On a Lévy-Khinchine type decomposition on universal quantum groups and the related cohomological properties

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joint work with:
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Classical Lévy processes

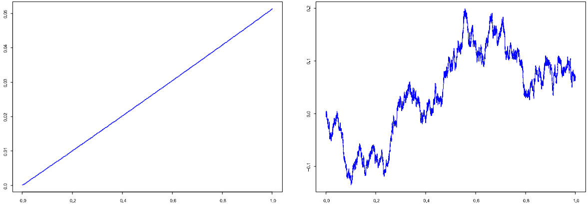

FIGURE 2.4. Examples of Lévy processes: linear drift (left) and Brownian motion.

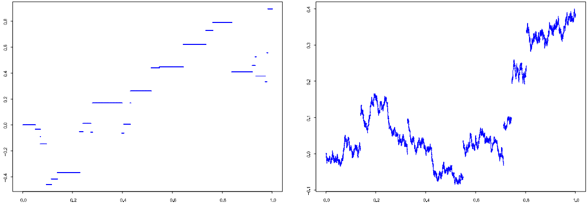

FIGURE 2.5. Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion.

Source: A. Papapantoleon, An Introduction to Lévy Processes with Applications in Finance.

Lévy processes on $\mathbb{R}^n$

Instead of a definition

Lévy processes = stationary and independent increments.

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Lévy-Khinchin Formula (1934/1937)

$X = (X_t)_t$ is a Lévy process on $\mathbb{R}^n$ iff the characteristic function

$$\phi_X(u) := \int_{\mathbb{R}^n} e^{\mathrm{i}\langle u,x\rangle} \mu_{X_1}(dx) = e^{\eta_X(u)},$$

where

$$\eta_X(u) = \mathrm{i}\langle b, u\rangle - \frac{1}{2}\langle u, \sigma u\rangle + \int_{\mathbb{R}^n} (e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle 1_{|y|\leq 1})\nu(dy).$$

for some $b \in \mathbb{R}^n$, $\sigma \in M(n, n)$ positive-definite and a 'Lévy measure' ν on $\mathbb{R}^n$.

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where

$$\eta_X(u) = \underbrace{\mathrm{i}\langle b, u\rangle - \frac{1}{2}\langle u, \sigma u\rangle}_{\text{Brownian motion with drift}} + \underbrace{\int_{\mathbb{R}^n} (e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle 1_{|y|\leq 1})\nu(dy)}_{\text{jump part}}.$$

for some $b \in \mathbb{R}^n$, $\sigma \in M(n, n)$ positive-definite and a 'Lévy measure' ν on $\mathbb{R}^n$.

Lévy processes on Lie groups

Let G be a Lie group, $\mathfrak{g}$ – the related Lie algebra.

- $(X_1, \ldots, X_n)$ basis in $\mathfrak{g}$
- $(e_1, \ldots, e^n)$ are canonical coordinates in a neighborhood of e,
- $(X_1^L, \ldots, X_n^L)$ derivations in the direction related to X_i

Hunt's Formula (1956)

Lévy process on G are in one-to-one correspondence with the generating functionals L of the form

$$\begin{aligned} Lf(x) &= \sum_i b_i X_i^L f(x) + \sum_{i,j} a_{ij} X_i^L X_j^L f(x) \\ &+ \int_{G \setminus \{e\}} \left[f(xy) - f(x) - \sum_i e^i(x) X_i^L(y) \right] \nu(dy) \end{aligned}$$

for some $b \in \mathbb{R}^n$, $a = (a_{ij})_{i,j} \in M_n(\mathbb{R})$ positive definite, symmetric and a Lévy measure ν on $G \setminus \{e\}$. The domain of L contains $C_c^\infty(G)$-functions.

Lévy process on $*$-bialgebra (Schürmann'1990)

Let $\mathcal{B}$ be a $*$-bialgebra with the counit ε.

