Optimization of Lyapunov Exponents

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Thermodynamic Formalism in Dynamical Systems
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General setting for the talk

- $X =$ compact metric space
- $T : X \to X$ continuous map
- $\mathcal{M}_T :=$ set of T-invariant Borel probability measures (compact convex)
- $\mathcal{M}_T^{\text{erg}} :=$ subset of ergodic measures $= \text{ext}(\mathcal{M}_T)$.

Part 1

Commutative ergodic optimization: Birkhoff averages

References: Surveys by O. Jenkinson.

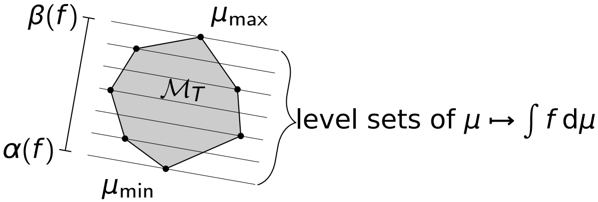
- *Ergodic Optimization*, Discrete and Cont. Dyn. Sys. A, vol. 15 (2006), pp. 197–224.
- *Ergodic Optimization in Dynamical Systems*, Ergodic Theory Dynam. Systems (2018; online)

Ergodic optimization of Birkhoff averages

Given a continuous function $f\colon X \to \mathbb{R}$ ("potential"),

$$\left\{ \int f \, d\mu \; ; \; \mu \in \mathcal{M}_T \right\} =: [\alpha(f), \beta(f)]$$

$\mu \in \mathcal{M}_T$ s.t. $\int f \, d\mu = \beta(f)$ is called a **maximizing measure**.



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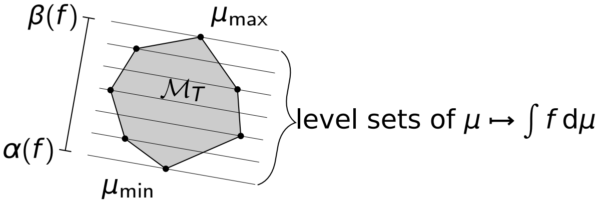

Note: **Ergodic** maximizing measures always exist. In particular, uniqueness $\Rightarrow$ ergodicity.

Expressing $\beta(f)$ in terms of Birkhoff averages

Birkhoff sum $f^{(n)} := f + f \circ T + \cdots + f \circ T^{n-1}$

$$\beta(f) = \sup_{x \in X} \limsup_{n \to \infty} \frac{f^{(n)}(x)}{n}$$
$$= \lim_{n \to \infty} \sup_{x \in X} \frac{f^{(n)}(x)}{n}$$

Ergodic optimization of Birkhoff averages

Meta-Problem

Describe maximizing measures.

Maximizing measures: Generic uniqueness

Theorem (Conze–Guivarch, Jenkinson, . . .)

Let $\mathcal{F}$ be any "reasonable"(*) space $\mathcal{F}$ of continuous functions.
For generic f in the maximizing measure is ***unique****.*

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Let $\mathcal{F}$ be any "reasonable"(*) space $\mathcal{F}$ of continuous functions.
*For generic f in the maximizing measure is **unique**.*

(*) a vector space $\mathcal{F}$ continuously and densely embedded in $C^0(X)$.

Generic set: intersection of a countable family of open and dense sets.

The inverse problem

Theorem (Jenkinson)

Given $\mu \in \mathcal{M}_T^{\mathrm{erg}}$, there exists $f \in C^0(X)$ such that μ is the unique maximizing measure for f.

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Given $\mu \in \mathcal{M}_T^{\mathrm{erg}}$, there exists $f \in C^0(X)$ such that μ is the unique maximizing measure for f.

If μ has finite support then f can be taken C^∞.

How regular f can be taken, in general? Not much. . .

Maximizing sets

Suppose:

- $T : X \to X$ is "**hyperbolic**" (e.g. uniformly expanding, Anosov);
- $f : X \to \mathbb{R}$ is "**regular**" (at least Hölder).

