Tiling Cohomology
and Quasiperiodic Baked Goods

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Three key questions

For every mathematical concept:

- What is it?
Three key questions

For every mathematical concept:

- What is it?
- How do you compute it?
Three key questions

For every mathematical concept:

- What is it?
- How do you compute it?
- Why in blazes should you care?
Puzzle 1: Mass transport
Musical chairs
Three different mass distributions

- $f_1$ puts 2 kg on every tile that sits in the standard L configuration, i.e. missing the northeast corner, and no mass on the other three kinds of tiles.
- $f_2$ puts 1 kg on every tile that is missing a NE or SW corner, and none on tiles that are missing NW or SE corners.
- $f_3$ puts 1 kg on every tile that is missing a NW or SE corner, and none on tiles that are missing NE or SW corners.
Three different mass distributions

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- All three distributions have overall density 0.5 kg/tile. Which are related by bounded/wPE/sPE transport?
2 kg on the NE chairs
1 kg on the NE and SW chairs
1 kg on the NW and SE chairs
Puzzle 2: Fibonacci shape changes

How are these tilings related? How do their diffraction patterns compare?
Puzzle 3: Penrose shape changes
Rational Penrose

180 Tiles
Squared off Penrose

180 Tiles
Puzzle 4: Ergodic averages

Thue-Morse tiling: $A \rightarrow AB$, $B \rightarrow BA$,

\[ \ldots ABBABAAABBAABBAABBAABBA \cdots \]

What are the maximum/minimum number of times that the pattern $ABA$ appears in a sub-word of length $N$? How does the variation scale with $N$?
Motivation

Tiling spaces

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F LC tiling metric

Idea for FLC tilings: Two tilings with the same set of tile types are $\epsilon$ close if they agree on $B_{1/\epsilon}$, up to an $\epsilon$ translation.
FLC tiling metric

- Idea for FLC tilings: Two tilings with the same set of tile types are $\epsilon$ close if they agree on $B_{1/\epsilon}$, up to an $\epsilon$ translation.
- If you want to allow rotations, shears, or an infinite variety of tile types, it’s a little more complicated.
Idea for FLC tilings: Two tilings with the same set of tile types are $\varepsilon$ close if they agree on $B_{1/\varepsilon}$, up to an $\varepsilon$ translation.

If you want to allow rotations, shears, or an infinite variety of tile types, it’s a little more complicated.

(We won’t go there)
Simplest way to build a tiling space:

- Start with an FLC tiling $T$. 

Continuous Hulls
Simplest way to build a tiling space:

- Start with an FLC tiling $T$.
- Consider the set $\{T - x\}$ of translates of $T$. 
Simplest way to build a tiling space:

- Start with an FLC tiling $T$.
- Consider the set $\{T - x\}$ of translates of $T$.
- $\Omega_T = \{T - x\}$. $T' \in \Omega_T$ iff every patch of $T'$ appears somewhere in $T$.
- Orbit closure of $T = \text{Tiling space of } T = \text{Continuous hull of } T$. 
Hulls of periodic tilings

What is $\Omega_T$?
Hulls of periodic tilings

What is $\Omega_T$?

A torus!
A non-periodic example

\[ T = \ldots AAAAA.BBBB\ldots = A^\infty.B^\infty. \]

What is \( \Omega_T \)?
A non-periodic example

\[ T = \ldots AAAAA.BBBB \ldots \text{“=” } A^\infty . B^\infty. \]

What is \( \Omega_T \)?

- Orbit of \( T \) is copy of \( \mathbb{R} \).
A non-periodic example

\[ T = \ldots AAAAA.BBBBB\ldots \quad "\Rightarrow\quad A^\infty.B^\infty.\]

What is \( \Omega_T \)?

