An equivariant pushout structure of Vaksman-Soibelman quantum spheres

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joint work with F. Arici, F. D'Andrea and P. M. Hajac

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Motivation: K-theory of complex projective spaces

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Initial goal

Geometric interpretation of generators of the K_0-group of $\mathbb{C}\mathrm{P}^n_q$ as Milnor vector bundles.

The Milnor connecting homomorphism

For any pullback unital algebras

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\
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The Milnor connecting homomorphism is given by

$$\partial_{10} : K_1^{alg}(A_{12}) \to K_0^{alg}(A) : [U] \mapsto [p_U] - [I_n],$$

where $U \in GL_n(A_{12})$ and p_U is an idempotent matrix in $M_{2n}(A)$ whose entries consist of liftings $c, d \in M_n(A_1)$ of U such that $\pi_1(c) = U^{-1}$ and $\pi_1(d) = U$.

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The Vaksman-Soibelman quantum spheres

Definition (Vaksman-Soibelman odd quantum spheres)

For any $0 < q < 1$, the C^*-algebra $C(S_q^{2n+1})$ of the Vaksman-Soibelman quantum sphere is the universal C^*-algebra generated by $z_0, z_1, \ldots, z_n$, subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j,$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*, \qquad \sum_{m=0}^{n} z_m z_m^* = 1.$$

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Note that $C(S_q^3) = C(SU_q(2))$. In the original approach of Vaksman and Soibelman, the algebra $C(S_q^{2n+1})$ was defined as the **quantum homogeneous space** $C(SU_q(n+1))/C(SU_q(n))$.

The Hong-Szymański even quantum balls

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For any $0 < q < 1$, the C^*-algebra $C(B_q^{2n})$ of the Hong-Szymański quantum ball is the universal C^*-algebra generated by $z_1, \ldots, z_n$, subject to the following relations:

$$z_i z_j = q^{1/2} z_j z_i \quad \text{for } i < j, \qquad z_i z_j^* = q^{-1/2} z_j^* z_i \quad \text{for } i \neq j,$$

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We have that

$$C(\partial B_q^{2n}) := C(B_q^{2n})/I \cong C(S_q^{2n-1}),$$

where I is the ideal generated by $1 - \sum_{m=1}^{n} z_m z_m^*$.

An equivariant pullback structure of $C(S_q^{2n+1})$

Theorem

$\forall\, n \in \mathbb{N} \setminus \{0\}$ $\exists$ *a* $U(1)$*-equivariant pullback of C*-algebras:*

$$\begin{tikzcd}
 & C(S_q^{2n+1}) \arrow[dl] \arrow[dr] & \\
C(S_q^{2n-1}) \arrow[dr] & & C(B_q^{2n}) \otimes C(U(1)) \arrow[dl] \\
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Bundles over quantum complex projective spaces

Corollary

$\forall\, n \in \mathbb{N} \setminus \{0\}$ $\exists$ *a pullback of C*-algebras:*

$$\begin{tikzcd} & C(\mathbb{C}P_q^{n}) \arrow[dl] \arrow[dr] & \\ C(\mathbb{C}P_q^{n-1}) \arrow[dr] & & C(B_q^{2n}) \arrow[dl] \\ & C(S_q^{2n-1}). & \end{tikzcd}$$

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$$\begin{tikzcd}
K_0(C(\mathbb{C}P_q^n)) \arrow[r] & K_0(C(\mathbb{C}P_q^{n-1})) \oplus K_0(C(B_q^{2n})) \arrow[r] & K_0(C(S_q^{2n-1})) \arrow[d, "\partial_{01}"] \\
K_1(C(S_q^{2n-1})) \arrow[u, "\color{red}\partial_{10}"] & K_1(C(\mathbb{C}P_q^{n-1})) \oplus K_1(C(B_q^{2n})) \arrow[l] & K_1(C(\mathbb{C}P_q^n)) \arrow[l]
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1. $\forall\ s_{e_i}, s_{e_j} \in Q_1 :\ s^*_{e_i} s_{e_j} = \delta_{ij} t(e_i)$,
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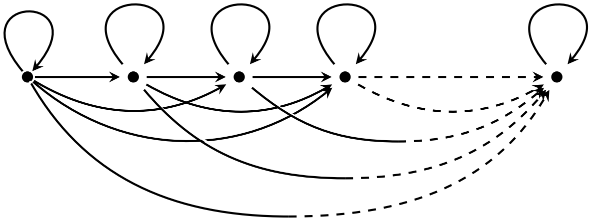
Gauge action: There is a natural $U(1)$-coaction on $C^*(Q)$ given by