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Theorem (Subordination principle)

In this good setting, there is a ***maximizing set****: a T-invariant compact set $K \subseteq X$ such that*

$$\mu \text{ is maximizing} \iff \operatorname{supp} \mu \subseteq K$$

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- It is **false** if f is only C^0 (by the previous theorem)
- It is a corollary of the **Mañé Lemma** (or **Revelation Lemma**). Several formulations: Mañé'92, Conze–Guivarc'h'93, Fathi'97, Savchenko'99, Bousch'00, Contreras–Lopes–Thieullen'01, Lopes–Thieullen'03, Pollicott–Sharp'04, Bousch'11).

Expected panorama for the good setting

Meta-Conjecture (∼ Hunt–Ott, Phys. Rev. 1996)

Suppose $T: X \to X$ is chaotic

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Then for typical

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Many results (Contreras, Lopes, Thieullen'01; Morris'08); the best one is:

Theorem (Contreras'16)

T unif. expanding $\Rightarrow$ for generic Lipschitz f's (actually all f's in an ***open*** *and dense subset), the maximizing measure is supported on a periodic orbit.*

Only result with a probabilistic notion of typicality (**prevalence**):

A nice example

Conze–Guivarch'93, Hunt–Ott'96, Jenkinson'96, Bousch'00

$T(x) = 2x \bmod 2\pi$ on the circle $X := {}^{\mathbb{R}}/_{2\pi\mathbb{Z}}$

$f =$ trigonometric polynomial of deg. 1
WLOG, $f(x) = f_\theta(x) = \cos(x - \theta)$

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Theorem (Bousch'00)

For every $\theta \in [0, 2\pi]$, the function f_θ has a unique maximizing measure μ_θ, and it has zero entropy (actually, Sturmian).

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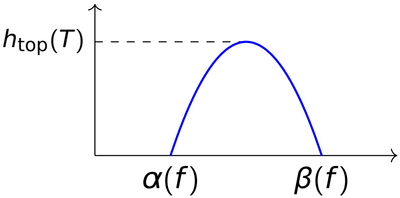
Theorem (Bousch'00)

For every $\theta \in [0, 2\pi]$, the function f_θ has a unique maximizing measure μ_θ, and it has zero entropy (actually, Sturmian).
Furthermore, for Lebesgue-a.e. θ (actually, all θ outside a set of Hausdorff dim. 0), μ_θ is supported on a periodic orbit.

A more complete picture: Multifractal analysis

Let (T, f) be in the setting of the meta-conjecture.
For $t \in [\alpha(f), \beta(f)]$, let:

$$\mathfrak{H}_f(t) := \sup \left\{ h(\mu, T) \, ; \, \mu \in \mathcal{M}_T, \ \int f \, \mathrm{d}\mu = t \right\}$$



(concave graph)

Part 2

Non-commutative ergodic optimization: (Top) Lyapunov exponent

Replace the scalar function f by a matrix-valued function ("cocycle"):

$$F\colon X \to \mathrm{Mat}(d \times d, \mathbb{R}) \text{ or } \mathrm{GL}(d, \mathbb{R})$$

The Birkhoff sums are replaced by products:

$$F^{(n)}(x) := F(T^{n-1}x) \cdots F(Tx)F(x)\,.$$

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Top Lyapunov exponent:

$$\lambda_1(F, x) := \lim_{n\to\infty} \frac{1}{n} \log \|F^{(n)}(x)\| \quad \text{(if it exists)}$$

For any $\mu \in \mathcal{M}_T$, the limit exists for μ-a.e. $x \in X$.

$$\lambda_1(F, \mu) := \int \lambda_1(F, x) \, \mathrm{d}\mu(x)$$

Optimization of the top Lyapunov exponent

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$

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Basic difficulty:
$\mu \in \mathcal{M}_T \mapsto \lambda_1(F, \mu)$ is **not continuous**, in general.
It is **upper semi-continuous**, at least.