- Orbit of \( T \) is copy of \( \mathbb{R} \).
- As \( x \to -\infty \), \( T - x \) approaches periodic \( \ldots AAAAA\ldots \) tiling. Limiting circle.
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- As \( x \to \infty \), \( T - x \) approaches periodic \( \ldots BBBBB \ldots \) tiling. Limiting circle.
- Hull = slinky! Connected but not path-connected.
If $T$ is a tiling, what does an $\epsilon$-neighborhood of $T$ in $\Omega_T$ look like?
If $T$ is a tiling, what does an $\epsilon$-neighborhood of $T$ in $\Omega_T$ look like?

- Restrict $T$ to $B_{1/\epsilon}$.
- Move $T$ by up to $\epsilon$: continuous degrees of freedom.
- Fill out near $\infty$. Discrete choices.
- Neighborhood $\sim B_\epsilon \times C$. 
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Inverse limits in general

If $X_0, X_1, \ldots$ are spaces and $\rho_n : X_n \to X_{n-1}$ are continuous maps,

$$X = \lim \leftarrow X_i := \{(x_0, x_1, \ldots) \in \prod X_n | \rho_n(x_n) = x_{n-1} \\forall n\}.$$
Inverse limits in general

If $X_0, X_1, \ldots$ are spaces and $\rho_n : X_n \to X_{n-1}$ are continuous maps,

$$X = \varprojlim X_i := \{(x_0, x_1, \ldots) \in \prod X_n | \rho_n(x_n) = x_{n-1} \forall n\}.$$

$X_n$ is called $n$-th approximant to $X$, since $x_n$ determines $(x_0, \ldots, x_n)$. 

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Tiling Cohomology
**Inverse limits in general**

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$$X = \lim \leftarrow X_i := \{(x_0, x_1, \ldots) \in \prod X_n \mid \rho_n(x_n) = x_{n-1} \forall n\}.$$ 

$X_n$ is called the $n$-th **approximant** to $X$, since $x_n$ determines $(x_0, \ldots, x_n)$.

$X$ has the product topology. $(x_0, x_1, \ldots)$ is close to $(y_0, y_1, \ldots)$ if $x_i \approx y_i$ for all $i \leq N$. I.e. if $x_N \approx y_N$. 

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**Motivation**

**Tiling spaces**

**Inverse limits**

**Pattern-Equivariant Cohomology**

**Shape changes**

**Topological conjugacies**

**Top cohomology, transport, and ergodic averages**

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Tiling Cohomology
Dyadic Solenoid

Example of inverse limit space. Take

\[ X_n = \mathbb{R}/(2^n \mathbb{Z}) \sim S^1. \]
Example of inverse limit space. Take

- $X_n = \mathbb{R}/(2^n \mathbb{Z}) \cong S^1$.
- $\rho_n$ induced by identity on $\mathbb{R}$. Winds $X_n$ twice around $X_{n-1}$. 
Example of inverse limit space. Take

- $X_n = \mathbb{R}/(2^n \mathbb{Z}) \simeq S^1$.
- $\rho_n$ induced by identity on $\mathbb{R}$. Winds $X_n$ twice around $X_{n-1}$. 

Diagram:

- $\Gamma_0$
- $\Gamma_1$
- $\Gamma_2$
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$. 
Tiling spaces are inverse limits

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- $\rho_n$ is forgetful map.
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- Many different schemes: different details, (mostly) same strategy.
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- $\lim\Gamma_n = $ consistent instructions for tiling bigger and bigger regions, i.e. instructions for a complete tiling.
Tiling spaces are inverse limits

- CW complex $\Gamma_n$ describes tiling out to distance that grows with $n$.
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- Many different schemes: different details, (mostly) same strategy.
- $\varprojlim \Gamma_n = $ consistent instructions for tiling bigger and bigger regions, i.e. instructions for a complete tiling.
- So how do instructions for partial tilings turn into a CW complex?!
Anderson-Putnam Complex

To place a tile at the origin, need:

Choice of tile type $t_i$.

Choice of point in $t_i$ to associate with origin.

What if origin is on boundary of 2 (or more tiles)? Identify!