- Lévy process on $\mathcal{B}$ is a family $(j_{st})_{0\le s\le t}$ of $*$-homomorphisms $\mathcal{B} \to (\mathcal{P}, \Phi)$ which are (tensor) independent and stationary. We also want to have $j_{st} \star j_{tu} = j_{su}$ (with $\phi \star \psi = (\phi \otimes \psi) \circ \Delta$).
- To every Lévy process one can associate $(\varphi_t)_{t\ge 0}$ on $\mathcal{B}$, $\varphi_t = \Phi \circ j_{0t}$, which form a semigroup of states, i.e.

$$\forall s,t \ge 0, \qquad \varphi_s \star \varphi_t = \varphi_{s+t},$$
$$\forall a \in \mathcal{B}, \qquad \lim_{t \searrow 0} \varphi_t(a) = \varphi_0(a) = \varepsilon(a).$$

- For the semigroup of states there exists a generating functional (GF)

$$L = \frac{d}{dt}\Big|_{t=0} \varphi_t.$$

- $L : \mathcal{B} \to \mathbb{C}$ is a generating functional iff
  - $L(\mathbf{1}) = 0$ • $L(a^*) = \overline{L(a)}$ • $L(a^* a) \ge 0$ $(a \in \ker \varepsilon)$.

Lévy processes $\leftrightarrow$ Generating functionals $\leftrightarrow$ **LK-formula ???**

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Definition (Schürmann'1990)

Let L_{G} be a generating functional, i.e. $L_{\mathrm{G}}(\mathbf{1}) = 0$, $L_{\mathrm{G}}(a^*) = \overline{L_{\mathrm{G}}(a)}$ and $L_{\mathrm{G}}(a^*a) \geq 0$ or $a \in \ker \varepsilon$. We say that L_{G} is:

- **Gaussian** if $L_{\mathrm{G}}(abc) = 0$ for $a, b, c \in \ker \varepsilon$;
- **Gaussian component** of a GF L if L_{G} is Gaussian and $L - L_{\mathrm{G}}$ is a generating functional (conditionally positive;
- **maximal Gaussian component** of L if $L_{\mathrm{G}} - L'_{\mathrm{G}}$ is conditionally positive for all Gaussian components L'_{G} of L.

Definition

We say that a generating functional L on $\mathcal{B}$ **admits a Lévy-Khintchine decomposition** if there exists a maximal Gaussian component L_{G} such that

$$L = L_{\mathrm{G}} + L_{\mathrm{R}}.$$

Lévy-Khintchine decomposition: maximal Gaussian component

Let $\mathcal{B}$ be a $*$-bialgebra (in fact: $*$-algebra with a distinguished character ε).

Question

- Can we always extract a maximal Gaussian component from a generating functional?
- Given an augmented $*$-algebra $(\mathcal{B}, \varepsilon)$, does any generating functional on it admits a Lévy-Khintchine decomposition?

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We say that $\mathcal{B}$ has the **property (LK)** if any generating functional on $\mathcal{B}$ admits a Lévy-Khintchine decomposition.

Lévy-Khintchine decomposition: known results

The following $*$-bialgebras has the property (GC) hence also (LK):

- "commutative" Lévy processes (Lévy, Khinchin, Hunt; 1930-1960)
- commutative $*$-bialgebras (Schürmann; 1990)
- the Brown-Glockner-von Waldenfels algebra $K\langle d\rangle$, i.e. the universal C*-algebra generated by the relations $\sum u_{jp}u^*_{kp} = \delta_{jk}1 = \sum u^*_{pj}u_{pk}$ (Schürmann; 1990)
- $SU_q(2)$ (Schürmann, Skeide; 1998)
- S_n^+, since no non-zero Gaussian cocycles (Franz, AK, Skalski; 2014)
- $S_n^+(D) = S_n^+/\langle uD = Du\rangle$, quantum reflexion groups, quantum automorphism groups of graphs (Bichon, Franz, Gerhold; 2017)

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Question

Does every $*$-bialgebra/compact quantum group have (LK)?

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Counterexample (Franz, Gerhold, Thom; CSA 2015)

no (LK): fundamental group of a closed oriented surface, genus ≥ 2
This is a cocommutative quantum group!

