On a Lévy-Khinchine type decomposition on universal quantum groups and the related cohomological properties

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joint work with:
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Quantum Homogeneous Spaces, Edinburgh 2018
Figure 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion.

Figure 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion.

Source: A. Papapantoleon, An Introduction to Lévy Processes with Applications in Finance.
Lévy processes on $\mathbb{R}^n$

Instead of a definition

Lévy processes = stationary and independent increments.
Lévy processes on $\mathbb{R}^n$

Instead of a definition

Lévy processes $=$ stationary and independent increments.

Lévy-Khinchin Formula (1934/1937)

$X = (X_t)_t$ is a Lévy process on $\mathbb{R}^n$ iff the characteristic function

$$\phi_X(u) := \int_{\mathbb{R}^n} e^{i \langle u, x \rangle} \mu_X(dx) = e^{\eta_X(u)},$$

where

$$\eta_X(u) = i \langle b, u \rangle - \frac{1}{2} \langle u, \sigma u \rangle + \int_{\mathbb{R}^n} (e^{i \langle u, y \rangle} - 1 - i \langle u, y \rangle 1_{|y| \leq 1}) \nu(dy).$$

for some $b \in \mathbb{R}^n$, $\sigma \in \mathcal{M}(n, n)$ positive-definite and a 'Lévy measure' $\nu$ on $\mathbb{R}^n$. 
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Brownian motion with drift

jump part

for some $b \in \mathbb{R}^n$, $\sigma \in M(n, n)$ positive-definite and a 'Lévy measure' $\nu$ on $\mathbb{R}^n$. 

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LK on CQG
Lévy processes on Lie groups

Let $G$ be a Lie group, $\mathfrak{g}$ – the related Lie algebra.

- $(X_1, \ldots, X_n)$ basis in $\mathfrak{g}$
- $(e_1, \ldots, e^n)$ are canonical coordinates in a neighborhood of $e$,
- $(X^L_1, \ldots, X^L_n)$ derivations in the direction related to $X_i$

Hunt’s Formula (1956)

Lévy process on $G$ are in one-to-one correspondence with the generating functionals $L$ of the form

$$Lf(x) = \sum_i b_i X^L_i f(x) + \sum_{i,j} a_{ij} X^L_i X^L_j f(x)$$

$$+ \int_{G \setminus \{e\}} \left[ f(xy) - f(x) - \sum_i e^i(x)X^L_i(y) \right] \nu(dy)$$

for some $b \in \mathbb{R}^n$, $a = (a_{ij})_{i,j} \in M_n(\mathbb{R})$ positive definite, symmetric and a Lévy measure $\nu$ on $G \setminus \{e\}$. The domain of $L$ contains $C^\infty_c(G)$-functions.
Lévy process on ∗-bialgebra (Schürmann'1990)

Let $B$ be a ∗-bialgebra with the counit $\varepsilon$.

- **Lévy process** on $B$ is a family $(j_{st})_{0 \leq s \leq t}$ of ∗-homomorphisms $B \to (\mathcal{P}, \Phi)$ which are (tensor) independent and stationary. We also want to have $j_{st} \ast j_{tu} = j_{su}$ (with $\phi \ast \psi = (\phi \otimes \psi) \circ \Delta$).

- To every Lévy process one can associate $(\varphi_t)_{t \geq 0}$ on $B$, $\varphi_t = \Phi \circ j_{0t}$, which form a semigroup of states, i.e.
  \[
  \forall s, t \geq 0, \quad \varphi_s \ast \varphi_t = \varphi_{s+t},
  \]
  \[
  \forall a \in B, \quad \lim_{t \searrow 0} \varphi_t(a) = \varphi_0(a) = \varepsilon(a).
  \]

- For the semigroup of states there exists a generating functional (GF)
  \[
  L = \left. \frac{d}{dt} \right|_{t=0} \varphi_t.
  \]

- $L : B \to \mathbb{C}$ is a generating functional iff
  \[
  \bullet L(1) = 0 \quad \bullet L(a^*) = \overline{L(a)} \quad \bullet L(a^*a) \geq 0 (a \in \ker \varepsilon).
  \]
Lévy-Khintchine decomposition: Gaussian gen. functionals

Lévy processes ↔ Generating functionals ↔ LK-formula ???
Lévy-Khintchine decomposition: Gaussian gen. functionals