$\Gamma_0 = \biguplus_{t_i} / \sim$ is the Anderson-Putnam complex.
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Collared tiles

- Start with a tiling $T$.
- Equivalent tiles have same label and same pattern of immediate neighbors.
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- Relabeling tiling with collared tiles is **local** operation. Does not change space.
Collared tiles

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- Equivalence classes are called **collared tiles**.
- Relabeling tiling with collared tiles is **local** operation. Does not change space.
- Can be repeated to get $n$-times collared tiles.
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Collared Fibonacci

Fibonacci sequence in 1D contains

\[ \ldots abaababaabaababaababa \ldots \]
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- Only one “b” collared tile: \( B = (a)b(a). \)
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Fibonacci sequence in 1D contains

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- Only one “b” collared tile: \( B = (a)b(a) \).
- Three “a” collared tiles: \( A_1 = (b)a(b) \), \( A_2 = (a)a(b) \), \( A_3 = (b)a(a) \).
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Fibonacci sequence in 1D contains

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- Three “a” collared tiles: \( A_1 = (b)a(b), A_2 = (a)a(b), A_3 = (b)a(a) \).
- Sequence becomes

\[ \ldots BA_3A_2BA_1BA_3A_2BA_3A_2BA_1BA_3A_2BA_1B \ldots \]
Collared Fibonacci

Fibonacci sequence in 1D contains

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\[ \ldots BA_3A_2BA_1BA_3A_2BA_3A_2BA_1BA_3A_2BA_1B \ldots \]

- Collared tiles have same size as regular tiles, but carry more info.
Gähler’s construction

Let $\Gamma^n$ be the Anderson-Putnam complex of $n$-collared tiles.
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- $\Omega = \varprojlim \Gamma^n$. 

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Tiling Cohomology
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- Conceptually very powerful idea. Great for proving theorems.
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- Edge identification can reduce that to $n - 1$. No sweat.
- $\Omega = \lim \Gamma^n$.
- Conceptually very powerful idea. Great for proving theorems.
- Calculationally not so much, since $\Gamma^n$’s are all different.
Substitution tilings

1-dimensional example (Fibonacci) : $a \rightarrow ab$, $b \rightarrow a$. 
Substitution tilings

1-dimensional example (Fibonacci): $a \rightarrow ab, b \rightarrow a$.

- $a$
- $ab$
- $ab.a$
- $ab.a.ab$
- $ab.a.ab.ab.a$
- $ab.a.ab.ab.a.ab.a.ab$
Substitution tilings

1-dimensional example (Fibonacci): \( a \rightarrow ab, \ b \rightarrow a \).

- \( a \)
- \( ab \)
- \( ab.a \)
- \( ab.a.ab \)
- \( ab.a.ab.ab.a \)
- \( ab.a.ab.ab.a.ab.a.ab \)

A word is **legal** if it sits inside one of these patterns.
A bi-infinite word is legal if every sub-word is legal.
Make into self-similar tilings by assigning length \((1 + \sqrt{5})/2\) to \( a \) tile and 1 to \( b \) tile.
Anderson-Putnam inverse limits

- Applies to substitutions that “force the border”.

Let $\Gamma_n$ be the Anderson-Putnam complex of $n$-supertiles. A point in $\Gamma_n$ describes the $n$-supertile containing the origin. All $\Gamma_n$'s are the same, up to scale. $\Omega = \lim \xleftarrow{-} (\Gamma_n, \sigma)$. One approximant. One expansive map. To get border forcing, collar once (if necessary).
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Other techniques

- Various tricks to collar as little as possible.
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- Bellissard-Benedetti-Gambaudo. Aggregate collared tiles into large patches.
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- (Forest-Hunton-Kellendonk have a different sort of inverse limit construction for cut-and-project tilings)
Other techniques

- Various tricks to collar as little as possible.
- Bellissard-Benedetti-Gambaudo. Aggregate collared tiles into large patches.
- (Forest-Hunton-Kellendonk have a different sort of inverse limit construction for cut-and-project tilings)
- Can express tilings with infinite local complexity as inverse limits, too. Details depend on setting.
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Pattern-equivariant functions and forms