Optimization of the top Lyapunov exponent

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F,\mu) \quad ☹ \text{ not necessarily attained}$$

$$\beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F,\mu) \quad ☺ \text{ always attained}$$

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Note: For step cocycles, $e^{\beta(F)}$ is called **joint spectral radius** – Rota, Strang'60; Daubechies, Lagarias'92, . . .

Example without λ_1-minimizing measure

Step cocycle $T\colon \{0,1\}^{\mathbb{N}} \hookleftarrow$ shift, $F(x) = A_{x_0}$ where $A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Claim

$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F,\mu) = -\log 2$, but the inf is not attained.

Proof.

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Proof.

$$\boxed{\mu_n := \delta_{(0^n 1)^\infty}} \Rightarrow \lambda_1(F,\mu_n) = \frac{1}{n+1} \log \text{spec. rad.} \begin{pmatrix} 0 & -2^{-3n} \\ 2^n & 0 \end{pmatrix}$$

$$= \frac{1}{n+1} \log \det(\dots) = \boxed{-\frac{n}{n+1} \log 2}$$

$$\searrow -\log 2.$$

So $\boxed{\alpha(F) \le -\log 2}$.

Discontinuity: $\lambda_1(F, \lim \mu_n) \ne \lim \lambda_1(F,\mu_n)$.

□

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Proof.

$$\lambda_1(F,\mu) \overset{(1)}{\geq} \frac{\lambda_1(F,\mu)+\lambda_2(F,\mu)}{2} = \int \frac{1}{2} \log \underbrace{|\det F(x)|}_{\geq 1/4} \, d\mu(x) \overset{(2)}{\geq} -\log 2 \, .$$

So $\boxed{\alpha(F) \geq -\log 2}$ and therefore $\boxed{\alpha(F) = -\log 2}$.
Moreover, (2) becomes "$=$" iff $\mu = \delta_{0^\infty}$, but then (1) is "$>$". So no μ attains $\lambda_1(F,\mu) = -\log 2$. $\square$

Expected panorama for λ_1-maximization

Meta-Conjecture

Suppose T is chaotic (unif. expanding / unif. hyperbolic / ...). Then for typical (topological sense / probabilistic sense) regular (Hölder / ... / analytic) cocycles F, the λ_1-maximizing measure has low complexity (zero topological entropy / ... / supported on a periodic orbit).

Some initial results

Similarly to the commutative **subordination principle**:

Theorem (Bochi–Garibaldi)

*Suppose T is a hyperbolic homeomorphism, and that F is a (strongly) fiber-bunched cocycle. Then there exists a **maximizing set**: a T-invariant compact set $K \subseteq X$ such that*

$$\mu \text{ is } \lambda_1\text{-maximizing} \iff \operatorname{supp} \mu \subseteq K$$

This is actually a corollary of a **version of Mañé Lemma for cocycles**:

J. Bochi, E. Garibaldi. Extremal norms for fiber bunched cocycles. ArXiv 1806.xxxxx

Part 3

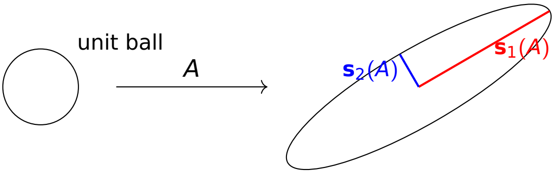
Non-commutative ergodic optimization: Full Lyapunov spectra

The other Lyapunov exponents

$T\colon X \to X$, $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ as before.
For each $i \in \{1, 2, \ldots, d\}$, and $x \in X$, let

$$\lambda_i(F,x) := \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{s}_i(F^{(n)}(x)) \quad \text{(if it exists)}$$

where $\mathbf{s}_i(\cdot) :=$ i-th singular value.