Lévy processes ↔ Generating functionals ↔ LK-formula

Definition (Schürmann’1990)

Let $L_G$ be a generating functional, i.e. $L_G(1) = 0$, $L_G(a^*) = \overline{L_G(a)}$ and $L_G(a^*a) \geq 0$ or $a \in \ker \varepsilon$. We say that $L_G$ is:

- **Gaussian** if $L_G(abc) = 0$ for $a, b, c \in \ker \varepsilon$;
- **Gaussian component** of a GF $L$ if $L_G$ is Gaussian and $L - L_G$ is a generating functional (conditionally positive);
- **maximal Gaussian component** of $L$ if $L_G - L'_G$ is conditionally positive for all Gaussian components $L'_G$ of $L$.

Definition

We say that a generating functional $L$ on $B$ **admits a Lévy-Khintchine decomposition** if there exists a maximal Gaussian component $L_G$ such that

$$L = L_G + L_R.$$
Let $\mathcal{B}$ be a $\ast$-bialgebra (in fact: $\ast$-algebra with a distinguished character $\varepsilon$).

**Question**

- Can we always extract a maximal Gaussian component from a generating functional?
- Given an augmented $\ast$-algebra $(\mathcal{B}, \varepsilon)$, does any generating functional on it admits a Lévy-Khintchine decomposition?
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**Question**

- Can we always extract a maximal Gaussian component from a generating functional?
- Given an augmented $\ast$-algebra $(\mathcal{B}, \varepsilon)$, does any generating functional on it admits a Lévy-Khintchine decomposition?

**Definition**

We say that $\mathcal{B}$ has the property (LK) if any generating functional on $\mathcal{B}$ admits a Lévy-Khintchine decomposition.
Lévy-Khintchine decomposition: known results

The following ∗-bialgebras has the property (GC) hence also (LK):

- “commutative” Lévy processes (Lévy, Khinchin, Hunt; 1930-1960)
- commutative ∗-bialgebras (Schürmann; 1990)
- the Brown-Glockner-von Waldenfels algebra $K\langle d \rangle$, i.e. the universal C*-algebra generated by the relations $\sum u_{jp} u_{kp}^* = \delta_{jk}1 = \sum u_{pj}^* u_{pk}$ (Schürmann; 1990)
- $SU_q(2)$ (Schürmann, Skeide; 1998)
- $S_n^+$, since no non-zero Gaussian cocycles (Franz, AK, Skalski; 2014)
- $S_n^+(D) = S_n^+ / \langle uD = Du \rangle$, quantum reflexion groups, quantum automorphism groups of graphs (Bichon, Franz, Gerhold; 2017)
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Question

Does every ∗-bialgebra/compact quantum group have (LK)?
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Counterexample (Franz, Gerhold, Thom; CSA 2015)

no (LK): fundamental group of a closed oriented surface, genus $\geq 2$
This is a cocommutative quantum group!
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The following ∗-bialgebras has the property (GC) hence also (LK):

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Counterexample (Franz, Gerhold, Thom; CSA 2015)

no (LK): fundamental group of a closed oriented surface, genus $\geq 2$ This is a cocommutative quantum group! So what goes wrong?
Schürmann triple associated to a generating functional

Generating functional $L \leftrightarrow$ Schürmann triple $(\pi, \eta, L)$

- $\pi : \mathcal{B} \rightarrow L(H)$ is a **unital $\ast$-representation** of $\mathcal{B}$ on some pre-Hilbert space $H$,
- $\eta : \mathcal{B} \rightarrow H$ is a **$\pi$-$\varepsilon$-cocycle**, i.e. a linear mapping satisfying
  \[ \eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b), \]
- $L : \mathcal{B} \rightarrow \mathbb{C}$ is a hermitian linear functional such that
  \[ L(ab) = \langle \eta(a^*), \eta(b) \rangle + \varepsilon(a)L(b) + L(a)\varepsilon(b). \]

**Remarks**
- Let $(\pi, \eta, L)$ be the Schürmann triple. Then:
  - $L$ is Gaussian, i.e. $L(abc) = 0$ if $a, b, c \in \ker \varepsilon$;
  - $\eta$ is Gaussian, i.e. $\eta(ab) = 0$ if $a, b \in \ker \varepsilon$;
  - $\pi$ is of the form $\pi(a) = \varepsilon(a)\text{id}_H$. 