- Given a tiling $T$, a function $f(x)$ on $\mathbb{R}^n$ is strongly pattern-equivariant (sPE) if $\exists R > 0$ s.t. $x$ depends only on tiling on $B_R(x)$. (Think: finite range potentials)
- That is, if $T - x$ and $T - y$ agree on $B_R(0)$, then $f(x) = f(y)$. 
Given a tiling $T$, a function $f(x)$ on $\mathbb{R}^n$ is strongly pattern-equivariant (sPE) if $\exists R > 0$ s.t. $x$ depends only on tiling on $B_R(x)$. (Think: finite range potentials)

That is, if $T - x$ and $T - y$ agree on $B_R(0)$, then $f(x) = f(y)$.

Weakly PE functions are uniform limits of sPE functions. For each $\epsilon > 0$ there is an $R_\epsilon$ s.t. $f(x)$ is determined to within $\epsilon$ by $T$ on $B_{R_\epsilon}(x)$.
Given a tiling $T$, a function $f(x)$ on $\mathbb{R}^n$ is strongly pattern-equivariant (sPE) if $\exists R > 0$ s.t. $x$ depends only on tiling on $B_R(x)$. (Think: finite range potentials)

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Weakly PE functions are uniform limits of sPE functions. For each $\epsilon > 0$ there is an $R_\epsilon$ s.t. $f(x)$ is determined to within $\epsilon$ by $T$ on $B_{R_\epsilon}(x)$.

Strongly/weakly PE forms are strongly/weakly PE functions times $dx^i \wedge dx^j \wedge \cdots$. 

If $\alpha$ is a PE form, so is $d\alpha$.

$H^k_{\text{PE}}(T) = \text{closed sPE}_k$-forms / $d$(sPE$_{k-1}$ forms).
Pattern-equivariant functions and forms

- Given a tiling $T$, a function $f(x)$ on $\mathbb{R}^n$ is **strongly pattern-equivariant** (sPE) if $\exists R > 0$ s.t. $x$ depends only on tiling on $B_R(x)$. (Think: finite range potentials)
- That is, if $T - x$ and $T - y$ agree on $B_R(0)$, then $f(x) = f(y)$.
- Weakly PE functions are uniform limits of sPE functions. For each $\epsilon > 0$ there is an $R_\epsilon$ s.t. $f(x)$ is determined to within $\epsilon$ by $T$ on $B_{R_\epsilon}(x)$.
- Strongly/weakly PE forms are strongly/weakly PE functions times $dx^i \wedge dx^j \wedge \cdots$.
- If $\alpha$ is a PE form, so is $d\alpha$. 
Pattern-equivariant functions and forms

- Given a tiling \( T \), a function \( f(x) \) on \( \mathbb{R}^n \) is strongly pattern-equivariant (sPE) if \( \exists R > 0 \) s.t. \( x \) depends only on tiling on \( B_R(x) \). (Think: finite range potentials)
- That is, if \( T - x \) and \( T - y \) agree on \( B_R(0) \), then \( f(x) = f(y) \).
- Weakly PE functions are uniform limits of sPE functions. For each \( \epsilon > 0 \) there is an \( R_\epsilon \) s.t. \( f(x) \) is determined to within \( \epsilon \) by \( T \) on \( B_{R_\epsilon}(x) \).
- Strongly/weakly PE forms are strongly/weakly PE functions times \( dx^i \wedge dx^j \wedge \cdots \).
- If \( \alpha \) is a PE form, so is \( d\alpha \).
- \( H_{PE}^k(T) = \text{closed sPE } k\text{-forms} / d(\text{sPE } k-1\text{ forms}) \).
A tiling $T$ gives a decomposition of $\mathbb{R}^n$ into vertices, edges, 2-cells, 3-cells, etc. Tiles are $n$-cells. Orient the cells arbitrarily.

A (real-valued) $k$-cochain assigns a real number to each oriented $k$-cell. A mass distribution is just an $n$-cochain.

$k$-cochains can be sPE or wPE.

