Convolution semigroups on quantum groups and non-commutative Dirichlet forms

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Quantum Homogeneous Spaces
ICMS, Edinburgh, 14.6.2018

Joint work with Adam Skalski
to appear in Journal de Mathématiques Pures et Appliquées

Locally compact quantum groups

Definition (Kustermans–Vaes, ’00)

A locally compact quantum group is a pair $\mathbb{G} = (M, \Delta)$ such that:

1. M is a von Neumann algebra
2. $\Delta : M \to M \overline{\otimes} M$ is a co-multiplication: a normal, faithful, unital $*$-homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

3. There are two n.s.f. weights φ, ψ on M (the Haar weights) with:
   - $\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\varphi(x) < \infty$
   - $\psi((\mathrm{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$ when $\omega \in M_*^+$, $x \in M^+$ and $\psi(x) < \infty$.

Denote $L^\infty(\mathbb{G}) := M$.
Have it act standardly on the Hilbert space $L^2(\mathbb{G})$.

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Main result

$\mathbb{G}$ – locally compact quantum group

Theorem (Skalski–V)

There exist $1-1$ *correspondences between:*

1. *w^*-continuous, symmetric, convolution semigroups of states on $\mathbb{G}$;*
2. *completely Dirichlet forms w.r.t. φ that are right-translation invariant;*
3. *completely Markov semigroups on $L^2(\mathbb{G})$ that are symmetric and contained in $L^\infty(\hat{\mathbb{G}})$;*
4. *completely Markov semigroups on $L^\infty(\mathbb{G})$ that are right-translation invariant and KMS-symmetric w.r.t. φ.*

G – locally compact group

Convolution of measures

For positive Borel measures μ, ν on G, their convolution $\mu \star \nu$ is given by

$$(\mu \star \nu)(A) := \int_G \left(\int_G I_A(gh)\, \mathrm{d}\mu(g) \right) \mathrm{d}\nu(h) \qquad (\forall \text{measurable } A).$$

Definition

A convolution semigroup of probability measures on G is a family $(\mu_t)_{t\geq 0}$ of probability measures on G satisfying

$$\mu_0 = \delta_e \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s, t \geq 0).$$

It is w^*-continuous if $\int_G f \,\mathrm{d}\mu_t \xrightarrow[t\to 0^+]{} \int_G f \,\mathrm{d}\mu_0 = f(e)$ for all $f \in C_0(G)$.

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Definition

Let $d \in \mathbb{N}$. A Lévy process with values in $\mathbb{R}^d$ is a family $X = (X_t)_{t\geq 0}$ of random variables from a probability space to $\mathbb{R}^d$ such that:

1. $X_0 = 0$;
2. X has independent and stationary increments;
3. X is continuous.

Given a Lévy process X, define $(\mu_t)_{t\geq 0}$ to be its family of distributions: for $t \geq 0$, μ_t is the probability measure on $\mathbb{R}^d$ defined by $\mu_t := \mathbb{P} \circ X_t^{-1}$.

Theorem

1. *$(\mu_t)_{t\geq 0}$ is a w^*-continuous convolution semigroup of probability measures on $\mathbb{R}^d$.*
2. *All such convolution semigroups arise this way.*

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~~Let $d \in \mathbb{N}$.~~ A Lévy process with values in G is a family $X = (X_t)_{t \geq 0}$ of random variables from a probability space to G such that:

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A (non-negative) quadratic form on a Hilbert space $\mathcal{H}$ is a semi-inner product $Q : D(Q) \times D(Q) \to \mathbb{C}$ on a subspace $D(Q)$ of $\mathcal{H}$.

- Densely defined if $D(Q)$ is dense in $\mathcal{H}$.
- Closedness.

More convenient to work with $Q' : \mathcal{H} \to [0, \infty]$ given by

$$Q'\zeta := \begin{cases} Q(\zeta, \zeta) & \zeta \in D(Q) \\ \infty & \text{else.} \end{cases}$$

closed, densely-defined quadratic forms	$\longleftrightarrow$	generally unbounded, positive selfadjoint operators	$\longleftrightarrow$	C_0-semigroups of selfadjoint contractions
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(X, m) – positive measure space

Definition

A map $S : L^2(X,m) \to L^2(X,m)$ is Markov if for all $f \in L^2(X,m)$,

$$0 \le f \le 1 \implies 0 \le Sf \le 1.$$

Definition (Based on Beurling–Deny, Acta Math., 1958)