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Remarks

Let $(\pi, \eta, L)$ be the Schürmann triple. TFAE:

- $L$ is Gaussian, i.e. $L(abc) = 0$ if $a, b, c \in \ker \varepsilon$;
- $\eta$ is **Gaussian**, i.e. $\eta(ab) = 0$ if $a, b \in \ker \varepsilon$;
- $\pi$ is of the form $\pi(a) = \varepsilon(a)\text{id}_H$. 
Let $L$ be a generating functional with the Schürmann triple $(H, \pi, \eta, L)$

- $H_G := \bigcap_{a \in \ker \varepsilon} \ker \pi(a) = \{ u \in H : \pi(a)u = \varepsilon(a)u, a \in \mathcal{B} \}$

is the maximal Gaussian subspace of $H$ which is reducing for $\pi$. Hence $\pi = \pi_G \oplus \pi_R$. 
Let $L$ be a generating functional with the Schürmann triple $(H, \pi, \eta, L)$

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  Hence $\pi = \pi_G \oplus \pi_R$.

- Let $P_G$ be the orthogonal projection onto $H_G$. Then $\eta_G := P_G \circ \eta$ is a Gaussian cocycle and $\eta_R = (I - P_G) \circ \eta$ is purely non-Gaussian, i.e. $(H_R)_G = \{0\}$. So $\eta = \eta_G \oplus \eta_R$ and
  $$(H, \pi, \eta) = (H_G, \pi_G, \eta_G) \oplus (H_R, \pi_R, \eta_R).$$
Let $L$ be a generating functional with the Schürmann triple $(H, \pi, \eta, L)$

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\[(H, \pi, \eta) = (H_G, \pi_G, \eta_G) \oplus (H_R, \pi_R, \eta_R).\]

- To get the decomposition $L = L_G + L_R$ we need to complete $(H_G, \pi_G, \eta_G)$ or $(H_R, \pi_R, \eta_R)$ by generating functionals.

- In general, it may be impossible to complete $((H_G, \pi_G), \eta_G)$ into the triple, since $L_G(x) = \langle \eta_G(a^*), \eta_G(b) \rangle$ for $x = ab$, $a, b \in \ker \varepsilon$. 

Example of a cocycle without the generating functional

- $\mathcal{B}$: free unital commutative $\ast$-algebra generated by $x$,
- counit: $\varepsilon(1) = 1$, $\varepsilon(x) = 0$.

Take $z, w \in \mathbb{C}$, $|z| \neq |w|$, and define

$$\eta(x) = z, \quad \eta(x^*) = w, \quad \eta(y) = 0$$

for $y$ any monomial with degree not equal to 1, and extend by linearity. Then $\eta$ is a Gaussian cocycle.

If there exists a conditionally positive functional $L$ related to $\eta$, then

$$L(xx^*) = \langle \eta(x^*), \eta(x^*) \rangle = |w|^2, \quad L(x^*x) = \langle \eta(x), \eta(x) \rangle = |z|^2.$$  

However, due to the commutativity of $\mathcal{B}$, $L(xx^*) = L(x^*x)$. Contradiction.
We say that $\mathcal{B}$ has:

- the **property (LK)** if any generating functional on $\mathcal{B}$ admits the Lévy-Khintchine decomposition.

- the **property (GC)** if any Gaussian cocycle $\eta : \mathcal{B} \to H$ can be completed to a Schürmann triples $(\varepsilon \text{id}, \eta, \psi)$.

- the **property (NC)** if any pair $(\rho, \eta)$ consisting of a $\ast$-representation $\rho : \mathcal{B} \to B(H)$ and a $\rho$-$\varepsilon$-cocycle $\eta : \mathcal{B} \to H$ with $H_G = \{0\}$ can be completed to a Schürmann triples $(\rho, \eta, \psi)$. 

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**Remark**

If for $(H, \pi, \eta) = (H_G, \pi_G, \eta_G) \oplus (H_R, \pi_R, \eta_R)$ there exists a generating functional $L_x$ such that $(\pi_x, \eta_x, L_x)$ $(x = G$ or $x = N)$, then $L_y = L - L_x$ is a generating functional to and $L = L_x + L_y$. 

$(\text{GC}) \lor (\text{NC}) \Rightarrow (\text{LK})$
We say that $B$ has:

- the **property (LK)** if any generating functional on $B$ admits the Lévy-Khintchine decomposition.