Coboundaries: If $\alpha$ is a $k$-cochain, and $c$ is a $(k+1)$-cell, then $(\delta \alpha)(c) := \alpha(\partial c)$.

If $\alpha$ is wPE/sPE, so is $\delta \alpha$.

Let $\Omega^k_w$ and $\Omega^k_s$ denote the weakly and strongly PE $k$-cochains on $T$. 

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Tiling Cohomology
A strongly PE cochain $\alpha$ is said to be

- Closed is $\delta \alpha = 0,$
A strongly PE cochain $\alpha$ is said to be

- Closed if $\delta \alpha = 0$,
- Exact if $\alpha = \delta \beta$ for some sPE cochain $\beta$, 

where $\delta$ denotes the coboundary operator.
A strongly PE cochain $\alpha$ is said to be

- Closed if $\delta \alpha = 0$,
- Exact if $\alpha = \delta \beta$ for some sPE cochain $\beta$,
- Weakly exact if $\alpha = \delta \gamma$ for some wPE cochain $\gamma$. 

$H^k_{\text{PE}}(T) =$ Closed $k$-cochains

Exact $k$-cochains (Same answer as with forms!) 

A cohomology class is asymptotically negligible (AN) if it can be represented by a weakly exact cochain/form.
**Strong PE cohomology**

A strongly PE cochain $\alpha$ is said to be

- Closed is $\delta \alpha = 0$,
- Exact if $\alpha = \delta \beta$ for some sPE cochain $\beta$,
- Weakly exact if $\alpha = \delta \gamma$ for some wPE cochain $\gamma$.

$$H^k_{PE}(T) = \frac{\text{Closed } k\text{-cochains}}{\text{Exact } k\text{-cochains}}$$

(Same answer as with forms!)
A strongly PE cochain $\alpha$ is said to be
- Closed is $\delta \alpha = 0$,
- Exact if $\alpha = \delta \beta$ for some sPE cochain $\beta$,
- Weakly exact if $\alpha = \delta \gamma$ for some wPE cochain $\gamma$.

$$H^k_{PE}(T) = \frac{\text{Closed } k\text{-cochains}}{\text{Exact } k\text{-cochains}}$$ (Same answer as with forms!)

A cohomology class is *asymptotically negligible (AN)* if it can be represented by a weakly exact cochain/form.
Theorem (Kellendonk-Putnam, S)

If $T$ is a repetitive tiling, then $H^k_{PE}$ is canonically isomorphic to the $k$-th real-valued Čech cohomology $\check{H}^k(\Omega_T)$, where $\Omega_T$ is the continuous hull of $T$. In particular, all tilings in $\Omega_T$ have the same PE cohomology.
What the heck is Čech cohomology?!

- Complicated definition involving combinatorics of open covers.
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- TMI! Just need 2 key properties:
What the heck is Čech cohomology?! 

- Complicated definition involving combinatorics of open covers.
- TMI! Just need 2 key properties:
  - If $X$ is a CW complex, $\check{H}^*(X) = H^*(X)$.
  - If $X = \lim X_i$, $\check{H}^*(X) = \lim \check{H}^*(X_i)$. 
What the heck is Čech cohomology?!

- Complicated definition involving combinatorics of open covers.
- TMI! Just need 2 key properties:
  - If $X$ is a CW complex, $\tilde{H}^*(X) = H^*(X)$.
  - If $X = \lim X_i$, $\tilde{H}^*(X) = \lim \tilde{H}^*(X_i)$.
- Strategy: Write tiling space $\Omega$ as inverse limit of CW complexes $\Gamma_i$. Then
What the heck is Čech cohomology?!

- Complicated definition involving combinatorics of open covers.
- TMI! Just need 2 key properties:
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- Strategy: Write tiling space $\Omega$ as inverse limit of CW complexes $\Gamma_i$. Then
  $$\check{H}^*(\Omega) = \lim \check{H}^*(\Gamma_i) = \lim H^*(\Gamma_i).$$
Complicated definition involving combinatorics of open covers.

