Substitutions and linear cellular automata

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Example (The Prouhet-Thue-Morse sequence)

The Prouhet-Thue–Morse sequence

\[ T(n)_{n \geq 0} = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots \]

is defined as

\[ T(n) = \text{(number of 1s in the base-2 representation of } n) \mod 2. \]
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Prouhet’s (1851) on multisets work implicitly used the P-T-M sequence. Thue’s work [1906, 1912] was the starting point of the branch of combinatorics on words. Morse [1921] used the P-T-M sequence to give the first example of a common situation in symbolic dynamics: the existence of objects that are uniformly recurrent without being periodic: every word occurring in that sequence occurs in any large enough window.
The Rudin-Shapiro sequence $a = (a_n)_{n \geq 0}$ is defined as $a_n := (-1)^{s_n}$ where $s_n =$ number of (possibly overlapping) occurrences of the block $11$ in the base-$2$ expansion of $n$. 
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![Diagram of the Rudin-Shapiro sequence automaton]

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![Diagram of the Rudin-Shapiro sequence]

Given a sequence \( a = (a_n)_{n \geq 0} \in \{-1, +1\}^N \), consider

\[
M_N(a) = \sup_{\theta \in [0,2\pi]} \left| \sum_{n=0}^{N} a_n e^{2\pi i n \theta} \right| ; \text{ then } \sqrt{N} \leq M_N(a) \leq N, \text{ and for random } a, \sqrt{N} \leq M_N(a) \leq \sqrt{N \log N}. \text{ Shapiro, 1951, showed that the R-S sequence also satisfied } \sqrt{N} \leq M_N(a) \leq 5\sqrt{N}. \]
The **Rudin-Shapiro sequence** \( a = (a_n)_{n \geq 0} \) is defined as \( a_n := (-1)^{s_n} \) where \( s_n \) is the number of (possibly overlapping) occurrences of the block 11 in the base-2 expansion of \( n \). The R-S sequence is an example of a 2-automatic sequence:

A sequence \( (a_n)_{n \geq 0} \) is **\( k \)-automatic** if there is a DFAO whose output is \( a_n \) when fed the base-\( k \) digits of \( n \).
Cobham [1972] was the first to systematically study $k$-automatic sequences.

Let $A$ be a finite alphabet. If $\theta : A \to A^k$, we call $\theta$ a length $k$ substitution. We can extend $\theta$ to act on any finite or infinite word, by concatenation.

**Example (Thue-Morse substitution)**

Let $\theta : \{0, 1\} \to \{0, 1\}^2$ be defined by $\theta(0) = 01$, and $\theta(1) = 10$; $\theta$ is the Thue-Morse substitution. Then $\theta^2(0) = \theta(01) = \theta(0)\theta(1) = 0110$.

$T = \lim_{n \to \infty} \theta^n(0) = 0110100110010110...$ satisfies $\theta(T) = T$.

A fixed point for $\theta$ is an infinite sequence $u$ such that $\theta(u) = u$.

With a $2$-DFAO with $\theta$, we see that $T_n = (\text{number of 1s in the binary representation of } n \text{ mod 2})$.
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\[
\begin{array}{c}
0 \\
\circ \\
\rightarrow \\
\downarrow \\
0 \\
\circ \\
\rightarrow \\
1
\end{array}
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Cobham’s characterisation

**Theorem (Cobham 1972)**

Let \( k \geq 2 \). Then \( a = (a_n)_{n \geq 0} \) is \( k \)-automatic if and only if \( a \) is the image, under a coding, of a fixed point of a substitution of length \( k \).

**Example**

The Rudin Shapiro sequence \( a \) is the *coding* of the fixed point of the substitution \( \theta(a) = ab, \theta(b) = ac, \theta(c) = bd \) and \( \theta(d) = da \), with the coding \( \tau(a) = \tau(b) = 1, \tau(c) = \tau(d) = -1 \).
Eilenberg’s characterization

Definition

The $k$-kernel of the sequence $a = (a_n)_{n \geq 0}$ is defined to be

$$Ker_k(a) := \{(a_{kn+\ell})_n : j \in \mathbb{N}, 0 \leq \ell < k^j\}.$$
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The sequence $a = (a_n)_{n \geq 0}$ is $k$-automatic if and only if $\text{Ker}_k(a)$ is finite.
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**Example**

The 2-kernel of the Thue-Morse sequence \( T = (T_n)_{n \geq 0} \) consists of two sequences: \( T \) and \( T + 1 \mod 2 \), since we have

\[
T_{2n} = T_n \quad \text{and} \quad T_{2n+1} = T_n + 1 \mod 2,
\]

and now apply recursion to show \( (T_{2jn+\ell})_n \) is either \( T \) or \( T + 1 \mod 2 \).
Characterisations if $k = p^n$, $p$ prime

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Let $\mathbb{F}_q$ denote the finite field with $q = p^n$ elements. Recall definitions of $\mathbb{F}_q[t], \mathbb{F}_q(t), \mathbb{F}_q[[t]],$ and $\mathbb{F}_q((t))$: polynomials, rational functions, formal power series, formal Laurent series with coefficients in $\mathbb{F}_q$ respectively.
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Given a sequence $a = (a_n)_{n \geq 0}$, let $A(x) = \sum_{n \geq 0} a_n x^n$.