- the **property (GC)** if any Gaussian cocycle $\eta : B \to H$ can be completed to a Schürmann triples $(\varepsilon \text{id}, \eta, \psi)$.

- the **property (NC)** if any pair $(\rho, \eta)$ consisting of a $\ast$-representation $\rho : B \to B(H)$ and a $\rho \varepsilon$-cocycle $\eta : B \to H$ with $H_G = \{0\}$ can be completed to a Schürmann triples $(\rho, \eta, \psi)$.

**Remark**

If for $(H, \pi, \eta) = (H_G, \pi_G, \eta_G) \oplus (H_R, \pi_R, \eta_R)$ there exists a generating functional $L_x$ such that $(\pi_x, \eta_x, L_x)$ ($x = G$ or $x = N$), then $L_y = L - L_x$ is a generating functional too and $L = L_x + L_y$.

$$(\text{GC}) \lor (\text{NC}) \Rightarrow (\text{LK})$$
Let $G$ be a compact matrix quantum group, i.e. $G = (A, u)$ with a unital C*-algebra $A$ and a unitary matrix $u = (u_{jk})_{j, k = 1}^d \in M_d(A)$ such that

- $A$ generated by $u_{jk}$ ($j, k = 1, \ldots, d$)
- $\Delta(u_{jk}) = \sum_p u_{jp} \otimes u_{pk}$ extends to a $*$-homomorphism on $A$,
- $\bar{u}$ is invertible.

Then the dense $B = Pol(G)$ is a $*$-bialgebra with $\varepsilon(u_{jk}) = \delta_{jk}$. 

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Compact quantum groups framework

Let $\mathbb{G}$ be a **compact matrix quantum group**, i.e. $\mathbb{G} = (A, u)$ with a unital C*-algebra $A$ and a unitary matrix $u = (u_{jk})_{j,k=1}^d \in M_d(A)$ such that

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Then the dense $B = \text{Pol}(\mathbb{G})$ is a $*$-bialgebra with $\varepsilon(u_{jk}) = \delta_{jk}$.

**Example: universal unitary and orthogonal quantum groups**

Let $d \in \mathbb{N}_2$, $F \in GL_d(\mathbb{C})$,

$$\text{Pol}(U_F^+) := \ast\text{-Alg}\langle u_{jk} \rangle_{j,k=1}^d / \langle uu^* = I = u^*u, F\bar{u}F^{-1}\text{-unitary} \rangle$$
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\textbf{Example: universal unitary and orthogonal quantum groups}

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$$

$$
\text{Pol}(O_F^+) := \text{-Alg} \langle u_{jk} \rangle_{j,k=1}^d / \langle uu^* = I = u^*u, u = F\bar{u}F^{-1} \rangle.
$$
Theorem (DFKS’2018)

(a) If a matrix $F \in GL_d(\mathbb{C})$ is such that $F^*F$ has pairwise distinct eigenvalues, then both $U_F^+$ and $O_F^+$ have the (GC) and the (LK) properties.

(b) The quantum groups $U_d^+ (d \geq 2)$ and $O_d^+ (d \geq 3)$ do have neither (GC), nor (NC), nor (LK) property.

Remarks:

- The first example of non-cocommutative quantum group without (LK)!
- $O_2^+$ has (GC) and (LK), but not (NC) (Skeide’1999 as $O_2^+ \cong SU_{-1}(2)$).
Strategy: quotients of $K\langle d \rangle$

$K\langle d \rangle$: the universal unital $\ast$-algebra generated by $x_{jk}$ ($j, k = 1, 2, \ldots d$) such that the matrix $x := (x_{jk})_{j,k=1}^d$ satisfies $xx^* = I = x^*x$.

**Definition**

$(\text{Pol}(G), u)$ is a quotient of $(K\langle d \rangle, x)$ if $\dim u = d$ and there is a unital (surjective) $\ast$-homomorphism $q : K\langle d \rangle \to \text{Pol}(G)$ such that

$$q(x_{jk}) = u_{jk}, \quad j, k = 1, \ldots, d.$$  

**Examples:**

- $U_F^+$ is a quotient of $K\langle d \rangle$ by $x^t Q\bar{x} Q^{-1} = I = Q\bar{x} Q^{-1} x^t$ ($Q = F^* F$);
- $O_F^+$ is a quotient of $K\langle d \rangle$ by $x = F\bar{x} F^{-1}$;
- $SU_q(d)$ is a quotient of $K\langle d \rangle$ by the twisted determinant condition.
Strategy: quotients of $K\langle d \rangle$