TMI! Just need 2 key properties:
- If $X$ is a CW complex, $\check{H}^*(X) = H^*(X)$.
- If $X = \varprojlim X_i$, $\check{H}^*(X) = \varprojlim \check{H}^*(X_i)$.

Strategy: Write tiling space $\Omega$ as inverse limit of CW complexes $\Gamma_i$. Then

$$\check{H}^*(\Omega) = \varprojlim \check{H}^*(\Gamma_i) = \varprojlim H^*(\Gamma_i).$$

But we already did that!
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Fibonacci

$H_1(\Gamma_n) = \mathbb{Z}_2$;
$H_1(\Omega) = \text{lim}(\mathbb{Z}_2, (1 1 1 0)) = \mathbb{Z}_2$. 

Lorenzo Sadun

Tiling Cohomology
Motivation
Tiling spaces
Inverse limits
Pattern-Equivariant Cohomology
Shape changes
Topological conjugacies
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Tiling Cohomology

**Fibonacci**

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$H^1(\Omega) = \mathbb{Z}[1/2]^2$,

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1D shape changes

Combinatorics of $T_1$ and $T_2$ are identical. Dynamics may be different. Some (but not all!) shape changes are topological conjugacies.
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- Dynamics may be different.
- Some (but not all!) shape changes are topological conjugacies.
The shape of an $n$-gon is determined by the $n$ vectors that describe the edges.
The shapes of all the tiles are given by:

- A vector for each edge of each species of tile, such that the vectors around a closed loop must add up to 0. If two tiles share an edge, their edge vectors must match. This is equivalent to a closed vector-valued 1-cochain on the Anderson-Putnam complex.
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Parametrizing shape

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  - The vectors around a closed loop must add up to 0.
  - If two tiles share an edge, their edge vectors must match.
- But that’s the same as a closed vector-valued 1-cochain on the Anderson-Putnam complex!
We are looking for results mod MLD.

- Can collar before assigning edge vectors, so different collared tiles can have different shape.
More generality with PE

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- Consider closed vector-valued cochains on AP complex of any tiling obtained by repeatedly collaring $T$. 
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- Can collar before assigning edge vectors, so different collared tiles can have different shape.
- Consider closed vector-valued cochains on AP complex of any tiling obtained by repeatedly collaring $T$.
- But that’s the same as a closed sPE cochain on $T$. 
MLD equivalence moves each vertex $x$ by $F(x)$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is an sPE function.
Modding out by MLD

- MLD equivalence moves each vertex $x$ by $F(x)$, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an sPE function.
- This changes vector of edge $e$ by $F(y) - F(x) = \delta F(e)$.
- MLD changes induced by adding **exact** 1-cochains to shape.
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MLD changes induced by adding exact 1-cochains to shape.

$$\frac{\text{Shape changes}}{\text{MLD}} = \frac{\text{Closed sPE 1-cochains}}{\delta(\text{sPE 0-cochains})}$$

$$= H^1_{\text{PE}}(T, \mathbb{R}^n) = \tilde{H}^1(\Omega_T, \mathbb{R}^n).$$
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Asymptotically negligible classes

Some sPE 1-cochains are not $\delta$ of sPE 0-cochains (functions), but are still $\delta$ of \textit{weakly} PE 0-cochains. These cochains are called asymptotically negligible (AN).
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- Moving points by wPE amounts induces topological conjugacies, so $H^1_{AN}$ describes shape changes that are topological conjugacies but not MLD.
Asymptotically negligible classes

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- Generate subspace $H^1_{AN}$ of $H^1$.
- Moving points by wPE amounts induces topological conjugacies, so $H^1_{AN}$ describes shape changes that are topological conjugacies but not MLD.
- Theorem (Gottschalk-Hedlund, Kellendonk-S): A closed sPE 1-cochain is AN if and only if its integral is bounded.
Fibonacci is rigid

- Fibonacci tiling has $\phi = (1 + \sqrt{5})/2$ “a” tiles for every “b” tile.
- If $\alpha(a) = 1$ and $\alpha(b) = -\phi$, $\alpha$ is AN.
- $H^1(\Omega_{Fib}, \mathbb{R}) = \mathbb{R}^2 = H^1_{AN} \oplus \mathbb{R}$.
- All shape changes for Fibonacci are a combination of topological conjugacy and overall rescaling.
- Dynamical properties of Fibonacci (e.g. pure point spectrum) unchanged by shape changes.
AN classes for substitutions

Setting: $\Omega$ is a substitution tiling space with a substitution map $\sigma : \Omega \rightarrow \Omega$.

