

Nonhyperbolicity

... is only seen in the *central* direction

E^{c} is a Oseledets subbundle and defines the central Lyapunov exponent

$$\chi^{\mathrm{c}}(\mu) = \int \varphi^{\mathrm{c}}\, d\mu, \quad \varphi^{\mathrm{c}} \stackrel{\mathrm{def}}{=} \log \|Df|_{E^{\mathrm{c}}}\|$$

and the spectrum of central exponents χ^{c} splits as

$$[\chi_{\min}, 0) \cup \{0\} \cup (0, \chi_{\max}].$$

and accordingly splits as

$$\mathcal{M}_{\mathrm{erg}} = \mathcal{M}^{-}_{\mathrm{erg}} \cup \mathcal{M}^{0}_{\mathrm{erg}} \cup \mathcal{M}^{+}_{\mathrm{erg}}.$$

$\mathcal{M}^{\mp}_{\mathrm{erg}} \stackrel{\mathrm{def}}{=} \{\mu \in \mathcal{M}_{\mathrm{erg}} \colon \chi^{\mathrm{c}}(\mu) \lessgtr 0\}$ hyperbolic

$\mathcal{M}^{0}_{\mathrm{erg}} \stackrel{\mathrm{def}}{=} \{\mu \in \mathcal{M}_{\mathrm{erg}} \colon \chi^{\mathrm{c}}(\mu) = 0\}$ nonhyperbolic

Theorem (Hyperbolic approximation of nonhyperbolicity)

For $\mu \in \mathcal{M}^0_{\mathrm{erg}}$ *with* $h(\mu) > 0$, *for* $\delta > 0$ *there exists a horseshoe* Γ^+ *s.t.*

$$h_{\mathrm{top}}(f|_{\Gamma^+}) \geq h(\mu) - \delta \quad \textit{(approximation in entropy)}$$

$$d_{w*}(\nu, \mu) < \delta \quad \forall \nu \in \mathcal{M}_{\mathrm{erg}}(f|_{\Gamma^+}) \quad \textit{and} \quad 0 < \chi^{\mathrm{c}}(\nu) < \delta \quad \textit{(weak*)}$$

For $\mu^- \in \mathcal{M}^-_{\mathrm{erg}}$ *with* $h(\mu^-) > 0$, *for* $\delta > 0$ *there exists a horseshoe* Γ^+ *s.t.*

$$h_{\mathrm{top}}(f|_{\Gamma^+}) \geq \frac{h(\mu^-)}{1 + C(|\chi(\mu^-)| + \delta)} \qquad \textit{(even from "the other side")}$$

with $< \chi^{\mathrm{c}}(\nu) < \delta$.

Analogously with Γ^- *and* $-\delta < \chi^{\mathrm{c}}(\nu) < 0$ *for* $\nu \in \mathcal{M}_{\mathrm{erg}}(f|_{\Gamma^-})$. *Analogously for* $\mu^+ \in \mathcal{M}^+_{\mathrm{erg}}$.

C^1 partially hyperbolic diffeomorphisms [Díaz-G-Santiago]
step skew-products [Díaz-G-Rams'17]
partial results [Yang-Zhang]
C^2 diffeomorphisms [Tahzibi-Yang]

To explain main ingredients: translate ...

... robustly transitive dynamics into step skew-products

Hypotheses (H'): step skew-product model with C^1 circle fiber maps

$$f\colon \Sigma_2 \times \mathbb{S}^1 \to \Sigma_2 \times \mathbb{S}^1, \quad (\xi, x) \mapsto f(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)).$$

- f is transitive
- satisfies Axioms Controlled Expanding Covering$\pm$ and Accessibility$\pm$

Motivated by: [Gorodetskii-Ilyashenko-Kleptsyn-Nalskii'05]

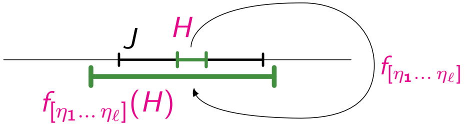
Notation for induced IFS:

$$f_{[\xi_0 \ldots \xi_n]} \stackrel{\text{def}}{=} f_{\xi_n} \circ \cdots \circ f_{\xi_0}$$