**Theorem (Schürmann’1990)**

Each cocycle $\eta'$ on $K\langle d \rangle$ admits a generating functional $L'$. The functional is uniquely defined by the rule:

$$L'(x_{jk}) = -\frac{1}{2} \sum_{n=1}^{d} \langle \eta'(x^*_{jn}), \eta'(x^*_{kn}) \rangle.$$
Strategy: quotients of $K\langle d \rangle$

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Let $\text{Pol}(G)$ be a quotient of $K\langle d \rangle$, $q : K\langle d \rangle \rightarrow \text{Pol}(G)$.

- $(\rho, \eta, L)$ are uniquely determined by the values on $u_{jk}$ and $u^*_{jk}$;
- $\eta$ - Gaussian cocycle on $G \Rightarrow \eta' = \eta \circ q$ - Gaussian on $K\langle d \rangle$;
- $\eta'$ admits a generating functional $L'$ on $K\langle d \rangle$;
- $L(q(a)) := L'(a)$ is a GF on $G$, provided it is well-defined, which happens when $L|_{\ker q} = 0$. 

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LK on CQG
Hochschild cohomology

$B$: an algebra, $M$: a $A$-bimodule with $v$. $b = \pi(a)v\rho(b)$ for $a, b, \in B$ and $v \in M$.

$n$-cochains:

$C_n(B, \pi M\rho) := \{\phi: B \otimes^n \rightarrow M, \text{linear}\}$

coboundary operator:

$\partial_n: C_n(B, \pi M\rho) \rightarrow C_{n+1}(B, \pi M\rho)$

$\partial_n \phi(a_1 \otimes a_{n+1}) = \pi(a_1)\phi(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{j=1}^{n}(-1)^j \phi(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1}\phi(a_1 \otimes \cdots \otimes a_n)\rho(a_{n+1})$.

$n$-cycles:

$Z_n(B, \pi M\rho) := \{\phi \in C_n(B, \pi M\rho): \partial_n \phi = 0\}$

$n$-coboundaries:

$B_n(B, \pi M\rho) := \partial_{n-1} - \partial_n C_{n-1}(B, \pi M\rho)$

$n$th cohomology group:

$H_n(B, \pi M\rho) := Z_n(B, \pi M\rho)/B_n(B, \pi M\rho)$.
Hochschild cohomology

$\mathcal{B}$ : an algebra,
$M$ : a $A$-bimodule with $a.v.b = \pi(a)\nu\rho(b)$ for $a, b, \in \mathcal{B}$ and $\nu \in M$

- $n$-cochains:

$$C^n(\mathcal{B}, \pi M_\rho) := \{\phi : \mathcal{B}^{\otimes n} \rightarrow M, \text{linear}\}$$

- coboundary operator: $\partial^n : C^n(\mathcal{B}, \pi M_\rho) \rightarrow C^{n+1}(\mathcal{B}, \pi M_\rho)$

\[
\partial^n \phi(a_1 \otimes a_{n+1}) = \pi(a_1)\phi(a_2 \otimes \ldots \otimes a_{n+1}) + \\
\sum_{j=1}^{n} (-1)^j \phi(a_1 \otimes \ldots a_j a_{j+1} \ldots \otimes a_{n+1}) + (-1)^{n+1} \phi(a_1 \otimes \ldots \otimes a_n)\rho(a_{n+1})
\]

- $n$-cocycles: $Z^n(\mathcal{B}, \pi M_\rho) := \{\phi \in C^n(\mathcal{B}, \pi M_\rho) : \partial^n \phi = 0\}$

- $n$-coboundaries: $B^n(\mathcal{B}, \pi M_\rho) := \partial^{n-1} C^{n-1}(\mathcal{B}, \pi M_\rho)$

- $n$th cohomology group:

$$H^n(\mathcal{B}, \pi M_\rho) := Z^n(\mathcal{B}, \pi M_\rho) / B^n(\mathcal{B}, \pi M_\rho)$$
Relation of (LK) with the Hochschild cohomology

Observation

For $\eta : B \to H$ linear we have

1. $\eta \in Z^1(B, \pi H_\varepsilon)$ iff $\eta$ is a $\pi$-$\varepsilon$-cocycle (from Schürmann triples).