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For any $\mu \in \mathcal{M}_T$, these limit exist for μ-a.e. $x \in X$.
If μ is **ergodic**, then $\lambda_i(F, \cdot)$ is μ-a.e. equal to some constant $\lambda_i(F, \mu)$.

Lyapunov spectrum of a cocycle

Given (T, F), the **Lyapunov vector** of $\mu \in \mathcal{M}_T^{\text{erg}}$ is:

$$\vec{\lambda}(F, \mu) := \big(\lambda_1(F, \mu), \dots, \lambda_d(F, \mu)\big)$$

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The **Lyapunov spectrum** of (T, F) is:

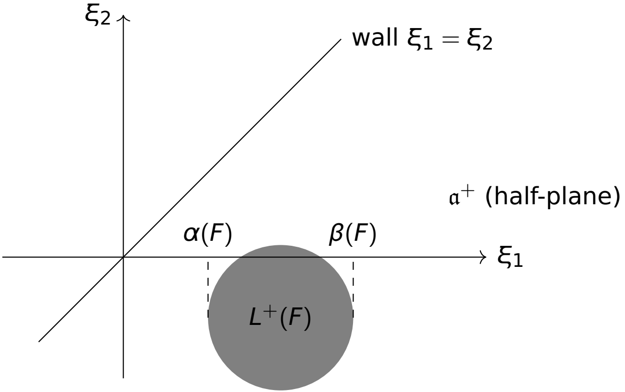
$$L^+(F) := \left\{\vec{\lambda}(F, \mu)\,;\, \mu \in \mathcal{M}_T^{\mathrm{erg}}\right\},$$

which is a subset of the **positive chamber**:

$$\mathfrak{a}^+ := \left\{(\xi_1, \dots, \xi_d) \in \mathbb{R}^d\,;\, \xi_1 \geq \cdots \geq \xi_d\right\}.$$

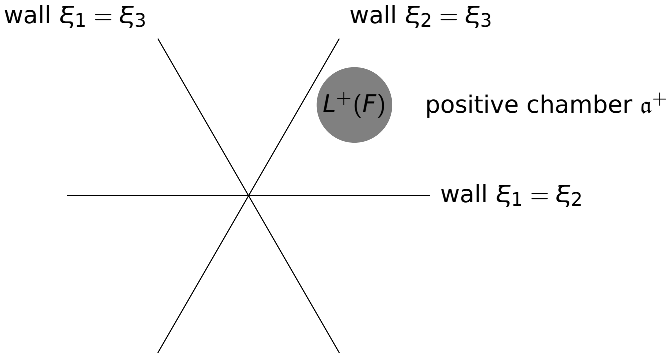
Lyapunov spectrum of a cocycle

$$L^+(F) := \left\{ \vec{\lambda}(F,\mu) \, ; \, \mu \in \mathcal{M}_T^{\mathrm{erg}} \right\}$$
$$\subset \qquad \mathfrak{a}^+ := \left\{ (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \, ; \, \xi_1 \geq \cdots \geq \xi_d \right\}.$$



If F takes values in $\mathrm{SL}(3,\mathbb{R})$ then the Lyapunov spectrum is also contained in the plane

$$\left\{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3\ ;\ \xi_1+\xi_2+\xi_3=0\right\}$$



Related: Sert's "Joint spectrum" – other groups; large deviation results.