Axiomatic approach: CEC± and Acc±

There is a (blending) closed interval $J \subset \mathbb{S}^1$ such that:

- **Transitivity:** Exists a point of $\mathbb{S}^1$ with dense forward orbit by the IFS
- **Controlled expanding covering:** there is $K > 1$ and for every interval H intersecting J there is $(\eta_1 \dots \eta_\ell)$, $\ell \simeq |\log |H||$:
  - (covering) $J \subset f_{[\eta_1 \dots \eta_\ell]}(H)$,
  - (expansion) $\log |(f_{[\eta_1 \dots \eta_\ell]})'(x)| \geq K \, \ell$ for $x \in H$



- **Accessibility:** The orbit by the IFS of J covers $\mathbb{S}^1$

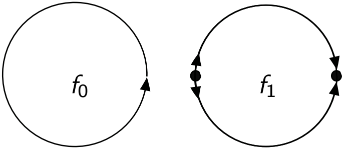
Similarly, backward properties

[Díaz-G-Rams'17]

Examples. Systems that satisfy Axioms

rotation-expansion-contraction

Motivated by: [Gorodetskii-Ilyashenko-Kleptsyn-Nalskii'05]



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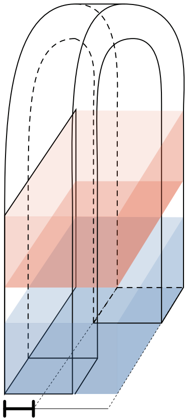
Examples:
induced projective action of $\mathrm{PSL}_2(\mathbb{R})$ matrix cocycle $\mathbb{A} = \{A_{\mathrm{ell}}, A_{\mathrm{hyp}}\}$.

Examples. Systems that satisfy Axioms

one-dimensional blenders

Motivated by: [Bonatti-Díaz'96], [Bonatti-Díaz-Ures'02]



IFS $\{f_i\}_{i=0,1}$, has *expanding blender* if: there are $[c,d] \subset [a,b] \subset \mathbb{S}^1$ so that

- (expansion) $f_0'(x) \ge \beta > 1 \ \forall x \in [a,b]$
- (boundary condition) $f_0(a) = f_1(c) = a$
- (covering and invariance) $f_0([a,d]) = [a,b]$ and $f_1([c,b]) \subset [a,b]$

It has a *contracting blender* if $\{f_i^{-1}\}_i$ does.

$\forall x \in \mathbb{S}^1$ there is some inside the expanding blender (a,b).

analogously: backward iterates

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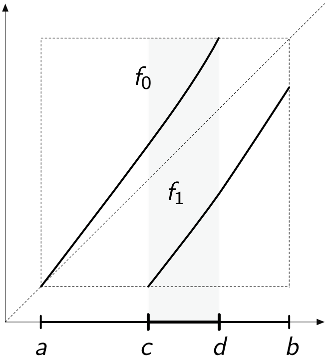

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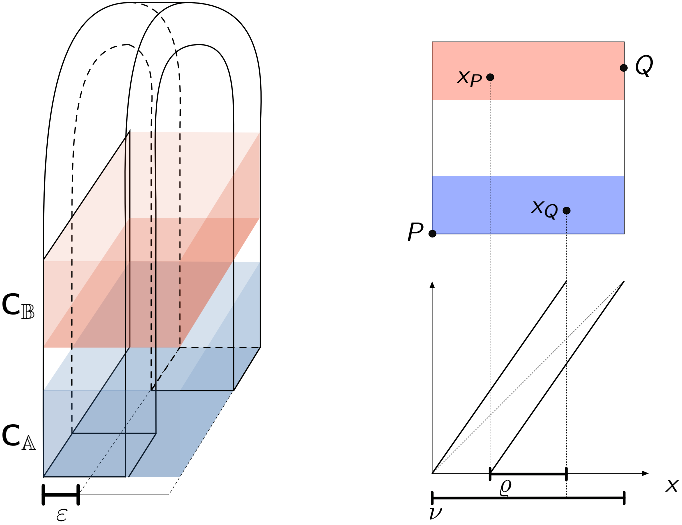

Figure: (affine) blender-horseshoe

$\Rightarrow$ [Bonatti-Díaz-Crovisier-Wilkinson'17 "What is … a blender?"]