2. $H^1(B, \varepsilon C_\varepsilon) = \{\mathbb{C}$-valued Gaussian cocycles$\}$. 
Relation of (LK) with the Hochschild cohomology

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We define

\[ c_\eta : B \otimes B \ni a \otimes b \mapsto -\langle \eta(a^*), \eta(b) \rangle \in \mathbb{C}. \]

3. If \( \eta \in Z^1(B, \pi H_\varepsilon) \), then \( c_\eta \in Z^2(B, \varepsilon C_\varepsilon) \).
4. In such case \( c_\eta \in B^2(B, \varepsilon C_\varepsilon) \) iff \( \eta \) admits a GF \( \psi \).
Relation of (LK) with the Hochschild cohomology

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For $\eta : \mathcal{B} \to H$ linear we have

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We define $c_\eta : \mathcal{B} \otimes \mathcal{B} \ni a \otimes b \mapsto -\langle \eta(a^*), \eta(b) \rangle \in \mathbb{C}$.

3. If $\eta \in Z^1(\mathcal{B}, \pi H_\varepsilon)$, then $c_\eta \in Z^2(\mathcal{B}, \varepsilon \mathbb{C}_\varepsilon)$.
4. In such case $c_\eta \in B^2(\mathcal{B}, \varepsilon \mathbb{C}_\varepsilon)$ iff $\eta$ admits a GF $\psi$.

1. $\eta \in Z^1(\mathcal{B}, \pi H_\varepsilon)$ iff $\partial^1 \phi(a \otimes b) = \pi(a)\phi(b) - \phi(ab) + \phi(a)\varepsilon(b) = 0$
2. $\phi \in B^1(\mathcal{B}, \varepsilon \mathbb{C}_\varepsilon)$ iff $\exists \psi : \mathbb{C} \to \mathbb{C}$ s.t.
$$\phi(a) = \partial^0 \psi(a) = \varepsilon(a)\psi - \psi\varepsilon(a) = 0.$$
3. Check that $\partial^3 c_\eta = 0$.
4. $c_\eta \in B^2(\mathcal{B}, \varepsilon \mathbb{C}_\varepsilon)$ iff there exists $\psi : \mathcal{B} \to \mathbb{C}$ s.t.
$$c_\eta(a \otimes b) = \partial^1 \psi(a \otimes b) = \varepsilon(a)\psi(b) - \psi(ab) + \psi(a)\varepsilon(b)$$
Observation

For $\eta : \mathcal{B} \rightarrow H$ linear we define

$$c_\eta : \mathcal{B} \otimes \mathcal{B} \ni a \otimes b \mapsto -\langle \eta(a^*), \eta(b) \rangle \in \mathbb{C}.$$ 

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Relation of (LK) with the Hochschild cohomology

**Observation**

For $\eta : B \to H$ linear we define

$$c_\eta : B \otimes B \ni a \otimes b \mapsto -\langle \eta(a^*), \eta(b) \rangle \in \mathbb{C}.$$  

1. If $\eta \in Z^1(B, \pi H_{\varepsilon})$, then $c_\eta \in Z^2(B, \varepsilon C_{\varepsilon})$.
2. In such case $c_\eta \in B^2(B, \varepsilon C_{\varepsilon})$ iff $\eta$ admits a GF $\psi$.

**Theorem (Franz, Gerhold, Thom; 2015)**

If $H^2(B, \varepsilon C_{\varepsilon}) = \{0\}$, then any pair $(\pi, \eta)$ can be completed to a Schürmann triple, hence $B$ has the property (LK).

- Let $\pi$ is a representation of $B$ and $\eta$ is a $\pi$-$\varepsilon$-cocycle
- $H^2(B, \varepsilon C_{\varepsilon}) = \{0\} \Rightarrow Z^2(B, \varepsilon C_{\varepsilon}) = B^2(B, \varepsilon C_{\varepsilon})$
- $\eta \in Z^1(B, \pi H_{\varepsilon}) \Rightarrow c_\eta \in Z^2(B, \varepsilon C_{\varepsilon}) = B^2(B, \varepsilon C_{\varepsilon}) \Rightarrow \eta$ admits a GF.
Remark

If $B$ does not have the property (LK) then $H^2(B, \varepsilon \mathbb{C}_\varepsilon) \neq \{0\}$

Theorem (Collons, Härtel, Thom ’2009; Bichon’2013)

