Substitutions and linear cellular automata

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$$T(n)_{n\geq 0} = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

is defined as
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Automatic sequences: History and uses. Example I

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Morse [1921] used the P-T-M sequence to give the first example of a common situation in symbolic dynamics: the existence of objects that are uniformly recurrent without being periodic: every word occuring in that sequence occurs in any large enough window.

Automatic sequences: History and uses. Example II

Example (The Rudin-Shapiro sequence)

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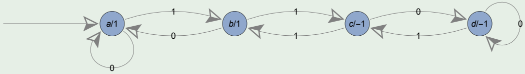

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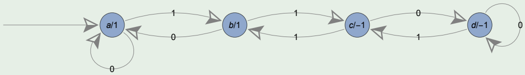

Given a sequence $\mathbf{a} = (a_n)_{n \geq 0} \in \{-1, +1\}^{\mathbb{N}}$, consider $M_N(\mathbf{a}) = \sup_{\theta \in [0, 2\pi)} \left| \sum_{n=0}^{N} a_n e^{2\pi i n \theta} \right|$; then $\sqrt{N} \leq M_N(\mathbf{a}) \leq N$, and for random $\mathbf{a}$, $\sqrt{N} \leq M_N(\mathbf{a}) \leq \sqrt{N log N}$. Shapiro, 1951, showed that the R-S sequence also satisfied $\sqrt{N} \leq M_N(\mathbf{a}) \leq 5\sqrt{N}$.

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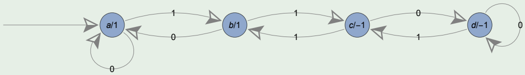

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A sequence $(a_n)_{n\geq 0}$ is *k*-automatic if there is a DFAO whose output is a_n when fed the base-k digits of n.

Characterisations of automatic sequences

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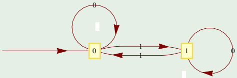

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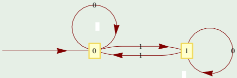


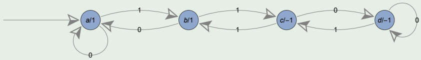
we see that $T_n =$ (number of 1s in the binary representation of n) mod 2.

Cobham's characterisation

Theorem (Cobham 1972)

Let $k \geq 2$. Then $\mathbf{a} = (a_n)_{n \geq 0}$ is k-automatic if and only if $\mathbf{a}$ is the image, under a coding, of a fixed point of a substitution of length k.

Example



The Rudin Shapiro sequence $\mathbf{a}$ is the *coding* of the fixed point of the substitution $\theta(a) = ab$, $\theta(b) = ac$, $\theta(c) = bd$ and $\theta(d) = da$, with the coding $\tau(a) = \tau(b) = 1, \tau(c) = \tau(d) = -1$.

Eilenberg's characterization

Definition

The k-kernel of the sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is defined to be

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Example

The 2-kernel of the Thue-Morse sequence $T = (T_n)_{n\geq 0}$ consists of two sequences: T and $T+1 \mod 2$, since we have

$$T_{2n} = T_n \text{ and } T_{2n+1} = T_n + 1 \mod 2,$$

and now apply recursion to show $(T_{2^j n+\ell})_n$ is either T or $T+1 \mod 2$.

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Theorem (Christol, Kamae, Mendès-France and Rauzy 1980)

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Cellular automata with memory

Definition

Let $\mathcal{A}$ a finite alphabet. The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined as $(\sigma(x))_n := x_{n+1}$. Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a closed, σ-invariant set. Then (X, σ) is called a *shift*.

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The Curtis–Hedlund–Lyndon theorem tells us that Φ is a cellular automaton with memory d iff there is a local rule $\phi : (\mathcal{A}^d)^{\ell+r+1} \to \mathcal{A}$ such that for all $R \in (\mathcal{A}^{\mathbb{Z}})^d$ and all $m \in \mathbb{Z}$,

$$(\Phi(R))(m) = \phi(R(m-\ell), R(m-\ell+1), \ldots, R(m+r)) . \tag{1}$$

Spacetime diagrams and LCA

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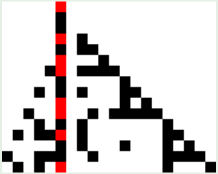

Figure : Initial portion of a spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.

Linear cellular automata and asymptotic randomization

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Theorem (Pivato, Y, 2001)

Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, and let $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a linear cellular automaton. Let μ be a fully supported N-step Markov measure. Then $\Phi^n\mu$ converges in density to Haar measure.

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Example

Let $d = 1$, $\ell = 0$, $r = 1$, $\mathcal{A} = \mathbb{F}_2$; then the local rule $\phi(a, b) = a + b$ defines the Ledrappier CA.

Suppose that we start with a "random" initial configuration $x = R_0$, and generate a spacetime diagam $y = (R_n)_{n\in\mathbb{Z}}$. As $n \to \infty$, what are the rows R_n random for?

Theorem (Pivato, Y, 2001)

Let $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, and let $\Phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be a linear cellular automaton. Let μ be a fully supported N-step Markov measure. Then $\Phi^n\mu$ converges in density to Haar measure.