Some good news

Theorem (Kalinin'11)

Suppose $T\colon X \to X$ is hyperbolic, and $F\colon X \to \mathrm{GL}(d, \mathbb{R})$ is a Hölder-continuous cocycle. Then the Lyapunov vectors of measures supported on ***periodic orbits*** *are dense in the Lyapunov spectra $L^+(F)$.*

Expected picture of $L^+(F)$

Meta-Conjecture (Typical spectra; part 1)

Suppose $T\colon X \to X$ is hyperbolic, and $F\colon X \to \mathrm{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

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Suppose $T : X \to X$ is hyperbolic, and $F : X \to \mathrm{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

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2. *Its boundary is **"fishy"**.*

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Suppose $T: X \to X$ is hyperbolic, and $F: X \to \mathrm{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:

1. *The Lyapunov spectrum $L^+(F)$ is a **convex** set.*
2. *Its boundary is **"fishy"**.*
3. *Every boundary point $\vec{\xi}$ **outside the walls** is attained as the Lyapunov vector of a unique ergodic measure $\mu_{\vec{\xi}}$; furthermore, $\mu_{\vec{\xi}}$ has low complexity (zero topological entropy).*

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2. *Its boundary is* ***"fishy"***.
3. *Every boundary point $\vec{\xi}$* ***outside the walls*** *is attained as the Lyapunov vector of a unique ergodic measure $\mu_{\vec{\xi}}$; furthermore, $\mu_{\vec{\xi}}$ has low complexity (zero topological entropy).*
4. *Subordination property: these $\mu_{\vec{\xi}}$ have uniquely ergodic supports.*

Partial result: Bochi–Rams'16.

Multifractal analysis

$$\mathfrak{H}_F(\vec{\xi}) := \sup\left\{h(\mu,T)\,;\,\mu\in\mathcal{M}_T^{\mathrm{erg}},\ \vec{\lambda}(F,\mu)=\vec{\xi}\right\}$$

Question

In the setting of the meta-conjecture:

- Is $\mathfrak{H}_F$ well defined in the interior of $L^+(F)$?
- Is $\mathfrak{H}_F$ continuous and concave there?
- Are the $\sup$'s attained? What can be said about the corresponding measures?

Related work: Díaz, Gelfert, Rams; Bárány, Jordan, Käenmäki, Rams.

A concrete example

"Step cocycle" $T\colon \{0,1\}^{\mathbb{N}} \hookleftarrow$ shift, $F(x) = A_{x_0}$ where $A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$. Then:

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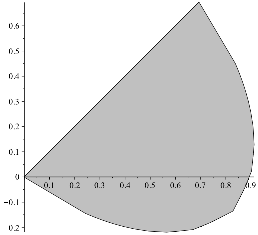
- $L^+(F)$ is convex.
- Its boundary is composed of a piece of the wall $\xi_1 = \xi_2$ and a curve with a **dense subset of corners** – "fishy".
- Every point in this curve is attained as the Lyapunov vector of a unique ergodic measure, which is Sturmian.



(Corollary of Hare, Morris, Sidorov, Theys'11; Morris, Sidorov'13.)

Commutativity regained

Suppose the matrices $F(x)$ are 2×2 and (entrywise) strictly **positive**.
Then there is a Hölder-continuous invariant splitting:

$$\mathbb{R}^2_x = V^1_x \oplus V^2_x \quad \text{such that}$$

$$\lambda_i(F,\mu) = \int \underbrace{\log \left\| F(x)|_{V^i_x} \right\|}_{=:f_i(x)} \, d\mu(x), \quad \forall i \in \{1,2\}, \ \forall \mu$$

Therefore:

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- $\mu \mapsto \vec{\lambda}(F,\mu)$ is continuous.
- $L^+(F) = \left\{ \int \vec{f} \, d\mu \,;\, \mu \in \mathcal{M}_T \right\}$, where $\vec{f} = (f_1, f_2)$ (the Lyapunov spectrum is a "rotation set".)
- $L^+(F)$ is away from the wall $\xi_1 = \xi_2$.

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- $L^+(F)$ is away from the wall $\xi_1 = \xi_2$.

Generalization of positivity: strictly invariant fields of cones ⇒ "dominated splittings".

Interlude:
Vectorial ergodic optimization

A step back: vectorial ergodic optimization

The **rotation set** of a continuous $\vec{f}: X \to \mathbb{R}^d$ is:

$$R(\vec{f}) := \left\{ \int f \, \mathrm{d}\mu \; ; \mu \in \mathcal{M}_T \right\}$$

It is compact and convex subset of $\mathbb{R}^d$ (a d-dimensional projection of $\mathcal{M}_T$).