Dictionary

... translating from step skew-products to partially hyperbolic diffeomorphisms

step skew-product map	C^1-robustly transitive nonhyperbolic diffeomorphism
• contracting blender • expanding blender	• center-stable blender-horseshoe • center-unstable blender-horseshoe [Bonatti-Díaz'12, Bonatti-Díaz-Ures'02]
• every point has forward iterate in interior of blender domain	• minimality of unstable foliations
• every point has backward iterate in interior of blender domain	• minimality of stable foliations
	extra difficulty: E^{c} may not be integrable, no dynamical coherence $\Rightarrow$ fake invariant foliations tangent to cone field about E^{c} [Burns-Wilkinson'10]

Dictionary was described in [Díaz-G-Rams'17].
Translation was done in [Díaz-G-Santiago].

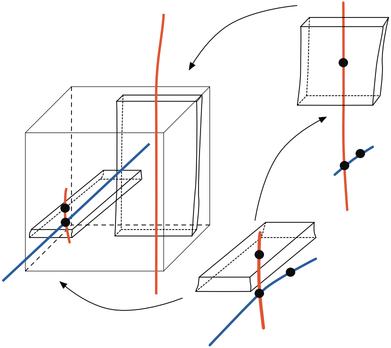

Figure: blender-horseshoe

difficulty:
E^c may not be integrable
no dynamical coherence
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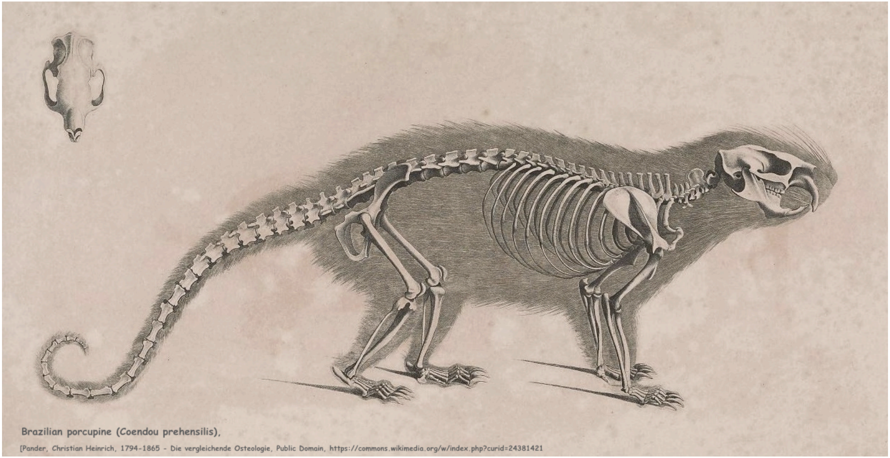
tangent to cone field about E^c

Skeletons

... ingredients to prove Theorem (Hyperbolic approximation of nonhyperbolicity)



Skeletons

... ingredients to prove Theorem (Hyperbolic approximation of nonhyperbolicity)

F has the skeleton property relative to $J \subset \mathbb{S}^1$, $h \geq 0$, $\alpha \geq 0$ if:

There exist *connecting times* $m_{\rm b}, m_{\rm f} \in \mathbb{N}$:
$\forall m \geq n_0$ exists a finite set $\mathfrak{X} = \{(\xi^i, x_i)\}$ of points:

(i) $\mathrm{card}(\mathfrak{X}) \asymp e^{mh}$,

(ii) the sequences $(\xi^i_0 \ldots \xi^i_{m-1})$ are all different,

(iii) $\dfrac{1}{n} \log |(f_{[\xi^i_0 \ldots \xi^i_{n-1}]})'(x_i)| \asymp \alpha \; \forall n = 0, \ldots, m.$

connecting sequences $(\theta^i_1 \ldots \theta^i_{r_i})$, $r_i \leq m_{\rm f}$, $(\beta^i_1 \ldots \beta^i_{s_i})$, $s_i \leq m_{\rm b}$, and $x_i' \in J$:

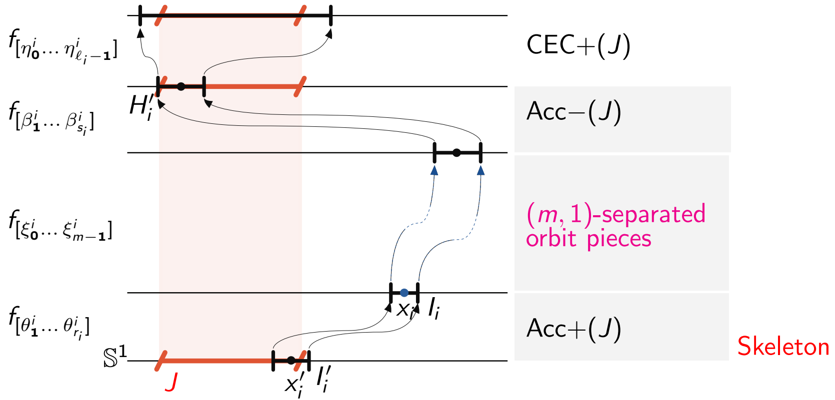
(iv) $f_{[\theta^i_1 \ldots \theta^i_{r_i}]}(x_i') = x_i$,

(v) $f_{[\xi^i_0 \ldots \xi^i_{m-1} \beta^i_1 \ldots \beta^i_{s_i}]}(x_i) \in J$.

Skeletons

... ingredients to prove Theorem (Hyperbolic approximation): Let μ be nonhyperbolic

$$\mathrm{card}\{(\xi^i, x_i)\} \asymp e^{mh(\mu)} \qquad \text{and} \qquad \frac{1}{m}\log|(f_{[\xi_0^i \dots \xi_{m-1}^i]})'(x_i)| \asymp 0 = \chi^{\mathrm{c}}(\mu)$$



Theorem (Topology of space of ergodic measures)

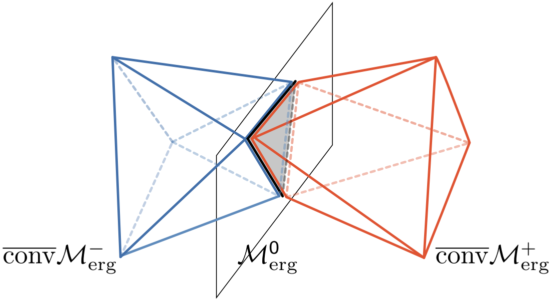
Assuming the Hypotheses.

- $\mathcal{M}^0_{\mathrm{erg}} \subset \overline{\mathrm{conv}}\mathcal{M}^-_{\mathrm{erg}} \cap \overline{\mathrm{conv}}\mathcal{M}^+_{\mathrm{erg}}$.
- *Each of the sets* $\mathcal{M}^-_{\mathrm{erg}}$ *and* $\mathcal{M}^+_{\mathrm{erg}}$ *is arcwise connected.*

C^1 partially hyperbolic diffeomorphisms [Díaz-G-Santiago]

step skew-products [Díaz-G-Rams'17]

$C^{1+\alpha}$ diffeomorphisms [Gorodetski-Pesin'17]



$\Rightarrow$ No unconnected component.

Which type of hyperbolicity prevails

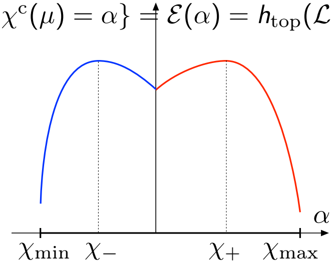
... in terms of entropy, for example?

Study the restricted variational principle for entropy and the level set

$$\sup\{h(\mu)\colon \mu \in \mathcal{M}_{\mathrm{erg}}, \chi^{\mathrm{c}}(\mu) = \alpha\}, \quad \mathcal{L}(\alpha) \stackrel{\mathrm{def}}{=} \{x\colon \chi^{\mathrm{c}}(x) = \alpha\}.$$

Theorem (Multifractal analysis for entropy of Lyapunov exponents)

$$\sup\{h(\mu)\colon \mu \in \mathcal{M}_{\mathrm{erg}}, \chi^{\mathrm{c}}(\mu) = \alpha\} = \mathcal{E}(\alpha) = h_{\mathrm{top}}(\mathcal{L}(\alpha)) \text{ for } \alpha \neq 0$$



spectrum $[\chi_{\min}, \chi_{\max}]$, $\mathcal{E}(\alpha)$ = *Legendre-Fenchel transform of variational pressures*

done for step skew-products in [Díaz-G-Rams'17]