**Theorem (Christol, Kamae, Mendès-France and Rauzy 1980)**

Let $(a_n)_{n \geq 0}$ be a sequence of elements in $\mathbb{F}_q$. Then $(a_n)_{n \geq 0}$ is $p$-automatic if and only if $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is algebraic over $\mathbb{F}_q(x)$. 

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**Example**

The Thue-Morse sequence's generating function is a root of

$$P(x, y) = (1 + x)^3 y^2 + (1 + x)^2 y + x.$$
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Let $(a_n)_{n \geq 0}$ be a sequence in $\mathbb{F}_q$. Then $A(x)$ is algebraic if and only if $A(x)$ is the diagonal of a rational function $A(x) = D\left(\frac{P(x,y)}{Q(x,y)}\right)$ in $\mathbb{F}_q(x, y)$. 

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Definition

Let $A$ a finite alphabet. The shift map $\sigma : A^\mathbb{Z} \to A^\mathbb{Z}$ is defined as $(\sigma(x))_n := x_{n+1}$. Let $X \subset A^\mathbb{Z}$ be a closed, $\sigma$-invariant set. Then $(X, \sigma)$ is called a shift.
A cellular automaton is a continuous map $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ such that $\Phi \circ \sigma = \sigma \circ \Phi$. 

A cellular automaton with memory $d$ is a continuous, $\sigma$-commuting map $\Phi : (\mathcal{A}^\mathbb{Z})^d \rightarrow \mathcal{A}^\mathbb{Z}$. Here by memory we mean a time memory. 

The Curtis–Hedlund–Lyndon theorem tells us that $\Phi$ is a cellular automaton with memory $d$ iff there is a local rule $\phi : ((\mathcal{A}^d)^\ell + r + 1) \rightarrow \mathcal{A}$ such that for all $R \in (\mathcal{A}^\mathbb{Z})^d$ and all $m \in \mathbb{Z}$, $(\Phi(R))(m) = \phi(R(m - \ell), R(m - \ell + 1), ..., R(m + r))$. (1)
Cellular automata with memory

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Spacetime diagrams and LCA

Definition

If $\Phi : (A^\mathbb{Z})^d \rightarrow A^\mathbb{Z}$ is a cellular automaton with memory $d$, then a spacetime diagram for $\Phi$ with initial conditions $R_0, \ldots, R_{d-1}$ is the sequence $(R_n)_{n \geq 0}$ where we inductively define $R_n := \Phi(R_{n-d}, \ldots, R_{n-1})$ for $n \geq d$. 

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Figure: Initial portion of a spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.
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Figure: Initial portion of a spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.
If $\mathcal{A} = \mathbb{F}_q$, then $(\mathbb{F}_q^d)^{\ell+r+1}$ and $\mathbb{F}_q$ are $\mathbb{F}_q$-vector spaces, we say that the CA $\Phi : \mathbb{F}_q^\mathbb{Z} \rightarrow \mathbb{F}_q^\mathbb{Z}$ is linear if $\Phi$ is an $\mathbb{F}_q$-linear map.
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Let $d = 1$, $\ell = 0$, $r = 1$, $\mathcal{A} = \mathbb{F}_2$; then the local rule $\phi(a, b) = a + b$ defines the Ledrappier CA.
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**Theorem (Pivato, Y, 2001)**

Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, and let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a linear cellular automaton. Let $\mu$ be a fully supported $N$-step Markov measure. Then $\Phi^n \mu$ converges in density to Haar measure.
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What happens if we start with a nonrandom initial configuration?
Theorem (Litow and Dumas, 1993 Rowland and Y 2014)

A sequence of elements in $\mathbb{F}_q$ is $p$-automatic if and only if it is a column in the spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions.
Theorem (Litow and Dumas, 1993 Rowland and Y 2014)

A sequence of elements in $\mathbb{F}_q$ is $p$-automatic if and only if it is a column in the spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions.

