

# Convolution semigroups on quantum groups and non-commutative Dirichlet forms

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Quantum Homogeneous Spaces  
ICMS, Edinburgh, 14.6.2018

Joint work with Adam Skalski  
to appear in Journal de Mathématiques Pures et Appliquées

## Definition (Kustermans–Vaes, '00)

A **locally compact quantum group** is a pair  $\mathbb{G} = (M, \Delta)$  such that:

- 1  $M$  is a **von Neumann algebra**
- 2  $\Delta : M \rightarrow M \bar{\otimes} M$  is a **co-multiplication**: a normal, faithful, unital  $*$ -homomorphism which is co-associative, i.e.,

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$$

- 3 There are two n.s.f. weights  $\varphi, \psi$  on  $M$  (the **Haar weights**) with:
  - ▶  $\varphi((\omega \otimes \text{id})\Delta(x)) = \omega(\mathbb{1})\varphi(x)$  when  $\omega \in M_*^+$ ,  $x \in M^+$  and  $\varphi(x) < \infty$
  - ▶  $\psi((\text{id} \otimes \omega)\Delta(x)) = \omega(\mathbb{1})\psi(x)$  when  $\omega \in M_*^+$ ,  $x \in M^+$  and  $\psi(x) < \infty$ .

Denote  $L^\infty(\mathbb{G}) := M$ .

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Have it act standardly on the Hilbert space  $L^2(\mathbb{G})$ .

$G$  – locally compact quantum group

## Theorem (Skalski–V)

There exist 1 – 1 correspondences between:

- 1  $w^*$ -continuous, symmetric, **convolution semigroups** of states on  $G$ ;
- 2 completely **Dirichlet forms** w.r.t.  $\varphi$  that are right-translation invariant;
- 3 completely **Markov semigroups** on  $L^2(G)$  that are symmetric and contained in  $L^\infty(\hat{G})$ ;
- 4 completely **Markov semigroups** on  $L^\infty(G)$  that are right-translation invariant and KMS-symmetric w.r.t.  $\varphi$ .

# Classical players

## 1. Convolution semigroups of probability measures

$G$  – locally compact group

### Convolution of measures

For positive Borel measures  $\mu, \nu$  on  $G$ , their **convolution**  $\mu \star \nu$  is given by

$$(\mu \star \nu)(A) := \int_G \left( \int_G I_A(gh) d\mu(g) \right) d\nu(h) \quad (\forall \text{measurable } A).$$

### Definition

A **convolution semigroup of probability measures** on  $G$  is a family  $(\mu_t)_{t \geq 0}$  of probability measures on  $G$  satisfying

$$\mu_0 = \delta_e \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \quad (\forall s, t \geq 0).$$

It is  **$w^*$ -continuous** if  $\int_G f d\mu_t \xrightarrow{t \rightarrow 0^+} \int_G f d\mu_0 = f(e)$  for all  $f \in C_0(G)$ .

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## 2. Lévy processes

### Definition

Let  $d \in \mathbb{N}$ . A **Lévy process** with values in  $\mathbb{R}^d$  is a family  $X = (X_t)_{t \geq 0}$  of random variables from a probability space to  $\mathbb{R}^d$  such that:

- 1  $X_0 = 0$ ;
- 2  $X$  has **independent** and **stationary increments**;
- 3  $X$  is **continuous**.

Given a Lévy process  $X$ , define  $(\mu_t)_{t \geq 0}$  to be its family of **distributions**: for  $t \geq 0$ ,  $\mu_t$  is the probability measure on  $\mathbb{R}^d$  defined by  $\mu_t := \mathbb{P} \circ X_t^{-1}$ .

### Theorem

- 1  $(\mu_t)_{t \geq 0}$  is a  $w^*$ -continuous **convolution semigroup** of probability measures on  $\mathbb{R}^d$ .
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## 3. Dirichlet forms and Markov semigroups

### Definition

A (non-negative) **quadratic form** on a Hilbert space  $\mathcal{H}$  is a semi-inner product  $Q : D(Q) \times D(Q) \rightarrow \mathbb{C}$  on a subspace  $D(Q)$  of  $\mathcal{H}$ .

- **Densely defined** if  $D(Q)$  is dense in  $\mathcal{H}$ .
- **Closedness**.