Exhausting families

... ingredients to prove Theorem (Multifractal analysis for entropy of Lyapunov exponents)

Given $\mathcal{N} \subset \mathcal{M}_{\mathrm{erg}}(X)$ define $\varphi(\mathcal{N}) \stackrel{\mathrm{def}}{=} \left\{ \int \varphi \, d\mu \colon \mu \in \mathcal{N} \right\}$ spectrum of Lyapunov

$$\mathcal{P}_{\mathcal{N}}(\varphi) \stackrel{\mathrm{def}}{=} \sup_{\mu \in \mathcal{N}} \left(h(\mu) + \int \varphi \, d\mu \right)$$ restricted variational pressure

$X_1 \subset \ldots \subset X_i \subset \ldots \subset X$ of compact f-invariant sets are $\mathcal{N}$-exhausting if

- $\mathcal{N}_i \stackrel{\mathrm{def}}{=} \mathcal{M}_{\mathrm{erg}}(f|_{X_i}) \subset \mathcal{N}$, $f|_{X_i}$ has specification property (X_i basic),
- convex conjugates on X_i:

$$\mathcal{E}_i(\alpha) \stackrel{\mathrm{def}}{=} \sup\{h(\mu) \colon \mu \in \mathcal{N}_i, \varphi(\mu) = \alpha\} = \inf_{q \in \mathbb{R}} \left(\mathcal{P}_{\mathcal{N}_i}(\varphi) - q\alpha \right)$$

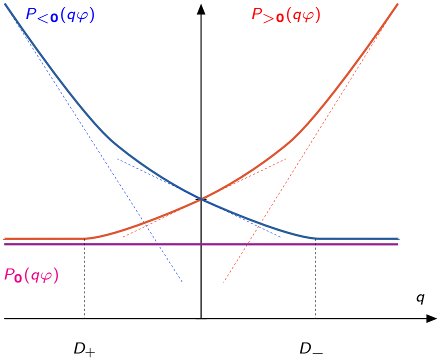
$$\mathcal{P}_{\mathcal{N}}(q\varphi) = \lim_{i \to \infty} P_{f|X_i}(q\varphi) \quad \forall q \in \mathbb{R}$$

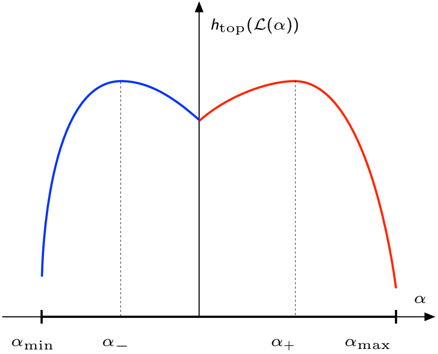
- spectrum on X_i exhausts all: $\overline{\bigcup_i \mathcal{N}_i} = \mathcal{N}$

Apply to $\mathcal{N} = \mathcal{M}^{\mp}_{\mathrm{erg}}$ by finding exhausting horseshoes.

Nonhyperbolic step skew products

All results combined – $h_{\mathrm{top}}(\mathcal{L}(\alpha))$





Nonhyperbolic step skew products

The still missing piece: $\alpha = 0$. Concavity indeed is a useful property.

Proposition (continuity of spectrum & smaller entropy at $\alpha = 0$)

$$h_{\text{top}}(\mathcal{L}(0)) = \mathcal{E}(0) \stackrel{\text{def}}{=} \limsup_{\beta \to 0} \mathcal{E}_{\mathcal{N}}(\beta) < h_{\text{top}}(f)$$

Nonhyperbolic step skew products

The still missing piece: $\alpha = 0$. Concavity indeed is a useful property.