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no (LK): fundamental group of a closed oriented surface, genus ≥ 2
This is a cocommutative quantum group! So what goes wrong?

Schürmann triple associated to a generating functional

Generating functional $L \leftrightarrow$ Schürmann triple (π, η, L)

- $\pi : \mathcal{B} \to L(H)$ is a **unital $*$-representation** of $\mathcal{B}$ on some pre-Hilbert space H,
- $\eta : \mathcal{B} \to H$ is a π-ε-**cocycle**, i.e. a linear mapping satisfying
$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b),$$
- $L : \mathcal{B} \to \mathbb{C}$ is a hermitian linear functional such that
$$L(ab) = \langle \eta(a^*), \eta(b) \rangle + \varepsilon(a)L(b) + L(a)\varepsilon(b).$$

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Remarks

Let (π, η, L) be the Schürmann triple. TFAE:

- L is Gaussian, i.e. $L(abc) = 0$ if $a, b, c \in \ker \varepsilon$;
- η is **Gaussian**, i.e. $\eta(ab) = 0$ if $a, b \in \ker \varepsilon$;
- π is of the form $\pi(a) = \varepsilon(a)\mathrm{id}_H$.

Lévy-Khintchine decomposition in terms of Schürmann triple

Let L be a generating functional with the Schürmann triple (H, π, η, L)

- $H_{\mathrm{G}} := \bigcap_{a \in \ker \varepsilon} \ker \pi(a) = \{u \in H : \pi(a)u = \varepsilon(a)u, a \in \mathcal{B}\}$ is the maximal Gaussian subspace of H which is reducing for π. Hence $\pi = \pi_{\mathrm{G}} \oplus \pi_{\mathrm{R}}$.

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- Let P_{G} be the orthogonal projection onto H_{G}. Then $\eta_{\mathrm{G}} := P_{\mathrm{G}} \circ \eta$ is a Gaussian cocycle and $\eta_{\mathrm{R}} = (I - P_{\mathrm{G}}) \circ \eta$ is purely non-Gaussian, i.e. $(H_{\mathrm{R}})_{\mathrm{G}} = \{0\}$. So $\eta = \eta_{\mathrm{G}} \oplus \eta_{\mathrm{R}}$ and

$$(H, \pi, \eta) = (H_{\mathrm{G}}, \pi_{\mathrm{G}}, \eta_{\mathrm{G}}) \oplus (H_{\mathrm{R}}, \pi_{\mathrm{R}}, \eta_{\mathrm{R}}).$$

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- To get the decomposition $L = L_{\mathrm{G}} + L_{\mathrm{R}}$ we need to complete $(H_{\mathrm{G}}, \pi_{\mathrm{G}}, \eta_{\mathrm{G}})$ or $(H_{\mathrm{R}}, \pi_{\mathrm{R}}, \eta_{\mathrm{R}})$ by generating functionals.
- In general, it may be impossible to complete $((H_{\mathrm{G}}, \pi_{\mathrm{G}}), \eta_{\mathrm{G}})$ into the triple, since $L_{\mathrm{G}}(x) = \langle \eta_{\mathrm{G}}(a^*), \eta_{\mathrm{G}}(b) \rangle$ for $x = ab$, $a, b \in \ker \varepsilon$.

Example of a cocycle without the generating functional

- $\mathcal{B}$: free unital commutative $*$-algebra generated by x,
- counit : $\varepsilon(1) = 1$, $\varepsilon(x) = 0$.

Take $z, w \in \mathbb{C}$, $|z| \neq |w|$, and define

$$\eta(x) = z, \quad \eta(x^*) = w, \quad \eta(y) = 0$$

for y any monomial with degree not equal to 1, and extend by linearity.
Then η is a Gaussian cocycle.
If there exists a conditionally positive functional L related to η, then

$$L(xx^*) = \langle \eta(x^*), \eta(x^*) \rangle = |w|^2, \qquad L(x^*x) = \langle \eta(x), \eta(x) \rangle = |z|^2.$$

However, due to the commutativity of $\mathcal{B}$, $L(xx^*) = L(x^*x)$. Contradition.