More convenient to work with  $Q' : \mathcal{H} \rightarrow [0, \infty]$  given by

$$Q'\zeta := \begin{cases} Q(\zeta, \zeta) & \zeta \in D(Q) \\ \infty & \text{else.} \end{cases}$$

closed,  
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$$\|A^{1/2}\cdot\|^2$$



generally  
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$C_0$ -semigroups of  
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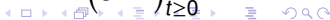
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$(X, m)$  – positive measure space

### Definition

A map  $S : L^2(X, m) \rightarrow L^2(X, m)$  is **Markov** if for all  $f \in L^2(X, m)$ ,

$$0 \leq f \leq 1 \implies 0 \leq Sf \leq 1.$$

### Definition (Based on Beurling–Deny, Acta Math., 1958)

A **Dirichlet form** on  $(X, m)$  is a closed, densely defined, quadratic form  $Q$  on  $L^2(X, m)$  such that for all  $\mathbb{R}$ -valued  $f \in L^2(X, m)$ ,

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### Theorem (Beurling–Deny)

*Dirichlet forms*  
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### Easy fact

$\mu$  – probability measure on  $G$  that is **symmetric** ( $\mu(B) = \mu(B^{-1}), \forall B$ ).  
The operator  $f \mapsto \mu \star f$  on  $L^2(G)$  is a **symmetric Markov operator**.

### Corollary

*$w^*$ -continuous convolution semigroup of symmetric probability measures on  $G$*

*$\rightsquigarrow$  symmetric Markov semigroup on  $L^2(G) \leftrightarrow$  Dirichlet form on  $L^2(G)$ .*

Which Dirichlet forms on  $L^2(G)$  arise this way?

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*Precisely the right-translation invariant ones.*

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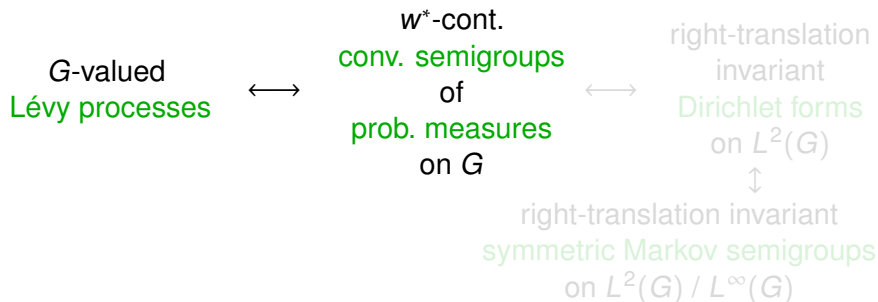
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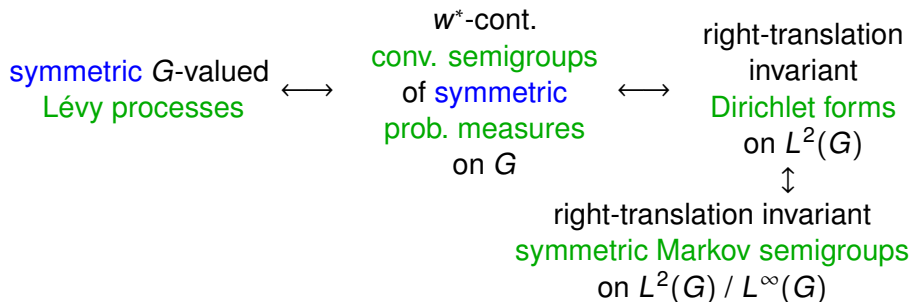
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## Interim summary



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## 4. Conditionally negative-definite functions

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### Definition

A **semigroup of normalized positive-definite functions** on  $G$  is a family  $(\varphi_t)_{t \geq 0}$  of positive-definite functions on  $G$  mapping  $e$  to 1 and satisfying

$$\varphi_0 \equiv 1 \quad \text{and} \quad \varphi_s \cdot \varphi_t = \varphi_{s+t} \quad (\forall s, t \geq 0).$$

Adjectives:  $w^*$ -continuous

**symmetric** = invariant under inversion (  $\iff$  real valued).

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### Example

$G = \hat{\Gamma}$ , where  $\Gamma$  – locally compact abelian group

$(\mu_t)_{t \geq 0}$  –  $w^*$ -cont. convolution semigroup of prob. measures on  $\Gamma$

$\rightsquigarrow$  Define  $\varphi_t := \hat{\mu}_t$  (the Fourier–Stieltjes transform) for all  $t \geq 0$ .

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Assume  $w^*$ -continuity and symmetry.

These families are of the form  $(e^{-t\theta})_{t \geq 0}$ . “Who” are their generators  $\theta$ ?