Proposition (continuity of spectrum & smaller entropy at $\alpha = 0$)

$$h_{\mathrm{top}}(\mathcal{L}(0)) = \mathcal{E}(0) \stackrel{\mathrm{def}}{=} \limsup_{\beta \to 0} \mathcal{E}_{\mathcal{N}}(\beta) < h_{\mathrm{top}}(f)$$

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By contradiction: $h_{\mathrm{top}}(\mathcal{L}(0)) = 0 \Rightarrow \quad \mathcal{E}(0) = 0 \Rightarrow D_R\mathcal{E}(0) = 0$
$\Rightarrow h_{\mathrm{top}}(\mathcal{L}(0)) = \text{maximum}$

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By contradiction: $h_{\rm top}(\mathcal{L}(0)) = 0 \Rightarrow \quad \mathcal{E}(0) = 0 \Rightarrow D_R\mathcal{E}(0) = 0$
$\Rightarrow h_{\rm top}(\mathcal{L}(0)) = \text{maximum} \Rightarrow \text{global maximum}$

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By contradiction: $h_{\rm top}(\mathcal{L}(0)) = 0 \Rightarrow \quad \mathcal{E}(0) = 0 \Rightarrow D_R\mathcal{E}(0) = 0$

$\Rightarrow h_{\rm top}(\mathcal{L}(0)) = \text{maximum} \Rightarrow \text{global maximum} \Rightarrow h_{\rm top}(\mathcal{L}(0)) = \log N$

$\mathrm{SL}(2,\mathbb{R})$ – classification of its elements

Given $\mathbb{A} := \{A_1, \dots, A_N\} \in (\mathrm{SL}(2,\mathbb{R}))^N$, $\xi^+ \in \Sigma_N^+$, $n \geq 1$

$$\mathbb{A}^n(\xi^+) := A_{\xi_{n-1}} \dots A_{\xi_0}$$

Elements in $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ are classified by trace:

hyperbolic ($|\mathrm{tr}\, A| > 2$), parabolic ($|\mathrm{tr}\, A| = 2$), elliptic ($|\mathrm{tr}\, A| < 2$), which each are conjugate into one of three subgroups, respectively:

$$A := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha > 0 \right\},\ N := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\},\ K := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}.$$

Consider the semi-group $\langle \mathbb{A} \rangle := \langle A_1, \dots, A_N \rangle$ and the sets

$\mathcal{H} = \{\mathbb{A} \in (\mathrm{SL}(2,\mathbb{R}))^N \colon \mathbb{A}$ is hyperbolic, i.e. $\langle \mathbb{A} \rangle$ hyperbolic$\}$
$\mathcal{E} = \{\mathbb{A} \in (\mathrm{SL}(2,\mathbb{R}))^N \colon \mathbb{A}$ is elliptic, i.e. $\langle \mathbb{A} \rangle$ has elliptic element$\}$

Theorem (Yoccoz(-Avila)'04)

$\mathcal{E} \cup \mathcal{H}$ is open and dense in $(SL(2,\mathbb{R}))^N$, more precisely $\mathcal{H}^c = \overline{\mathcal{E}}$.

$\mathrm{SL}(2,\mathbb{R})$ – classification of its elements

elliptic with some hyperbolicity

Study action of $A \in \mathrm{SL}(2,\mathbb{R})$ on projective line $\mathbb{P}^1$ by diffeomorphism f_A.

For A hyperbolic, there are one attracting and one repelling fixed point and strictly absorbing intervals $I_A^- = \{v \colon |f_A'(v)| < 1\}$, $I_A^+ = \{v \colon |f_A'(v)| > 1\}$:

$$f_A(\overline{I_A^-}) \subset I_A^-, \quad f_A^{-1}(\overline{I_A^+}) \subset I_A^+.$$

Consider $\mathcal{E}_{\mathrm{shyp}} := \{\mathbb{A} \in \mathcal{E}$ with "some hyperbolicity"$\}$:

- There exists $A \in \langle \mathbb{A} \rangle$ hyperbolic.
- There exists $M \geq 1$ such that $\forall v \in \mathbb{P}^1\ \exists \theta^+, \beta^+ \in \Sigma_N^+$: for some $s, r \leq M$

$$f_{A_{\theta^+_{s-1}}} \circ \ldots \circ f_{A_{\theta^+_0}}(v) \in I_A^+, \quad f_{A_{\beta^+_{r-1}}} \circ \ldots \circ f_{A_{\beta^+_0}}(v) \in I_A^-$$

Lemma (consequence of Avila-Bochi-Yoccoz'10)

$\mathcal{E}_{\mathrm{shyp}}$ is an open and dense subset of $\mathcal{E}$ (in $\mathcal{E}$).