A Dirichlet form on (X, m) is a closed, densely defined, quadratic form Q on $L^2(X,m)$ such that for all $\mathbb{R}$-valued $f \in L^2(X,m)$,

$$Q\Big(\min\left(\max(f,0),1\right)\Big) \le Q(f).$$

Theorem (Beurling–Deny)

Dirichlet forms on (X, m) $\longleftrightarrow$ symmetric Markov semigroups on $L^2(X,m)\,/\,L^\infty(X,m)$

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Dirichlet forms on (X, m) $\longleftrightarrow$ *symmetric Markov semigroups on $L^2(X, m)$ / $L^\infty(X, m)$*

G – locally compact group

Easy fact

μ – probability measure on G that is symmetric ($\mu(B) = \mu(B^{-1})$, $\forall B$).
The operator $f \mapsto \mu \star f$ on $L^2(G)$ is a symmetric Markov operator.

Corollary

w^*-continuous convolution semigroup of symmetric probability measures on G
$\rightsquigarrow$ symmetric Markov semigroup on $L^2(G)$ $\leftrightarrow$ Dirichlet form on $L^2(G)$.

Which Dirichlet forms on $L^2(G)$ arise this way?

Theorem

Precisely the right-translation invariant ones.

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Lévy processes

$\longleftrightarrow$

w^*-cont.
conv. semigroups
of
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right-translation invariant
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symmetric G-valued Lévy processes $\longleftrightarrow$ w^*-cont. conv. semigroups of symmetric prob. measures on G $\longleftrightarrow$ right-translation invariant Dirichlet forms on $L^2(G)$

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right-translation invariant symmetric Markov semigroups on $L^2(G)$ / $L^\infty(G)$

G – locally compact group

Definition

A semigroup of normalized positive-definite functions on G is a family $(\varphi_t)_{t\geq 0}$ of positive-definite functions on G mapping e to 1 and satisfying

$$\varphi_0 \equiv 1 \quad \text{and} \quad \varphi_s \cdot \varphi_t = \varphi_{s+t} \qquad (\forall s, t \geq 0).$$

Adjectives: w^*-continuous
symmetric = invariant under inversion ($\iff$ real valued).

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Example

$G = \hat{\Gamma}$, where Γ – locally compact abelian group
$(\mu_t)_{t\geq 0}$ – w^*-cont. convolution semigroup of prob. measures on Γ
$\rightsquigarrow$ Define $\varphi_t := \hat{\mu}_t$ (the Fourier–Stieltjes transform) for all $t \geq 0$.

Classical players

4. Conditionally negative-definite functions

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Assume w^*-continuity and symmetry.
These families are of the form $(e^{-t\theta})_{t\geq 0}$. "Who" are their generators θ?

Definition

A continuous $\theta : G \to \mathbb{R}$ is conditionally negative definite if:

1. $\theta(e) = 0$;
2. $\theta(g^{-1}) = \theta(g)$ for all $g \in G$;
3. $\big(\theta(g_i) + \theta(g_j) - \theta(g_j^{-1} g_i)\big)_{1 \le i,j \le n}$ is positive definite for all $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in G$.

Schönberg's Theorem

A continuous $\theta : G \to \mathbb{R}$ satisfying 1 and 2

is CND $\iff$ $e^{-t\theta}$ is positive definite for all $t \geq 0$.

Corollary

The w^*-cont. semigroups of symmetric, normalized, positive-definite functions on G are exactly $(e^{-t\theta})_{t \geq 0}$ for a CND function $\theta : G \to \mathbb{R}$.

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Example

For $d \in \mathbb{N}$ and $0 \leq \alpha \leq 2$, the function $\mathbb{R}^d \to [0, \infty)$ given by $x \mapsto \|x\|^\alpha$ is conditionally negative definite.

Example (Haagerup, Invent. Math., 1978/79)

Let $n \in \mathbb{N}$. The function $\mathbb{F}_n \to [0, \infty)$ given by

$$s \mapsto |s|$$

is conditionally negative definite.

Consequently:

1. the C^*-algebra $C^*_r(\mathbb{F}_n)$ has the metric approximation property;
2. $\mathbb{F}_n$ is weakly amenable;
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symmetric G-valued Lévy processes $\longleftrightarrow$ w^*-cont. conv. semigroups of symmetric prob. measures on G $\longleftrightarrow$ right-translation invariant Dirichlet forms on $L^2(G)$

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right-translation invariant symmetric Markov semigroups on $L^2(G)$ / $L^\infty(G)$

Duality

w^*-cont. semigroups of symmetric, normalized, positive-definite functions on G $\xleftrightarrow{\varphi_t = e^{-t\theta}}$ conditionally negative-definite functions on G

Duality $\mathbb{G} \mapsto \hat{\mathbb{G}}$ within the category satisfying $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.