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A continuous  $\theta : G \rightarrow \mathbb{R}$  is **conditionally negative definite** if:

- 1  $\theta(e) = 0$ ;
- 2  $\theta(g^{-1}) = \theta(g)$  for all  $g \in G$ ;
- 3  $(\theta(g_i) + \theta(g_j) - \theta(g_j^{-1}g_i))_{1 \leq i, j \leq n}$  is positive definite for all  $n \in \mathbb{N}$  and  $g_1, \dots, g_n \in G$ .

### Schönberg's Theorem

A continuous  $\theta : G \rightarrow \mathbb{R}$  satisfying 1 and 2

is **CND**  $\iff e^{-t\theta}$  is positive definite for all  $t \geq 0$ .

### Corollary

The *w\*-cont. semigroups of symmetric, normalized, positive-definite functions on  $G$*  are exactly  $(e^{-t\theta})_{t \geq 0}$  for a **CND** function  $\theta : G \rightarrow \mathbb{R}$ .

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For  $d \in \mathbb{N}$  and  $0 \leq \alpha \leq 2$ , the function  $\mathbb{R}^d \rightarrow [0, \infty)$  given by  $x \mapsto \|x\|^\alpha$  is conditionally negative definite.

Example (Haagerup, Invent. Math., 1978/79)

Let  $n \in \mathbb{N}$ . The function  $\mathbb{F}_n \rightarrow [0, \infty)$  given by

$$s \mapsto |s|$$

is conditionally negative definite.

Consequently:

- 1 the  $C^*$ -algebra  $C_r^*(\mathbb{F}_n)$  has the metric approximation property;
- 2  $\mathbb{F}_n$  is weakly amenable;
- 3  $\mathbb{F}_n$  has the Haagerup property.

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## 4. Conditionally negative-definite functions

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For  $d \in \mathbb{N}$  and  $0 \leq \alpha \leq 2$ , the function  $\mathbb{R}^d \rightarrow [0, \infty)$  given by  $x \mapsto \|x\|^\alpha$  is conditionally negative definite.

### Example (Haagerup, Invent. Math., 1978/79)

Let  $n \in \mathbb{N}$ . The function  $\mathbb{F}_n \rightarrow [0, \infty)$  given by

$$s \mapsto |s|$$

is conditionally negative definite.

Consequently:

- 1 the  $C^*$ -algebra  $C_r^*(\mathbb{F}_n)$  has the metric approximation property;
- 2  $\mathbb{F}_n$  is weakly amenable;
- 3  $\mathbb{F}_n$  has the Haagerup property.

# Classical players

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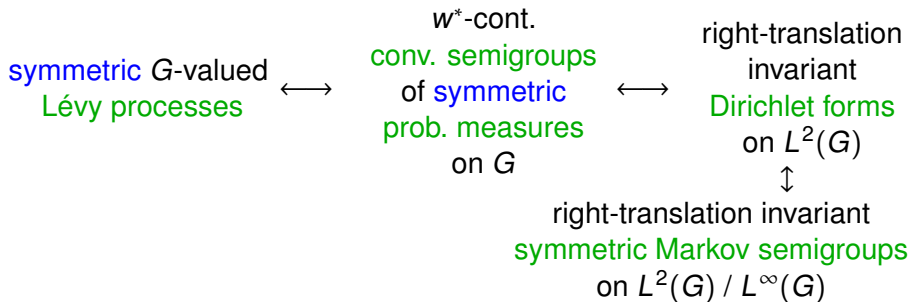
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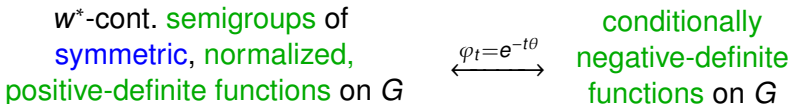
# Classical players

## Summary



## Duality

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# Back to locally compact quantum groups

Features and two basic examples

**Duality**  $G \mapsto \hat{G}$  within the category satisfying  $\hat{\hat{G}} = G$ .

-----  
G – locally compact group

G	$L^\infty(G)$	$C_0(G)$	$C_0^u(G)$	(states of $C_0^u(G), \star$ )	antipode
G					
$\hat{G}$					

# Back to locally compact quantum groups

## Features and two basic examples

### Three “faces” (algebras):

- the von Neumann algebra  $L^\infty(G)$ ;
- the “reduced”  $C^*$ -algebra  $C_0(G)$ ;
- the “universal”  $C^*$ -algebra  $C_0^u(G)$ .