What happens if we start with a nonrandom initial configuration?

Spacetime diagrams of LCA with periodic initial conditions

Theorem (Litow and Dumas, 1993 Rowland and Y 2014)

A sequence of elements in $\mathbb{F}_q$ is p-automatic if and only if it is a column in the spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions.

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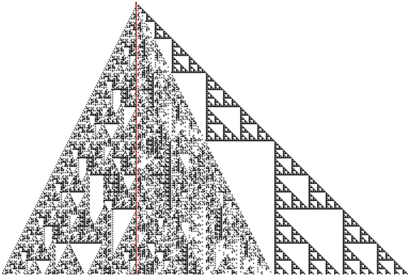

Figure : Spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.

Example: The Rudin-Shapiro sequence

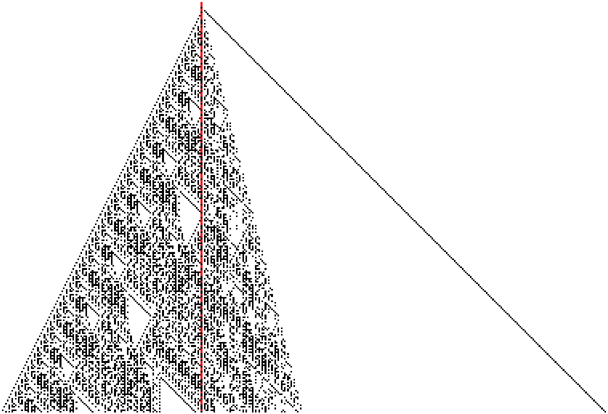

Figure : Spacetime diagram of a linear cellular automaton with memory 20 containing the Rudin–Shapiro sequence as a column.

Example: The Baum-Sweet sequence

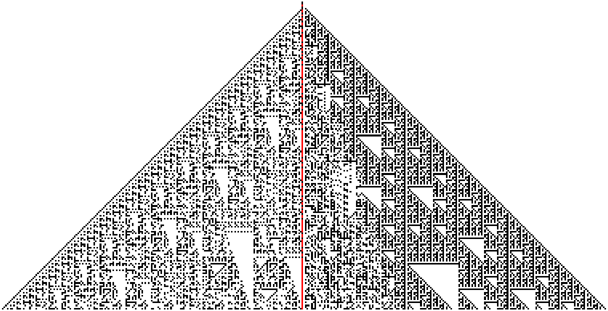

Figure : Spacetime diagram of a linear cellular automaton with memory 27 containing the Baum–Sweet sequence as a column.

Ingredients in the proof

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Thus if (u_n) is p-automatic, then (u_n) can be realized as the diagonal of a quarter-lattice array of elements in $\mathbb{F}_q$ which is the formal power series expansion of $E(t,x) = \frac{P(t,x)}{Q(t,x)}$, where $P,\ Q \in \mathbb{F}_q[t,x]$.

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Heuristic: Rotate this quarter array clockwise so that (u_n) shows up as a column in this diagram, and, under suitable choice of the polynomials, show that you end up with the space-time diagram of a linear cellular automaton with memory.

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In particular the proof of Furstenberg's theorem implies that if (u_n) is automatic, $u_0 = 0$, $P(t, F(t))$=0 and $P_x(0,0) = \frac{\partial P(t,x)}{\partial x}|_{(0,0)} \neq 0$, then $F(t)$ is the "-2 column" of

$$\frac{P_x(t,x)}{P(t,x)}.$$

A corollary

Definition

If $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$, define $X_{\mathbf{u}} := \overline{\{\sigma^n(\mathbf{u}) : \mathbf{n} \in \mathbb{N}\}}$. The dynamical system $(X_{\mathbf{u}}, \sigma)$ is called the (one-sided) subshift associated with $\mathbf{u}$.

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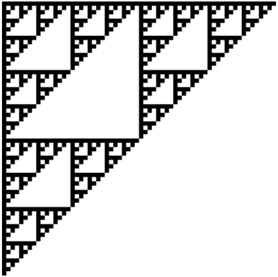
Corollary

Let $\mathbf{u}$ *be p-automatic. Then* $(X_{\mathbf{u}}, \sigma)$ *is a factor of a subsystem of some linear cellular automaton* $((\mathbb{F}_q^d)^{\mathbb{Z}}, \Phi)$.

A helpful referee remark

Theorem (basic model, Allouche, von Haeseler, Peitgen, Skordev 1992-2003)

Suppose that we start with a sequence u in $\mathbb{F}_p^{\mathbb{Z}}$ which has finitely many nonzero entries. Let $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ be a linear cellular automaton with no right radius. Then the spacetime diagram $\mathrm{ST}_\Phi(u)$ is a p-automatic two dimensional configuration.



Jacking up the complexity

Litow-Dumas:Starting with a finite initial sequence and applying a linear cellular automaton, any column in the spacetime diagram is one-dimensional p-automatic.

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Allouche, von Haeseler, Peitgen, Skordev: Any finite initial condition from $\mathbb{F}_p$ and a zero-anticipation linear cellular automaton generates a two-dimensional p-automatic sequence.