We say that $\mathcal{B}$ has:

- the **property (LK)** if any generating functional on $\mathcal{B}$ admits the Lévy-Khintchine decomposition.
- the **property (GC)** if any Gaussian cocycle $\eta : \mathcal{B} \to H$ can be completed to a Schürmann triples $(\varepsilon \mathrm{id}, \eta, \psi)$.
- the **property (NC)** if any pair (ρ, η) consisting of a $*$-representation $\rho : \mathcal{B} \to B(H)$ and a ρ-ε-cocycle $\eta : \mathcal{B} \to H$ with $H_{\mathrm{G}} = \{0\}$ can be completed to a Schürmann triples (ρ, η, ψ).

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Remark

If for $(H, \pi, \eta) = (H_{\mathrm{G}}, \pi_{\mathrm{G}}, \eta_{\mathrm{G}}) \oplus (H_{\mathrm{R}}, \pi_{\mathrm{R}}, \eta_{\mathrm{R}})$ there exists a generating functional L_{x} such that $(\pi_{\mathrm{x}}, \eta_{\mathrm{x}}, L_{\mathrm{x}})$ $(\mathrm{x} = \mathrm{G}$ or $\mathrm{x} = \mathrm{N})$, then $L_{\mathrm{y}} = L - L_{\mathrm{x}}$ is a generating functional too and $L = L_{\mathrm{x}} + L_{\mathrm{y}}$.

$$(\mathrm{GC}) \vee (\mathrm{NC}) \Rightarrow (\mathrm{LK})$$

Compact quantum groups framework

Let $\mathbb{G}$ be a compact matrix quantum group, i.e. $\mathbb{G} = (\mathsf{A}, u)$ with a unital C*-algebra A and a unitary matrix $u = (u_{jk})_{j,k=1}^{d} \in M_d(\mathsf{A})$ such that

- A generated by u_{jk} $(j, k = 1, \ldots, d)$
- $\Delta(u_{jk}) = \sum_p u_{jp} \otimes u_{pk}$ extends to a $*$-homomorphism on A,
- $\bar{u}$ is invertible.

Then the dense $\mathcal{B} = \mathrm{Pol}(\mathbb{G})$ is a $*$-bialgebra with $\varepsilon(u_{jk}) = \delta_{jk}$.

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Example: universal unitary and orthogonal quantum groups

Let $d \in \mathbb{N}_2$, $F \in GL_d(\mathbb{C})$,

$$\mathrm{Pol}(U_F^+) := *\text{-Alg}\langle u_{jk}\rangle_{j,k=1}^{d} / \langle uu^* = I = u^*u, F\bar{u}F^{-1}\text{-unitary}\rangle$$

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$$\begin{aligned}\mathrm{Pol}(U_F^+) &:= *\text{-Alg}\langle u_{jk}\rangle_{j,k=1}^d / \langle uu^* = I = u^*u, F\bar{u}F^{-1}\text{-unitary}\rangle \\ &= *\text{-Alg}\langle u_{jk}\rangle_{j,k=1}^d / \langle uu^* = I = u^*u, u^tQ\bar{u}Q^{-1} = I = Q\bar{u}Q^{-1}u^t\rangle,\end{aligned}$$

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$$\mathrm{Pol}(O_F^+) := *\text{-Alg}\langle u_{jk}\rangle_{j,k=1}^{d} / \langle uu^* = I = u^*u, u = F\bar{u}F^{-1}\rangle.$$

(LK) for universal quantum groups

Theorem (DFKS'2018)

(a) If a matrix $F \in GL_d(\mathbb{C})$ is such that F^*F has pairwise distinct eigenvalues, then both U_F^+ and O_F^+ have the (GC) and the (LK) properties.

(b) The quantum groups U_d^+ $(d \geq 2)$ and O_d^+ $(d \geq 3)$ do have neither (GC), nor (NC), nor (LK) property.

Remarks:

- The first example of non-cocommutative quantum group without (LK)!
- O_2^+ has (GC) and (LK), but not (NC) (Skeide'1999 as $O_2^+ \cong SU_{-1}(2)$).