Everyone's favorite example: the fish

$$T(x) = 2x \bmod 2\pi \text{ on } \mathbb{R}/2\pi\mathbb{Z}, \qquad \vec{f}(x) = (\cos x, \sin x).$$

$$T(z) = z^2 \text{ on } S^1 \subset \mathbb{C}, \qquad \vec{f}(z) = z \in \mathbb{C} = \mathbb{R}^2.$$

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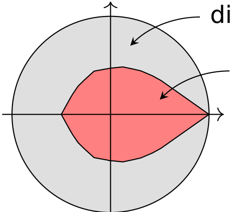
Theorem (Bousch'00, "Le poisson n'a pas d'arêtes")

$\partial R(\vec{f})$ has a dense set of corners. Each point in $\partial R(\vec{f})$ is attained by a unique measure, which is Sturmiann. The corners correspond to the periodic Sturmiann measures.

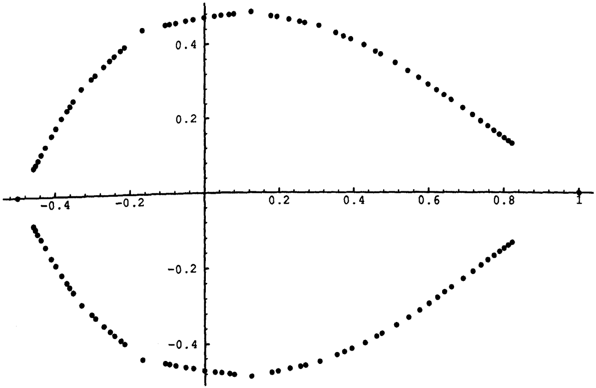


- All the curvature is concentrated on the corners.
- Sharper corners are more likely to be maximizing.

Everyone's favorite example: the fish

Appendix D of Jenkinson's PhD thesis (1996):

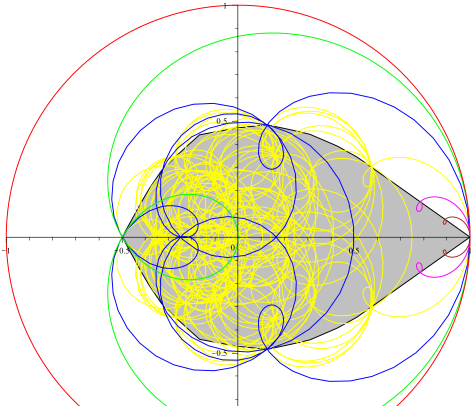
Figure 1. The 120 extremal points of Ω_{19}



Everyone's favorite example: the fish

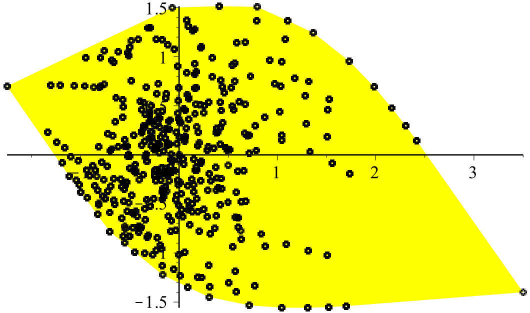
The Birkhoff averages form a sequence of curves that converges to the fish:

$$z \in S^1 \mapsto \frac{\vec{f}^{(n)}(z)}{n} = \frac{z + z^2 + z^4 + \cdots + z^{2^{n-1}}}{n}.$$



Exercise

Formulate a meta-conjecture for ergodic optimization of vectorial functions $\vec{f}: X \to \mathbb{R}^d$.