The conjugate space  $C_0^u(G)^*$  carries a convolution  $\star$  turning it into a Banach algebra with unit  $\epsilon$  (the co-unit).

---

### $G$ – locally compact group

$G$	$L^\infty(G)$	$C_0(G)$	$C_0^u(G)$	(states of $C_0^u(G), \star$ )	antipode
$G$	$L^\infty(G)$	$C_0(G)$	$C_0(G)$	regular probability measures, $\star =$ convolution	
$\hat{G}$	$VN(G)$	$C_r^*(G)$	$C^*(G)$	normalized positive definite functions, $\star =$ product	

# Back to locally compact quantum groups

## Features and two basic examples

The **antipode**: an (unbounded) operator on  $L^\infty(G) / C_0(G) / C_0^u(G)$ .  
Decomposes into:

- a “bounded part”: the **unitary antipode**, an anti-automor. of  $L^\infty(G)$ ;
- an “unbounded part”: the **scaling group**.

-----  
G – locally compact group

G	$L^\infty(G)$	$C_0(G)$	$C_0^u(G)$	(states of $C_0^u(G), \star$ )	antipode
G	$L^\infty(G)$	$C_0(G)$	$C_0(G)$	regular probability measures, $\star = \text{convolution}$	composition with inverse
$\hat{G}$	$VN(G)$	$C_r^*(G)$	$C^*(G)$	normalized positive definite functions, $\star = \text{product}$	$\lambda_g \mapsto \lambda_{g^{-1}}$ for $g \in G$



# Convolution semigroups

$\mathbb{G}$  – locally compact quantum group.

## Definition

A **convolution semigroup of states on  $\mathbb{G}$**  is a family  $(\mu_t)_{t \geq 0}$  of states of  $C_0^u(\mathbb{G})$  such that

$$\mu_0 = \epsilon \quad \text{and} \quad \mu_s \star \mu_t = \mu_{s+t} \quad (\forall s, t \geq 0).$$

Adjectives:

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## Examples

of  $w^*$ -continuous, symmetric, convolution semigroups of states on  $G$ :

- $G = G$ :  $w^*$ -continuous, symmetric, convolution semigroups of probability measures on  $G$ .
- $G = \hat{G}$ :  $w^*$ -continuous, symmetric, semigroups of normalized positive-definite functions on  $G$ .

These capture the two types of semigroups discussed earlier.

# Non-commutative Dirichlet forms and Markov semigroups

- ★ **Commutative** = “classical” = on a **positive measure space** with a **reference measure**.
- ★ **Non-commutative** = on a **von Neumann algebra** with a **reference weight**.
  - ▶ Developed by many people: Albeverio–Høegh-Krohn, Sauvageot, Davies–Lindsay, Guido–Isola–Scarlatti, Cipriani–Sauvageot, Cipriani, Goldstein–Lindsay, ...
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# Main result

$\mathbb{G}$  – locally compact quantum group

(Recall:  $L^\infty(\mathbb{G})$  – underlying von Neumann algebra,  $L^2(\mathbb{G})$  – a standard Hilbert space;  $\varphi$  – left Haar weight.)

## Theorem (Skalski–V)

*There exist 1 – 1 correspondences between:*

- 1  $w^*$ -continuous, symmetric, **convolution semigroups** of states on  $\mathbb{G}$ ;
- 2 completely **Dirichlet forms** w.r.t.  $\varphi$  that are right-translation invariant;
- 3 completely **Markov semigroups** on  $L^2(\mathbb{G})$  that are symmetric and contained in  $L^\infty(\hat{\mathbb{G}})$ ;
- 4 completely **Markov semigroups** on  $L^\infty(\mathbb{G})$  that are right-translation invariant and KMS-symmetric w.r.t.  $\varphi$ .

Our main theorem is definitive.

It unifies and extends:

- $\mathbb{G} = G$  or  $\hat{G}$ : the **classical** work;
- $\mathbb{G}$  – **compact** quantum group: Cipriani–Franz–Kula (JFA, 2014).

Such  $\mathbb{G}$  has a canonical Hopf  $*$ -algebra  $\text{Pol}(\mathbb{G})$ .

Its embedding in  $L^2(\mathbb{G})$  is a core of all Dirichlet forms  $\rightsquigarrow$  the problem becomes more algebraic.

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# Approximation properties for groups

$G$  – locally compact group.

## Definition

$G$  has the **Haagerup property** if it admits a **mixing** representation with **almost-invariant vectors**.

## Examples

Amenable groups; free groups;  $SL(2, \mathbb{Z})$ ; groups that act properly on a tree (more generally: on a space with “walls”); Coxeter groups.