G – locally compact group

$\mathbb{G}$	$L^\infty(\mathbb{G})$	$C_0(\mathbb{G})$	$C_0^{\mathrm{u}}(\mathbb{G})$	(states of $C_0^{\mathrm{u}}(\mathbb{G})$,$\star$)	antipode
G					
$\hat{G}$					

Three "faces" (algebras):

- the von Neumann algebra $L^\infty(\mathbb{G})$;
- the "reduced" C^*-algebra $C_0(\mathbb{G})$;
- the "universal" C^*-algebra $C_0^{\mathrm{u}}(\mathbb{G})$.
  The conjugate space $C_0^{\mathrm{u}}(\mathbb{G})^*$ carries a convolution $\star$ turning it into a Banach algebra with unit ϵ (the co-unit).

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G	$L^\infty(G)$	$C_0(G)$	$C_0(G)$	regular probability measures, $\star$ = convolution	
$\hat{G}$	$\mathrm{VN}(G)$	$C_\mathrm{r}^*(G)$	$C^*(G)$	normalized positive definite functions, $\star$ = product	

Back to locally compact quantum groups

Features and two basic examples

The antipode: an (unbounded) operator on $L^\infty(\mathbb{G})$ / $C_0(\mathbb{G})$ / $C_0^{\mathrm{u}}(\mathbb{G})$. Decomposes into:

- a "bounded part": the unitary antipode, an anti-automor. of $L^\infty(\mathbb{G})$;
- an "unbounded part": the scaling group.

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G	$L^\infty(G)$	$C_0(G)$	$C_0(G)$	regular probability measures, $\star$ = convolution	composition with inverse
$\hat{G}$	$\mathrm{VN}(G)$	$C_{\mathrm{r}}^*(G)$	$C^*(G)$	normalized positive definite functions, $\star$ = product	$\lambda_g \mapsto \lambda_{g^{-1}}$ for $g \in G$

Convolution semigroups

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Definition

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$$\mu_0 = \epsilon \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s, t \geq 0).$$

Adjectives:

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A convolution semigroup of states on $\mathbb{G}$ is a family $(\mu_t)_{t\geq 0}$ of states of $C_0^{\mathrm{u}}(\mathbb{G})$ such that

$$\mu_0 = \epsilon \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \qquad (\forall s, t \geq 0).$$

Adjectives:

- w^*-continuous
- symmetric = invariant under the unitary antipode.

Convolution semigroups

Examples

of w^*-continuous, symmetric, convolution semigroups of states on $\mathbb{G}$:

- $\mathbb{G} = G$: w^*-continuous, symmetric, convolution semigroups of probability measures on G.
- $\mathbb{G} = \hat{G}$: w^*-continuous, symmetric, semigroups of normalized positive-definite functions on G.

These capture the two types of semigroups discussed earlier.

Non-commutative Dirichlet forms and Markov semigroups

- ⋆ Commutative = "classical" = on a positive measure space with a reference measure.
- ⋆ Non-commutative = on a von Neumann algebra with a reference weight.
  - Developed by many people: Albeverio–Høegh-Krohn, Sauvageot, Davies–Lindsay, Guido–Isola–Scarlatti, Cipriani–Sauvageot, Cipriani, Goldstein–Lindsay, ...
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Main result

$\mathbb{G}$ – locally compact quantum group

(Recall: $L^\infty(\mathbb{G})$ – underlying von Neumann algebra, $L^2(\mathbb{G})$ – a standard Hilbert space; φ – left Haar weight.)

Theorem (Skalski–V)

There exist $1-1$ correspondences between:

1. *w^*-continuous, symmetric, convolution semigroups of states on $\mathbb{G}$;*
2. *completely Dirichlet forms w.r.t. φ that are right-translation invariant;*
3. *completely Markov semigroups on $L^2(\mathbb{G})$ that are symmetric and contained in $L^\infty(\hat{\mathbb{G}})$;*
4. *completely Markov semigroups on $L^\infty(\mathbb{G})$ that are right-translation invariant and KMS-symmetric w.r.t. φ.*

Earlier results

Our main theorem is definitive.
It unifies and extends:

- $\mathbb{G} = G$ or $\hat{G}$: the classical work;
- $\mathbb{G}$ – compact quantum group: Cipriani–Franz–Kula (JFA, 2014).