Final part:
Back to cocycles

Back to cocycles: Dominated splittings

Let $F\colon X \to \mathrm{GL}(d, \mathbb{R})$ be a cocycle. Consider an ***F*-invariant** splitting:

$$\mathbb{R}^d_x = \underbrace{V_x}_{\dim = i} \oplus \underbrace{W_x}_{\dim = d-i} \qquad F(x)(V_x) = V_{Tx},\ F(x)(V_x) = W_{Tx}\,.$$

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It is **dominated** if $\exists\ \varepsilon > 0$ s.t. (changing the norm if necessary)

$$\|F(x)w\| < e^{-\varepsilon}\|F(x)v\| \quad \forall x,\ \forall \text{unit vectors } v \in V_x,\ w \in W_x\,.$$

- In this case, the Lyapunov spectrum $L^+(F)$ is (ε-)away from the wall $\xi_i = \xi_{i+1}$.

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- In this case, the Lyapunov spectrum $L^+(F)$ is (ε-)away from the wall $\xi_i = \xi_{i+1}$.
- The converse is false (but maybe true for typical cocycles).

Finest dominated splitting

Every cocycle admits a **finest dominated splitting** $\mathbb{R}^d = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ (maybe **trivial** ($k = 1$)).

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Possible strategy for the convexity of $L^+(F)$: use subsystems with simple dominated splitting?

Compare with Bárány, Jordan, Käenmäki, Rams.

Extra convexity properties of $L^+(F)$?

Let's add an item:

Meta-Conjecture (Typical Lyapunov spectra)

Suppose $T\colon X \to X$ is hyperbolic, and $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ is a typical regular cocycle. Then:

1. *The Lyapunov spectrum $L^+(F)$ is a **convex** set.*
2. *Its boundary is **"fishy"**.*
3. *Every boundary point $\vec{\xi}$ **outside the walls** is attained as the Lyapunov vector of a unique ergodic measure $\mu_{\vec{\xi}}$; furthermore, $h(\mu_{\vec{\xi}}, T) = 0$.*
4. *Subordination property: these $\mu_{\vec{\xi}}$ have uniquely ergodic supports.*
5. *$L^+(F)$ touches the wall $\xi_i = \xi_{i+1}$ iff $\exists$ a dominated splitting with dominating bundle of dim. i. Furthermore, . . .*

Extra convexity properties of $L^+(F)$?

Meta-Conjecture (Typical Lyapunov spectra)

bla bla . . .

- *$L^+(F)$ touches a chamber wall $\xi_i = \xi_{i+1}$ iff $\exists$ a dominated splitting with dominating bundle of dim. i.*
  Furthermore, there exists a (larger) ***convex*** *set $M^+(F) \subset \mathbb{R}^d$ (**Morse set**) such that $\boxed{M^+(F) \cap \mathfrak{a}^+ = L^+(F)}$ and $M^+(F)$ is invariant by reflections across the walls it touches.*

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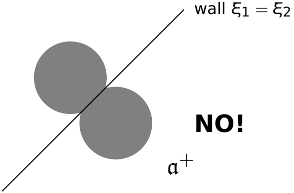

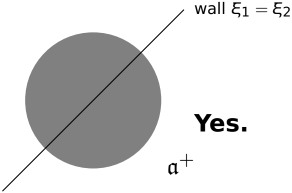

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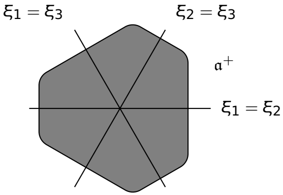

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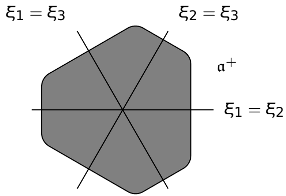

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Philosophy: Lack of domination should allow us to mix (make convex combinations) of Lyapunov exponents λ_i and λ_{i+1}. (Compare with Bochi, Viana'05; Bochi, Bonatti'12.)

Remark: The terminology **Morse set** comes from Control Theory: Colonius, Kliemann'96.