## Definition

$G$  does **not** have **property (T)** if it admits an **ergodic** representation with **almost-invariant vectors**.

## Examples (of property (T))

$SL(n, \mathbb{Z})$ ,  $n \geq 3$ ;  $SL(n, \mathbb{K})$ ,  $n \geq 3$ ;  $Sp(2n, \mathbb{K})$ ,  $n \geq 2$  ( $\mathbb{K}$  – local field).

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# Approximation properties for groups

## Geometric characterizations

Theorem (Guichardet, '72 + Delorme, '77; Akemann–Walter, '81)

Assume that  $G$  is  $\sigma$ -compact.

- 1  $G$  does **not** have **property (T)**  
 $\iff$  it has an **unbounded** conditionally negative-definite function  
 $\iff$  it has a  $w^*$ -cont. semigroup of symm. normalized pos.-def. functions that is **not norm continuous**.
- 2  $G$  has the **Haagerup property**  
 $\iff$  it has a **proper** conditionally negative-definite function  
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(proper = goes to  $\infty$  at  $\infty$ ).

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The CND function  $s \mapsto |s|$  on  $\mathbb{F}_n$  presented earlier!

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# Approximation properties for quantum groups

Geometric characterizations: the discrete case

$\mathbb{G}$  – second countable, **discrete** quantum group.

## Definition

A **conditionally negative-definite** function on  $\mathbb{G}$  is a map  $\theta : \text{Pol}(\hat{\mathbb{G}}) \rightarrow \mathbb{C}$  satisfying:

- 1  $\theta(\mathbb{1}) = 0$ ;
- 2  $\theta(a^*) = \overline{\theta(a)}$  for all  $a \in \text{Pol}(\hat{\mathbb{G}})$ ;
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A version of Schönberg's Theorem for finite-dim. co-algebras gives:

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*The correspondence  $\theta \leftrightarrow e_{\star}^{-t\theta}$  is 1 – 1 between the **CND** functions on  $\mathbb{G}$  and  **$w^*$ -cont. convolution semigroups of states** on  $\hat{\mathbb{G}}$ .*

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$\mathbb{G}$  – second countable, **discrete** quantum group.

**Theorem (Kyed, JFA, 2011)**

If  $\mathbb{G}$  is unimodular, then it does **not** have **property (T)**

$\iff$  it has an **unbounded** conditionally negative-definite function.

**Theorem (Daws–Fima–Skalski–White, Crelle’s Journal, 2016)**

$\mathbb{G}$  has the **Haagerup property**

$\iff$  it has a **proper** conditionally negative-definite function.

- Many applications, e.g. Daws–Skalski–V, Comm. Math. Phys., 2017.

# Approximation properties for quantum groups

Geometric characterizations: the general case

$\mathbb{G}$  – second countable, locally compact quantum group.

## Theorems (Skalski–V)

- 1  $\mathbb{G}$  does **not** have **property (T)**
  - $\iff \hat{\mathbb{G}}$  has a completely Dirichlet form w.r.t.  $\hat{\varphi}$  that is right-translation invariant and **unbounded**
  - $\iff \hat{\mathbb{G}}$  has a  $w^*$ -continuous, symmetric, convolution semigroup of states that is **not norm continuous**.
- 2  $\mathbb{G}$  has the **Haagerup property**
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# Convolution semigroups via cocycle twisting

$\mathbb{G}$  – locally compact quantum group

$U \in L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$  – 2-cocycle: a unitary satisfying

$$(U \otimes \mathbb{1})(\Delta \otimes \text{id})(U) = (\mathbb{1} \otimes U)(\text{id} \otimes \Delta)(U).$$

Theorem (De Commer, '09)

*Replacing  $\Delta$  by  $U\Delta(\cdot)U^*$  one gets a new LCQG,  $\mathbb{G}_U$ .*

$(\mu_t)_{t \geq 0}$  –  $w^*$ -cont. (symmetric) convolution semigroup of states on  $\mathbb{G}$ .

Proposition

If  $U$  is invariant under the Markov semigroup  $(T_t)_{t \geq 0}$  associated with  $(\mu_t)_{t \geq 0}$  when applied to its left leg, then  $(T_t)_{t \geq 0}$  is also induced by a convolution semigroup on  $\mathbb{G}_U$ .

Example

conv. semigroup on  $\mathbb{R}^n \rightsquigarrow$  conv. semigroup on  $\mathbb{H}_n^q(\mathbb{R})$ .

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Thank you for your attention!