Strategy: quotients of $K\langle d\rangle$

$K\langle d\rangle$: the universal unital $*$-algebra generated by x_{jk} $(j,k=1,2,...d)$ such that the matrix $x := (x_{jk})_{j,k=1}^{d}$ satisfies $xx^* = I = x^*x$.

Definition

$(\mathrm{Pol}(\mathbb{G}), u)$ is a quotient of $(K\langle d\rangle, x)$ if $\dim u = d$ and there is a unital (surjective) *-homomorphism $q : K\langle d\rangle \to \mathrm{Pol}(\mathbb{G})$ such that

$$q(x_{jk}) = u_{jk}, \quad j,k = 1,\ldots,d.$$

Examples:

- U_F^+ is a quotient of $K\langle d\rangle$ by $x^t Q\overline{x}Q^{-1} = I = Q\overline{x}Q^{-1}x^t$ $(Q = F^*F)$;
- O_F^+ is a quotient of $K\langle d\rangle$ by $x = F\bar{x}F^{-1}$;
- $SU_q(d)$ is a quotient of $K\langle d\rangle$ by the twisted determinant condition.

Strategy: quotients of $K\langle d\rangle$

Theorem (Schürmann'1990)

Each cocycle η' on $K\langle d\rangle$ admits a generating functional L'. The functional is uniquely defined by the rule:

$$L'(x_{jk}) = -\frac{1}{2}\sum_{n=1}^{d}\langle\eta'(x_{jn}^*),\eta'(x_{kn}^*)\rangle.$$

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Let $\mathrm{Pol}(\mathbb{G})$ be a quotient of $K\langle d\rangle$, $q : K\langle d\rangle \twoheadrightarrow \mathrm{Pol}(\mathbb{G})$.

- (ρ, η, L) are uniquely determined by the values on u_{jk} and u_{jk}^*;
- η - Gaussian cocycle on $\mathbb{G} \Rightarrow \eta' = \eta \circ q$ - Gaussian on $K\langle d\rangle$;
- η' admits a generating functional L' on $K\langle d\rangle$;
- $L(q(a)) := L'(a)$ is a GF on $\mathbb{G}$, provided it is well-defined, which happens when $L|_{\ker q} = 0$.

Hochschild cohomology

Hochschild cohomology

$\mathcal{B}$: an algebra,
M : a A-bimodule with $a.v.b = \pi(a)v\rho(b)$ for $a, b, \in \mathcal{B}$ and $v \in M$

- n-cochains:

$$C^n(\mathcal{B}, {}_\pi M_\rho) := \{\phi : \mathcal{B}^{\otimes n} \to M, \text{ linear}\}$$

- coboundary operator: $\partial^n : C^n(\mathcal{B}, {}_\pi M_\rho) \to C^{n+1}(\mathcal{B}, {}_\pi M_\rho)$

$$\partial^n \phi(a_1 \otimes a_{n+1}) = \pi(a_1)\phi(a_2 \otimes \ldots \otimes a_{n+1}) +$$
$$\sum_{j=1}^{n} (-1)^j \phi(a_1 \otimes \ldots a_j a_{j+1} \ldots \otimes a_{n+1}) + (-1)^{n+1}\phi(a_1 \otimes \ldots \otimes a_n)\rho(a_{n+1})$$

- n-cocycles: $Z^n(\mathcal{B}, {}_\pi M_\rho) := \{\phi \in C^n(\mathcal{B}, {}_\pi M_\rho) : \partial^n \phi = 0\}$
- n-coboundaries: $B^n(\mathcal{B}, {}_\pi M_\rho) := \partial^{n-1} C^{n-1}(\mathcal{B}, {}_\pi M_\rho)$
- nth cohomology group:

$$H^n(\mathcal{B}, {}_\pi M_\rho) := Z^n(\mathcal{B}, {}_\pi M_\rho)/B^n(\mathcal{B}, {}_\pi M_\rho)$$

Relation of (LK) with the Hochschild cohomology

Observation

For $\eta : \mathcal{B} \to H$ linear we have

1. $\eta \in Z^1(\mathcal{B}, {}_\pi H_\varepsilon)$ iff η is a π-ε-cocycle (from Schürmann triples).
2. $H^1(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) = \{\mathbb{C}\text{-valued Gaussian cocycles}\}$.