$$\delta(p_v) = p_v \otimes 1, \quad v \in Q_0,$$

$$\delta(s_e) = s_e \otimes u, \quad e \in Q_1, \quad u \in C(U(1)).$$

Quantum spheres and balls as graph algebras



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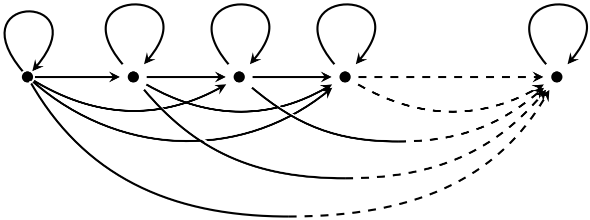

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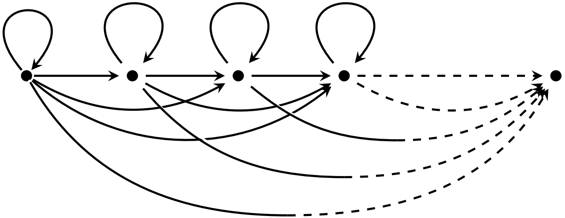
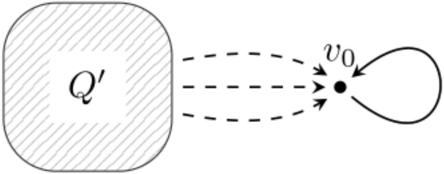


$C(B_q^{2n})$

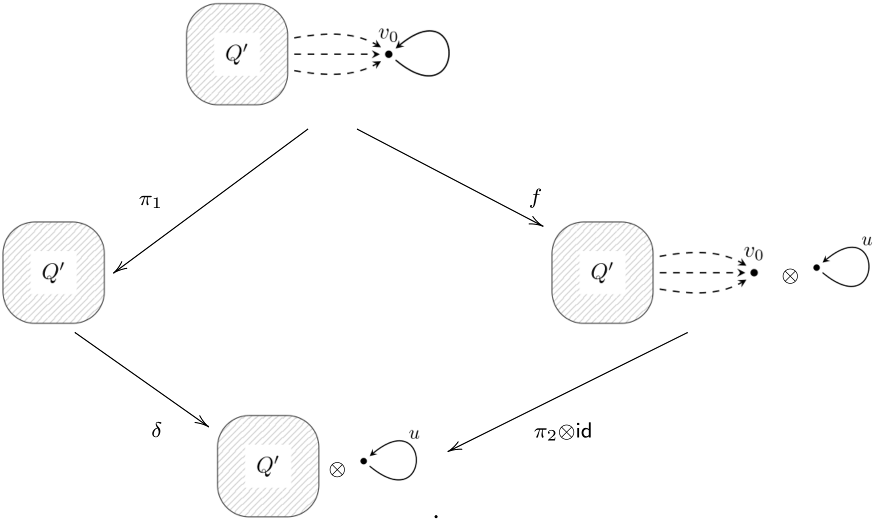
Trimmable graph C*-algebras

Definition (Trimmable graph)

Let Q be a finite graph consisting of a sub-graph Q' emitting at least one edge to an external vertex v_0 whose only outgoing edge e_0 is a loop. We call such a graph (Q', v_0)-*trimmable* iff all edges from Q' to v_0 begin in a vertex emitting an edge that ends inside Q'.



Pullback structure of trimmable graph C*-algebras (I)



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Towards q-deformed CW-complexes

Definition

Let A and D be C*-algebras. We say that A is obtained from D by *attaching a q-n-cell*, if there is a pullback diagram of C*-algebras

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