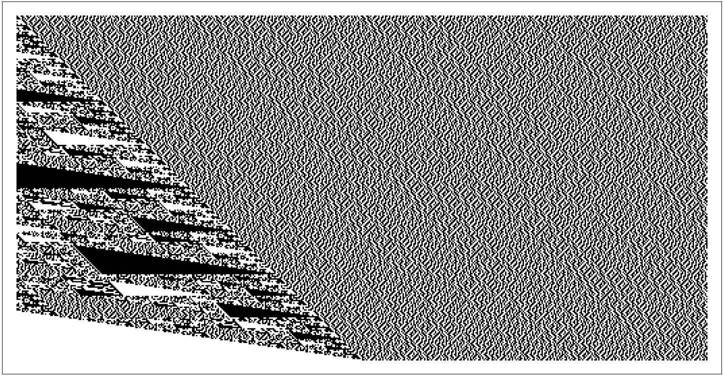
What happens if we move up a level of complexity in the initial conditions, and we take an initial configuration which is not eventually periodic, but p-automatic?

Theorem (Rowland-Y. 2018)

Let $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ *be a linear cellular automaton. If* $u \in \mathbb{F}_p^{\mathbb{Z}}$ *is such that* $(u_m)_{m\in\mathbb{N}}$ *is p-automatic and* $u_m = 0$ *for* $m \leq -1$*, then* $\mathrm{ST}_\Phi(u)$ *is two-dimensional p-automatic.*

Theorem (Rowland-Y. 2018)

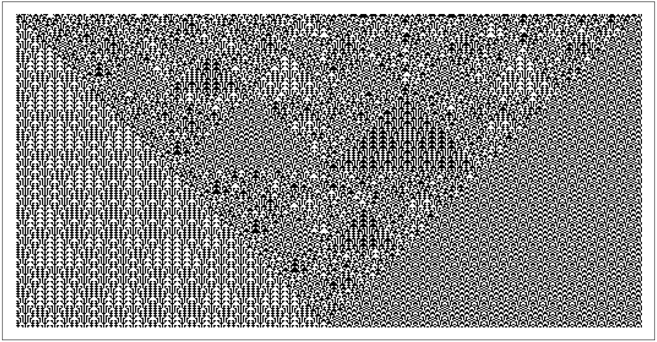
Let $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ be a linear cellular automaton. If $u \in \mathbb{F}_p^{\mathbb{Z}}$ is such that $(u_m)_{m\in\mathbb{N}}$ is p-automatic and $u_m = 0$ for $m \leq -1$, then $\mathrm{ST}_\Phi(u)$ is two-dimensional p-automatic.



The 2-automatic $\mathrm{ST}_\Phi(u)$ for a cellular automaton with generating polynomial $\phi(x) = x^{-1} + x^{-3} + x^{-7}$. The left half of initial condition $u \in \mathbb{F}_2^{\mathbb{Z}}$ is identically 0, and the right half is the Thue–Morse sequence.

Corollary (Rowland-Y. 2018)

Let $\Phi : \mathbb{F}_p^{\mathbb{Z}} \to \mathbb{F}_p^{\mathbb{Z}}$ be a linear cellular automaton. If $u \in \mathbb{F}_p^{\mathbb{Z}}$ is such that both $(u_m)_{m\in\mathbb{N}}$ and $(u_{-m})_{m\in\mathbb{N}}$ are p-automatic, then the left half and the right half of $\mathrm{ST}_\Phi(u)$ are each two-dimensional p-automatic.



$\mathrm{ST}_\Phi(u)$ for a cellular automaton with generating polynomial $\phi(x) = x + 1 + x^{-1}$, where the left half of initial condition $u \in \mathbb{F}_2^{\mathbb{Z}}$ is the reflection of the Toeplitz sequence, and the right half is the Thue–Morse sequence.

Work in Progress

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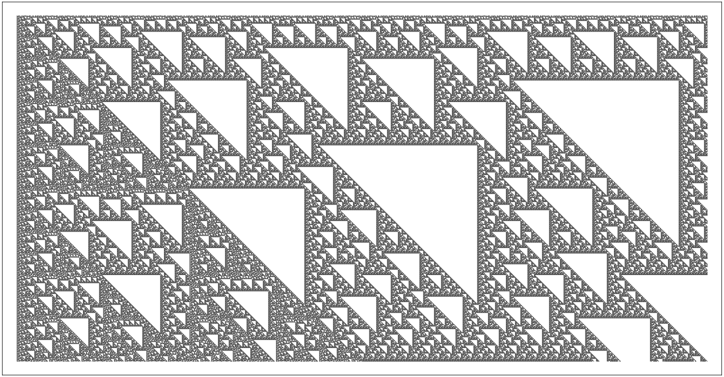
When is $\mathrm{ST}_\Phi(u)$ uniformly, or not uniformly recurrent? (conditions on Φ and u)

Which 2-dimensional automatic sequences can be realised as $\mathrm{ST}_\Phi(u)$?

When are the "solid triangles" purely a facet of the Lucas-p property?



What are conditions that give nontrivial measures μ which are asymptotically randomised by Φ?