Relation of (LK) with the Hochschild cohomology

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$$c_\eta : \mathcal{B} \otimes \mathcal{B} \ni a \otimes b \mapsto -\langle \eta(a^*), \eta(b) \rangle \in \mathbb{C}.$$

3. If $\eta \in Z^1(\mathcal{B}, {}_\pi H_\varepsilon)$, then $c_\eta \in Z^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon)$.
4. In such case $c_\eta \in B^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon)$ iff η admits a GF ψ.

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1. $\eta \in Z^1(\mathcal{B}, {}_\pi H_\varepsilon)$ iff $\partial^1\phi(a \otimes b) = \pi(a)\phi(b) - \phi(ab) + \phi(a)\varepsilon(b) = 0$
2. $\phi \in B^1(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon)$ iff $\exists_{\psi:\mathbb{C}\to\mathbb{C}}\ \phi(a) = \partial^0\psi(a) = \varepsilon(a)\psi - \psi\varepsilon(a) = 0$.
3. Check that $\partial^3 c_\eta = 0$.
4. $c_\eta \in B^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon)$ iff there exists $\psi : \mathcal{B} \to \mathbb{C}$ s.t.
   $c_\eta(a \otimes b) = \partial^1\psi(a \otimes b) = \varepsilon(a)\psi(b) - \psi(ab) + \psi(a)\varepsilon(b)$

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Theorem (Franz, Gerhold, Thom; 2015)

If $H^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) = \{0\}$, then any pair (π, η) can be completed to a Schürmann triple, hence $\mathcal{B}$ has the property (LK).

- Let π is a representation of $\mathcal{B}$ and η is a π-ε-cocycle
- $H^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) = \{0\} \Rightarrow Z^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) = B^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon)$
- $\eta \in Z^1(\mathcal{B}, {}_\pi H_\varepsilon) \Rightarrow c_\eta \in Z^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) = B^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) \Rightarrow \eta$ admits a GF.

Hochschild cohomology for algebras without (LK)

Remark

If $\mathcal{B}$ does not have the property (LK) then $H^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) \neq \{0\}$

Theorem (Collons, Härtel, Thom '2009; Bichon'2013)

- For $F = I_d$:

$$H^1(O_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = H^2(O_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \{M \in M_d(\mathbb{C}),\ M + M^t = 0\}.$$

- For general F (by Poincaré duality):

$$H^1(O_F^+, {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \frac{\{M \in M_d(\mathbb{C}),\ M + \bar{F}M^t\bar{F}^{-1} = 0\}}{\{\lambda\left(\bar{F}(F^*)^{-1} - F^*\bar{F}^{-1}\right)), \lambda \in \mathbb{C}\}},$$

$$H^2(O_F^+, {}_\varepsilon\mathbb{C}_\varepsilon) \simeq \frac{M_d(\mathbb{C})}{\{M + \bar{F}M^t\bar{F}^{-1} : M \in M_d(\mathbb{C})\}}$$

Hochschild cohomology for U_F^+ ?

Let $F = I_d$.

- $\mathbb{C}$-valued Gaussian cocycle $\eta \leftrightarrow V \in M_d(\mathbb{C})$; $H^1(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = M_d(\mathbb{C})$

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- $\eta \in Z^1(U_d^+, {}_\pi H_\varepsilon)$ admits a GF if and only if $c_\eta \in Z^2(U_d^+, {}_\pi H_\varepsilon)$ satisfies

$$\sum_{p=1}^{d} \underbrace{\langle \eta(u_{pj}), \eta(u_{pk})\rangle}_{c_\eta(u_{pj}^* \otimes u_{pk})} = \sum_{p=1}^{d} \underbrace{\langle \eta(u_{kp}), \eta(u_{jp})\rangle}_{c_\eta(u_{kp}^* \otimes u_{jp})}, \quad j,k = 1,\dots,d.$$