- For $F = I_d$:
  $$H^1(O_d^+, \varepsilon \mathbb{C}_\varepsilon) = H^2(O_d^+, \varepsilon \mathbb{C}_\varepsilon) \simeq \{ M \in M_d(\mathbb{C}), \ M + M^t = 0 \}.$$  

- For general $F$ (by Poincaré duality):
  $$H^1(O_F^+, \varepsilon \mathbb{C}_\varepsilon) \simeq \left\{ \frac{M \in M_d(\mathbb{C}), \ M + \bar{F} M^t \bar{F}^{-1} = 0}{\lambda (\bar{F}(F^*)^{-1} - F^* ar{F}^{-1})}, \lambda \in \mathbb{C} \right\},$$  
  $$H^2(O_F^+, \varepsilon \mathbb{C}_\varepsilon) \simeq \frac{M_d(\mathbb{C})}{\{ M + \bar{F} M^t \bar{F}^{-1} : M \in M_d(\mathbb{C}) \}}.$$
Hochschild cohomology for $U_F^+$?

Let $F = I_d$.

- $\mathbb{C}$-valued Gaussian cocycle $\eta \leftrightarrow V \in M_d(\mathbb{C})$; $H^1(U_d^+, \mathbb{C}\varepsilon) = M_d(\mathbb{C})$
Let $F = I_d$.

- $\mathbb{C}$-valued Gaussian cocycle $\eta \leftrightarrow V \in M_d(\mathbb{C})$; $H^1(U^+_d, \varepsilon \mathbb{C}_\varepsilon) = M_d(\mathbb{C})$
- $\eta \in Z^1(U^+_d, \pi H_\varepsilon)$ admits a GF if and only if $c_\eta \in Z^2(U^+_d, \pi H_\varepsilon)$ satisfies

\[
\sum_{p=1}^d \left( \langle \eta(u_{pj}), \eta(u_{pk}) \rangle - c_\eta(u_{pj}^* \otimes u_{pk}) \right) = \sum_{p=1}^d \left( \langle \eta(u_{kp}), \eta(u_{jp}) \rangle - c_\eta(u_{kp}^* \otimes u_{jp}) \right), \quad j, k = 1, \ldots, d.
\]
Hochschild cohomology for $U_F^+$?

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\sum_{p=1}^{d} \left\langle \eta(u_{pj}), \eta(u_{pk}) \right\rangle = \sum_{p=1}^{d} \left\langle \eta(u_{kp}), \eta(u_{jp}) \right\rangle, \quad j, k = 1, \ldots, d.
\]

- we show that any $c \in B^2(U_d^+, \varepsilon \mathbb{C})$ iff

\[
\sum_{p=1}^{d} c(u_{pj}^* \otimes u_{pk}) = \sum_{p=1}^{d} c(u_{kp}^* \otimes u_{jp}), \quad j, k = 1, \ldots, d.
\]

- the matrix $\Delta(c)$ defined by

\[
\Delta(c)_{jk} := \sum_{p=1}^{d} (c(u_{pj}^* \otimes u_{pk}) - c(u_{kp}^* \otimes u_{jp}))
\]

measures “how far $c$ is from being” a 2-coboundary.
Hochschild cohomology for $U_F^+$ ? $F = I$

**Theorem (DFKS’2018)**

Define $\Delta : Z^2(U_d^+, \epsilon \mathbb{C}_\epsilon) \to M_d(\mathbb{C})$ by the formula

$$\Delta(c) = \left( \sum_{p=1}^d \left( c(u^*_p \otimes u_{pk}) - c(u^*_p \otimes u_{jp}) \right) \right)_{j,k=1}^d.$$

Then $\text{Ker} \Delta = B_2(U_d^+, \epsilon \mathbb{C}_\epsilon)$ and $\text{Im} \Delta = \mathcal{S}(d) := \{ M \in M_d(\mathbb{C}) : \text{Tr}(M) = 0 \}$.

Hence $H_2(U_d^+, \epsilon \mathbb{C}_\epsilon) = Z_2(U_d^+, \epsilon \mathbb{C}_\epsilon) / B_2(U_d^+, \epsilon \mathbb{C}_\epsilon) \cong \mathcal{S}(d)$.

In particular, $\dim H_2(U_d^+, \epsilon \mathbb{C}_\epsilon) = d^2 - 1$. 