Figure: Spacetime diagram of a linear cellular automaton with memory containing the Thue–Morse sequence as a column.
Example: The Rudin-Shapiro sequence

Figure: Spacetime diagram of a linear cellular automaton with memory 20 containing the Rudin–Shapiro sequence as a column.
Example: The Baum-Sweet sequence

Figure: Spacetime diagram of a linear cellular automaton with memory 27 containing the Baum–Sweet sequence as a column.
Ingredients in the proof

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**Furstenberg’s** theorem tells us that if a FPS is algebraic over $\mathbb{F}_q(x)$, then it is the diagonal of a rational Laurent series in two variables over that field. Thus if $(u_n)$ is $p$-automatic, then $(u_n)$ can be realized as the diagonal of a quarter-lattice array of elements in $\mathbb{F}_q$ which is the formal power series expansion of $E(t, x) = \frac{P(t,x)}{Q(t,x)}$, where $P, Q \in \mathbb{F}_q[t, x]$. 
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Heuristic: Rotate this quarter array clockwise so that $(u_n)$ shows up as a column in this diagram, and, under suitable choice of the polynomials, show that you end up with the space-time diagram of a linear cellular automaton with memory.
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In particular the proof of Furstenberg’s theorem implies that if \((u_n)\) is automatic, \( u_0 = 0, P(t, F(t))=0 \) and \( P_x(0, 0) = \frac{\partial P(t, x)}{\partial x}|_{(0,0)} \neq 0 \), then \( F(t) \) is the "-2 column" of

\[
\frac{P_x(t, x)}{P(t, x)}.
\]
A corollary

**Definition**

If \( u \in A^\mathbb{N} \), define \( X_u := \{ \sigma^n(u) : n \in \mathbb{N} \} \). The dynamical system \((X_u, \sigma)\) is called the (one-sided) subshift associated with \( u \).
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Corollary

Let \( u \) be \( p \)-automatic. Then \((X_u, \sigma)\) is a factor of a subsystem of some linear cellular automaton \(((\mathbb{F}_q^d)^\mathbb{Z}, \Phi)\).

Suppose that we start with a sequence $u$ in $\mathbb{F}_p^\mathbb{Z}$ which has finitely many nonzero entries. Let $\Phi : \mathbb{F}_p^\mathbb{Z} \rightarrow \mathbb{F}_p^\mathbb{Z}$ be a linear cellular automaton with no right radius. Then the spacetime diagram $\text{ST}_\Phi(u)$ is a $p$-automatic two dimensional configuration.
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Jacking up the complexity

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What happens if we move up a level of complexity in the initial conditions, and we take an initial configuration which is not eventually periodic, but $p$-automatic?
Theorem (Rowland-Y. 2018)

Let $\Phi : \mathbb{F}_p^\mathbb{Z} \rightarrow \mathbb{F}_p^\mathbb{Z}$ be a linear cellular automaton. If $u \in \mathbb{F}_p^\mathbb{Z}$ is such that $(u_m)_{m \in \mathbb{N}}$ is $p$-automatic and $u_m = 0$ for $m \leq -1$, then $ST_\Phi(u)$ is two-dimensional $p$-automatic.
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Let $\Phi : \mathbb{F}_p^\mathbb{Z} \to \mathbb{F}_p^\mathbb{Z}$ be a linear cellular automaton. If $u \in \mathbb{F}_p^\mathbb{Z}$ is such that $(u_m)_{m \in \mathbb{N}}$ is $p$-automatic and $u_m = 0$ for $m \leq -1$, then $\text{ST}_\Phi(u)$ is two-dimensional $p$-automatic.

The 2-automatic $\text{ST}_\Phi(u)$ for a cellular automaton with generating polynomial $\phi(x) = x^{-1} + x^{-3} + x^{-7}$. The left half of initial condition $u \in \mathbb{F}_2^\mathbb{Z}$ is identically 0, and the right half is the Thue–Morse sequence.
Corollary (Rowland-Y. 2018)

Let $\Phi : \mathbb{F}_p^\mathbb{Z} \to \mathbb{F}_p^\mathbb{Z}$ be a linear cellular automaton. If $u \in \mathbb{F}_p^\mathbb{Z}$ is such that both $(u_m)_{m \in \mathbb{N}}$ and $(u-m)_{m \in \mathbb{N}}$ are $p$-automatic, then the left half and the right half of $\text{ST}_\Phi(u)$ are each two-dimensional $p$-automatic.

$\text{ST}_\Phi(u)$ for a cellular automaton with generating polynomial $\phi(x) = x + 1 + x^{-1}$, where the left half of initial condition $u \in \mathbb{F}_2^\mathbb{Z}$ is the reflection of the Toeplitz sequence, and the right half is the Thue–Morse sequence.
When is $\Sigma^\infty (u)$ uniformly, or not uniformly recurrent? (conditions on $\Sigma$ and $u$)

Which 2-dimensional automatic sequences can be realised as $\Sigma^\infty (u)$?

When are the "solid triangles" purely a facet of the Lucas-$p$ property?

What are conditions that give nontrivial measures $\mu$ which are asymptotically randomised by $\Sigma$?
When is $ST_{\Phi}(u)$ uniformly, or not uniformly recurrent? (conditions on $\Phi$ and $u$)

Which 2-dimensional automatic sequences can be realised as $ST_{\Phi}(u)$?

When are the “solid triangles” purely a facet of the Lucas-$p$ property?

What are conditions that give nontrivial measures $\mu$ which are asymptotically randomised by $\Phi$?