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- we show that any $c \in B^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon)$ iff

$$\sum_{p=1}^{d} c(u_{pj}^* \otimes u_{pk}) = \sum_{p=1}^{d} c(u_{kp}^* \otimes u_{jp}), \quad j,k = 1,\ldots,d.$$

- the matrix $\Delta(c)$ defined by

$$\Delta(c)_{jk} := \sum_{p=1}^{d} \left(c(u_{pj}^* \otimes u_{pk}) - c(u_{kp}^* \otimes u_{jp}) \right)$$

measures “how far c is from being” a 2-coboundary.

Hochschild cohomology for U_F^+ ? $F = I$

Theorem (DFKS'2018)

Define $\Delta : Z^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) \to M_d(\mathbb{C})$ by the formula

$$\Delta(c) = \left(\sum_{p=1}^{d} \left(c(u_{pj}^* \otimes u_{pk}) - c(u_{kp}^* \otimes u_{jp}) \right) \right)_{j,k=1}^{d} .$$

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Then

- $\mathrm{Ker}\ \Delta = B^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon)$
- $\mathrm{Im}\ \Delta = sl(d) := \{M \in M_d(\mathbb{C}) : \mathrm{Tr}(M) = 0\}$.

Hence

$$H^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = Z^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon)/B^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) \cong sl(d).$$

In particular, $\dim H^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = d^2 - 1$.

Hochschild cohomology for U_F^+ ?

Let $Q = F^*F$ be a positive diagonal matrix, $Q = \sum_{i=1}^n \lambda_i P_{d_i}$, λ_i's: (different) eigenvalues, P_{d_i}'s : d_i-dimensional projections on eigenspaces.

Theorem to be...

Define $\Delta : Z^2(U_F^+, {}_\varepsilon\mathbb{C}_\varepsilon) \to M_d(\mathbb{C})$ by the formula

$$D(c) = \sum_{p=1}^{d} \left(c(u_{pj}^* \otimes u_{pk}) - \frac{Q_k}{Q_p} c(u_{kp}^* \otimes u_{jp}) \right), \quad \Delta(c) = \sum_{i=1}^{n} P_{d_i} D P_{d_i}$$

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- $\text{Ker } \Delta = B^2(U_F^+, {}_\varepsilon\mathbb{C}_\varepsilon)$,
- $H^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) = Z^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon)/B^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) \cong \text{Im } \Delta$,
- $sl(d_1) \oplus \ldots \oplus sl(d_n) \subset \text{Im } \Delta \subset sl_Q(d)$,
  $sl_Q(d) = \{M \in M_d(\mathbb{C}) : MQ = QM, \mathrm{Tr}(QM) = \mathrm{Tr}(Q^{-1}M) = 0\}$.

In particular,

$$\sum_{i=1}^{n} d_i^2 - n \leq \dim H^2(U_d^+, {}_\varepsilon\mathbb{C}_\varepsilon) \leq \sum_{i=1}^{n} d_i^2 - 2 \quad (n \geq 2).$$

Final remarks

Open problem

Find a characterisation of $*$-bialgebras which have the property (LK).

We know that:

- if $H^2(\mathcal{B}, {}_\varepsilon\mathbb{C}_\varepsilon) = \{0\}$ then $\mathcal{B}$ has the property (LK);
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Observation

Neither (LK), nor (GC), nor their negations transfer to quotients.

- (GC): ${O_2^+}_{\mathrm{(GC)}} \subset {U_2^+}_{\mathrm{no\,(GC)}}$, $SU_q(2)_{\mathrm{(GC)}} \subset SU_q(3)_{\mathrm{no\,(GC)}} \subset U_3^+(F)_{\mathrm{(GC)}}$
- (LK): ${O_2^+}_{\mathrm{(LK)}} \subset {O_3^+}_{\mathrm{no\,(LK)}} \subset K\langle 3\rangle_{\mathrm{(LK)}}$.