- $\tilde{H}^1(\Omega, \mathbb{R}) = \tilde{H}^1(\Omega) \otimes \mathbb{R}$ is a vector space.
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- $H^1_{AN}(\Omega, \mathbb{R})$ is contracting subspace of $\tilde{H}^1(\Omega, \mathbb{R})$.
- $H^1_{AN}(\Omega, \mathbb{R}^n) = H^1_{AN}(\Omega, \mathbb{R}) \otimes \mathbb{R}^n$. 
Penrose is almost rigid

\[ H^1(\Omega_{pen}) = \mathbb{Z}^5, \text{ so } H^1(\Omega_{pen}, \mathbb{R}) = \mathbb{R}^5. \]
Penrose is almost rigid

- \( H^1(\Omega_{\text{pen}}) = \mathbb{Z}^5 \), so \( H^1(\Omega_{\text{pen}}, \mathbb{R}) = \mathbb{R}^5 \).
- Eigenvalues of \( \sigma^* : H^1 \rightarrow H^1 \) are \( \phi \), \( \phi \), \( 1 - \phi \), \( 1 - \phi \), and -1.
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- Shape changes parametrized by $H^1(\Omega, \mathbb{R}^2) = \mathbb{R}^{10}$.
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  - 4-dimensional family, corresponding to e-val $\phi$, that are rigid linear transformations.
  - 4-dimensional family, corresponding to e-val $1 - \phi$, that are topological conjugacies.
  - 2-dimensional family, corresponding to e-val $-1$. These break 180-degree rotational symmetry.
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  - 2-dimensional family, corresponding to e-val $-1$. These break 180-degree rotational symmetry.
- All shape changes that preserve 180 degree rotational symmetry are combinations of rigid linear transformations and topological conjugacies, and preserve dynamics.
Rational Penrose

180 Tiles
Theorem (Kellendonk-S)

If $T$ is a cut-and-project tiling of dimension $n$ and codimension $k$, and if the “window” is a finite union of polyhedra, then

$$H_{AN}^1(\Omega_T, \mathbb{R}) = \mathbb{R}^k.$$
AN classes for cut-and-project

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Roughly speaking, shape conjugacies come from phasons and nothing else.
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Theorem (Kellendonk-S)

Shape conjugacies of cut-and-project sets with polygonal windows are MLD to “reprojections”. Same total space, lattice, same window, different projection to $\mathbb{R}^n$. 
Cohomology and ergodic averages

- Counting a patch $P$ is the same thing as integrating a cochain (or bump form) that gives 1 every time $P$ appears.
- All $n$ cochains are closed, so this defines a cohomology class $[P]$. 

If $H^n(\Omega, R) = R^k$, there are $k$ patches $P_1, \ldots, P_k$ such that

$$\{[P_i]\}$$

generate $H^n$.

For any other patch $P$, $[P] = \sum c_j [P_j] + \delta\alpha$.

#($P$'s in a region $R$) $= \sum c_j$ #($P_j$'s in $R$) + boundary correction.
Cohomology and ergodic averages

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- $i_P = \sum c_j i_{P_j} + \delta \alpha$. 
Cohomology and ergodic averages

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- $\#(P's \text{ in a region } R) = \sum c_j \#(P_j's \text{ in } R) + \text{boundary correction}$
Frequency of \(aba\) in Thue-Morse