  Such $\mathbb{G}$ has a canonical Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G})$.
  Its embedding in $L^2(\mathbb{G})$ is a core of all Dirichlet forms $\rightsquigarrow$ the problem becomes more algebraic.

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Approximation properties for groups

G – locally compact group.

Definition

G has the Haagerup property if it admits a mixing representation with almost-invariant vectors.

Examples

Amenable groups; free groups; $\mathrm{SL}(2,\mathbb{Z})$; groups that act properly on a tree (more generally: on a space with "walls"); Coxeter groups.

Definition

G does not have property (T) if it admits an ergodic representation with almost-invariant vectors.

Examples (of property (T))

$\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$; $\mathrm{SL}(n,\mathbb{K})$, $n \geq 3$; $\mathrm{Sp}(2n,\mathbb{K})$, $n \geq 2$ ($\mathbb{K}$ – local field).

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Theorem (Guichardet, '72 + Delorme, '77; Akemann–Walter, '81)

Assume that G is σ-compact.

1. *G does not have property (T)*
   $\iff$ it has an unbounded conditionally negative-definite function
   $\iff$ it has a w^*-cont. semigroup of symm. normalized pos.-def. functions that is not norm continuous.
2. *G has the Haagerup property*
   $\iff$ it has a proper conditionally negative-definite function
   $\iff$ it has a w^*-cont. semigroup of symm. normalized pos.-def. functions that is C_0 in positive time.

(proper = goes to ∞ at ∞).

Example

The CND function $s \mapsto |s|$ on $\mathbb{F}_n$ presented earlier!

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$\mathbb{G}$ – second countable, discrete quantum group.

Definition

A conditionally negative-definite function on $\mathbb{G}$ is a map $\theta : \mathrm{Pol}(\hat{\mathbb{G}}) \to \mathbb{C}$ satisfying:

1. $\theta(\mathbb{1}) = 0$;
2. $\theta(a^*) = \overline{\theta(a)}$ for all $a \in \mathrm{Pol}(\hat{\mathbb{G}})$;
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A version of Schönberg's Theorem for finite-dim. co-algebras gives:

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$\mathbb{G}$ – second countable, discrete quantum group.

Theorem (Kyed, JFA, 2011)

If $\mathbb{G}$ is unimodular, then it does not have property (T)
$\iff$ *it has an unbounded conditionally negative-definite function.*

Theorem (Daws–Fima–Skalski–White, Crelle's Journal, 2016)

$\mathbb{G}$ has the Haagerup property
$\iff$ *it has a proper conditionally negative-definite function.*

- Many applications, e.g. Daws–Skalski–V, Comm. Math. Phys., 2017.

$\mathbb{G}$ – second countable, locally compact quantum group.

Theorems (Skalski–V)

1. $\mathbb{G}$ does not have property (T)
   $\iff$ $\hat{\mathbb{G}}$ has a completely Dirichlet form w.r.t. $\hat{\varphi}$ that is right-translation invariant and unbounded
   $\iff$ $\hat{\mathbb{G}}$ has a w^*-continuous, symmetric, convolution semigroup of states that is not norm continuous.
2. $\mathbb{G}$ has the Haagerup property
   $\iff$ $\hat{\mathbb{G}}$ has a completely Dirichlet form w.r.t. $\hat{\varphi}$ that is right-translation invariant and proper
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Convolution semigroups via cocycle twisting

$\mathbb{G}$ – locally compact quantum group
$U \in L^\infty(\mathbb{G}) \,\overline{\otimes}\, L^\infty(\mathbb{G})$ – 2-cocycle: a unitary satisfying

$$(U \otimes \mathbb{1})(\Delta \otimes \mathrm{id})(U) = (\mathbb{1} \otimes U)(\mathrm{id} \otimes \Delta)(U).$$

Theorem (De Commer, '09)

Replacing Δ by $U\Delta(\cdot)U^*$ one gets a new LCQG, $\mathbb{G}_U$.

$(\mu_t)_{t\geq 0}$ – w^*-cont. (symmetric) convolution semigroup of states on $\mathbb{G}$.

Proposition

If U is invariant under the Markov semigroup $(T_t)_{t\geq 0}$ associated with $(\mu_t)_{t\geq 0}$ when applied to its left leg, then $(T_t)_{t\geq 0}$ is also induced by a convolution semigroup on $\mathbb{G}_U$.

Example

conv. semigroup on $\mathbb{R}^n \rightsquigarrow$ conv. semigroup on $\mathbb{H}_n^q(\mathbb{R})$.

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Thank you for your attention!