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LK on CQG
Hochschild cohomology for $U_F^+$ ? $F = I$

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Define $\Delta : Z^2(U_d^+, \varepsilon \mathbb{C}_\varepsilon) \to M_d(\mathbb{C})$ by the formula

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\Delta(c) = \left( \sum_{p=1}^{d} (c(u_{pj}^* \otimes u_{pk}) - c(u_{kp}^* \otimes u_{jp})) \right)^d_{j,k=1}. 
$$

Then

- $\text{Ker } \Delta = B^2(U_d^+, \varepsilon \mathbb{C}_\varepsilon)$
- $\text{Im } \Delta = sl(d) := \{ M \in M_d(\mathbb{C}) : \text{Tr}(M) = 0 \}$.

Hence

$$
H^2(U_d^+, \varepsilon \mathbb{C}_\varepsilon) = Z^2(U_d^+, \varepsilon \mathbb{C}_\varepsilon) / B^2(U_d^+, \varepsilon \mathbb{C}_\varepsilon) \cong sl(d).
$$

In particular, $\dim H^2(U_d^+, \varepsilon \mathbb{C}_\varepsilon) = d^2 - 1.$
Hochschild cohomology for $U_F^+$?

Let $Q = F^* F$ be a positive diagonal matrix, $Q = \sum_{i=1}^n \lambda_i P_{d_i}$, $\lambda_i$'s: (different) eigenvalues, $P_{d_i}$'s: $d_i$-dimensional projections on eigenspaces.

**Theorem to be...**

Define $\Delta : Z^2(U_F^+, \epsilon \mathbb{C}) \to M_d(\mathbb{C})$ by the formula

$$D(c) = \sum_{p=1}^d \left( c(u^*_pj \otimes u_{pk}) - \frac{Q_k}{Q_p} c(u^*_kp \otimes u_{jp}) \right), \quad \Delta(c) = \sum_{i=1}^n P_{d_i} DP_{d_i}$$
Hochschild cohomology for $U_F^+$?

Let $Q = F^*F$ be a positive diagonal matrix, $Q = \sum_{i=1}^n \lambda_i P_{d_i}$, $\lambda_i$'s: (different) eigenvalues, $P_{d_i}$'s: $d_i$-dimensional projections on eigenspaces.

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Then

- $\text{Ker } \Delta = B^2(U_F^+, \varepsilon C_\varepsilon)$,
- $H^2(U_d^+, \varepsilon C_\varepsilon) = Z^2(U_d^+, \varepsilon C_\varepsilon) / B^2(U_d^+, \varepsilon C_\varepsilon) \cong \text{Im } \Delta$,
- $\text{sl}(d_1) \oplus \ldots \oplus \text{sl}(d_n) \subset \text{Im } \Delta \subset \text{sl}_Q(d)$,

$\text{sl}_Q(d) = \{ M \in M_d(\mathbb{C}) : MQ = QM, \text{Tr}(QM) = \text{Tr}(Q^{-1}M) = 0 \}$.

In particular, $\sum_{i=1}^n d_i^2 - n \leq \dim H^2(U_d^+, \varepsilon C_\varepsilon) \leq \sum_{i=1}^n d_i^2 - 2 \quad (n \geq 2)$. 

Anna Wysoczanska-Kula
LK on CQG
Open problem

Find a characterisation of $\ast$-bialgebras which have the property (LK).

We know that:

- if $H^2(B, \varepsilon C_\varepsilon) = \{0\}$ then $B$ has the property (LK);
- $U^+_F$ has the property (LK) iff $F^*F$ has all eigenvalues of multiplicity 1.
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Find a characterisation of $\ast$-bialgebras which have the property (LK).

We know that:

- if $H^2(\mathcal{B}, \mathbb{C}) = \{0\}$ then $\mathcal{B}$ has the property (LK);
- $U_F^+$ has the property (LK) iff $F^*F$ has all eigenvalues of multiplicity 1.

Observation

Neither (LK), nor (GC), nor their negations transfer to quotients.

- (GC): $O_2^+(\text{GC}) \subset U_2^+ \quad \text{no (GC)}, \quad SU_q(2)_{\text{GC}}^+ \subset SU_q(3)_{\text{no (GC)}} \subset U_3^+(F)_{\text{GC}}$
- (LK): $O_2^+(\text{LK}) \subset O_3^+ \quad \text{no (LK)} \subset K\langle 3 \rangle_{\text{(LK)}}$. 