- \(H^1(\Omega_{TM}, \mathbb{R}) = \mathbb{R}^2\). Substitution acts with eigenvalues 2 and \(-1\). \(H^1_{AN}\) is trivial.
Frequency of \textit{aba} in Thue-Morse

\begin{itemize}
\item $H^1(\Omega_{TM}, \mathbb{R}) = \mathbb{R}^2$. Substitution acts with eigenvalues 2 and $-1$. $H^1_{AN}$ is trivial.
\item $[i_{aba}]$ is a nontrivial linear combination of the two eigenvectors.
\end{itemize}
Frequency of $aba$ in Thue-Morse

- $H^1(\Omega_{TM}, \mathbb{R}) = \mathbb{R}^2$. Substitution acts with eigenvalues 2 and $-1$. $H^1_{AN}$ is trivial.
- $[i_{aba}]$ is a nontrivial linear combination of the two eigenvectors.
- $[i_{aba}] - c_1 dx$ is not AN.
- Deviations in count of $aba$ are unbounded. (Actually grow as $\ln(N)$.)
Frequency of $aba$ in Thue-Morse

- $H^1(\Omega_{TM}, \mathbb{R}) = \mathbb{R}^2$. Substitution acts with eigenvalues 2 and $-1$. $H^1_{AN}$ is trivial.
- $[i_{aba}]$ is a nontrivial linear combination of the two eigenvectors.
- $[i_{aba}] - c_1 dx$ is not AN.
- Deviations in count of $aba$ are unbounded. (Actually grow as $\ln(N)$.)
- Nothing special about $aba$. Same thing applies to almost any pattern. (Just not $a$ or $b$).
If $f_1$ and $f_2$ are mass distributions on $T$, then $f_1$ and $f_2$ are closed and define cohomology classes $[f_1]$ and $[f_2]$. Then

- **Theorem:** There is a bounded transport from $f_1$ to $f_2$ if and only if $[f_1 - f_2]$ is well-balanced. (I.e. $\left\| \int_R (f_1 - f_2) \right\| \leq c \| \partial R \|$.)

There is a wPE transport from $f_1$ to $f_2$ if and only if $f_1 - f_2$ is weakly exact.

There is a sPE transport from $f_1$ to $f_2$ if and only if $f_1 - f_2$ is exact, i.e. if and only if $[f_1] = [f_2]$. 
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- There is a wPE transport from $f_1$ to $f_2$ if and only if $f_1 - f_2$ is weakly exact.
Cohomological answers to transport questions

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2 kg on the NE chairs
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1 kg on the NE and SW chairs
1 kg on the NW and SE chairs
Chair answers

- For the chair tiling, $H^2_{AN}$ is trivial and $H^2(\Omega_T, \mathbb{R}) = \mathbb{R}^3$.
- One generator counts all tiles equally. Not well-balanced.
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NE + SW - SE - SW is cohomologically trivial. Every 1-supertile has exactly two (NE or SW) and two (NW + SE). To get sPE transport, just move mass within each 1-supertile.
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One generator counts NE minus SW. This is \( f_1 - f_2 \). Not weakly exact, so there is no wPE transport.

(Last generator counts NW minus SE.)
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(Last generator counts NW minus SE.)

Remaining question: Is $f_1 - f_2$ well-balanced?
Scaling properties

Under substitution, \( f_1 - f_2 \) doubles at each stage:

\[
\begin{array}{cccc}
1 & 0 & -1 & 0 \\
2 & 0 & -2 & 0 \\
4 & & & \\
\end{array}
\]
On triangle of side length $N = 2^m$, $f_1 - f_2$ goes as $m2^m$. 
Tiling spaces are inverse limits.
Summary

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- $H^1(\Omega, \mathbb{R}^n)$ parametrizes shape changes. $H^1_{AN}$ parametrizes shape conjugacies.

Counting patches is the same as evaluating a top-cochain. Up to boundary terms, the answer only depends on the cohomology class. If you understand the cohomology, you know how all ergodic averages behave.

Mass distributions define classes in $H^n$. Bounded/wPE/sPE transport correspond to properties of $f_1 - f_2$.

Lots of other applications of cohomology, but we're out of time (and sliced bread).
Summary

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Thank You!