Quantum groups acting on the nodal cubic

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Brief motivation

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<th>Classical notions</th>
<th>Quantum analogues</th>
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### Definition

An algebra embedding $B \subseteq A$ is **faithfully flat** when any chain complex

$$L \xrightarrow{f} M \xrightarrow{g} N, \quad g \circ f = 0$$

of $B$-modules is exact if and only if so is the induced complex of $A$-modules

$$A \otimes_B L \to A \otimes_B M \to A \otimes_B N.$$
Quantum homogeneous spaces

Definition

Let $A$ be a Hopf algebra. A quantum homogeneous space is a right coideal subalgebra $B \subseteq A$, such that $A$ is a faithfully flat $B$-module.
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- Given a right coideal subalgebra $B \subseteq A$,

  $$ C := A/AB^+, \quad \text{where } B^+ := B \cap \ker \varepsilon $$

  is a quotient coalgebra. It is Hopf iff $AB^+ = B^+A$. 

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  is a quotient coalgebra. It is Hopf iff $AB^+ = B^+ A$.
- The canonical map $\pi: A \to C$ induces a left coaction $\lambda: A \to C \otimes A$ (via $\Delta$). Faithfully flatness implies that
  \[ B = A^{CoC} := \{ a \in A \mid \lambda(a) = \pi(1) \otimes a \}. \]
\( B \subseteq A \) as a coalgebra Galois extension

- \( B \subseteq A \) is a **Galois C-extension**, since the Galois map

\[
\beta : A \otimes_B A \to C \otimes A, \quad a \otimes_B b \mapsto \pi(a_{(1)}) \otimes a_{(2)} b.
\]

is bijective with inverse given by \( \pi(a) \otimes b \mapsto a_1 \otimes_B S(a_2) b. \)
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- Classical: the orbit stabiliser theorem $\sim \to G/H \simeq X$, i.e., as a set $G \simeq X \times H$. However, $\alpha_X : G \to X$ is not necessarily a trivial $H$-principal bundle.
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- Classical: the orbit stabiliser theorem $\sim G/H \simeq X$, i.e., as a set $G \cong X \times H$. However, $\alpha_X : G \to X$ is not necessarily a trivial $H$-principal bundle.

- Quantum: the most trivial bundle is $A = C \otimes B$, followed by $C \ltimes B$, followed by **cleft extensions**.

**Definition**

The extension $B \subseteq A$ is **$C$-cleft** if there exists a $C$-colinear map $\gamma : C \to A$ that is convolution invertible.
Cleftness can be equivalently stated as $A \cong C \otimes B$ as left $C$-comodules right $B$-modules. Think $c \mapsto c \otimes 1$ in $C \otimes B$ under the identification.
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Faithful flatness implies that the functor

$$-\square_C A : \text{Mod}^C \to \text{Mod}^A_B$$

is an equivalence of categories. To finite-dimensional $C$-comodules $V$ correspond finitely generated projective $B$-modules $\rightsquigarrow K$-theory of $B$. 

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Cleftness and associated modules

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**Question:** can we generate $K_0(B)$ in this way?
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**Question**: can we generate $K_0(B)$ in this way?

$C$-cleft $\Rightarrow$ all associated modules are free since $V \square_C C \otimes B \cong V \otimes B$. $\Rightarrow$ answer is no, non-trivial elements in $K_0(B)$ are not reached.
The nodal cubic

Let $k$ be a field.

The **nodal cubic** is plane curve in $k^2$ given by the equation $y^2 = x^2 + x^3$:

![Graph of the nodal cubic over $\mathbb{R}$]

Its coordinate ring is $B = k[x, y]/\langle y^2 - x^2 - x^3 \rangle$.

Singular curves cannot be homogeneous, right?
Consider the algebra \( \tilde{A} \) generated by \( x, y \) and invertible elements \( a, b \) satisfying

\[
\begin{align*}
y^2 &= x^2 + x^3, \quad a^3 = b^2, \\
ba &= ab, \quad ya = ay, \quad bx = xb, \quad by = -yb, \\
xa^2 + axa + a^2 x - a^2 + a^3 &= 0, \quad x^2 a + xax + ax^2 + ax + xa = 0,
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\end{align*}

From these relations, it follows that $\tilde{A}$ is a right free $B$-module (hence faithfully flat), with vector space basis

$$\left\{ a^i b^j (xa)^k x^m y^n \mid i \in \mathbb{Z}, k, m \in \mathbb{N}, j, n \in \{0, 1\} \right\}.$$

However, in this presentation, the commutation relations are too complicated for extensive computations.
A Hopf algebra structure in $\tilde{A}$ is given by

\[
\Delta(x) = 1 \otimes x + x \otimes a, \quad \Delta(y) = 1 \otimes y + y \otimes b,
\]

\[
\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b,
\]

\[
\varepsilon(x) = \varepsilon(y) = 0, \quad \varepsilon(a) = \varepsilon(b) = 1,
\]

\[
S(x) = -xa^{-1}, \quad S(y) = -yb^{-1}, \quad S(a) = a^{-1}, \quad S(b) = b^{-1},
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from where it is clear that $\Delta(B) \subseteq B \otimes A$. 
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$$

from where it is clear that $\Delta(B) \subseteq B \otimes A$.

- $\tilde{A}$ is pointed but not connected. What else can we say about its properties, both as an algebra and as a coalgebra?
- Coproducts on powers of $x$ and $(xa)$ are also hard to compute, because of the commutation relations.
Playing around with the relations

**Idea**: find a smaller Hopf algebra which still admits a faithfully flat embedding of $B$ as a right coideal subalgebra.

**How**: look for central and grouplike/primitive elements in $\tilde{A}$, generating (Hopf) ideals that intersect $B$ trivially. E.g., $\langle a^3 - 1 \rangle$, since $a^3$ is grouplike and central.
Idea: find a smaller Hopf algebra which still admits a faithfully flat embedding of $B$ as a right coideal subalgebra.

How: look for central and grouplike/primitive elements in $\tilde{A}$, generating (Hopf) ideals that intersect $B$ trivially. E.g., $\langle a^3 - 1 \rangle$, since $a^3$ is grouplike and central.

If the field contains a primitive 3rd root $r$ of unity, then one can define

$$F := xa + (r + 1)ax + \frac{r + 2}{3} (a - a^2).$$

Think of $F$ as a change of the variable ($xa$) that yields a nicer presentation.
Playing around with relations

Lemma

The class of $F$ in $\tilde{A}/\langle a^3 - 1 \rangle$ satisfies

$$aF = r^2 Fa, \quad bF = Fb, \quad yF = Fy,$$

$$xF = rFx + \frac{r + 2}{3} aF + \frac{r - 1}{3} F + \frac{1}{3} (a - 1),$$

and

$$\Delta(F) = a \otimes F + F \otimes a^2.$$

Furthermore, the class of $F^3$ is central and primitive.

$\implies$ It is also “safe” to add the relation $F^3 = 0$. 

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Lemma

The class of $F$ in $\tilde{A}/\langle a^3 - 1 \rangle$ satisfies

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\[ xF = rFx + \frac{r + 2}{3} aF + \frac{r - 1}{3} F + \frac{1}{3}(a - 1), \]

and

\[ \Delta(F) = a \otimes F + F \otimes a^2. \]

Furthermore, the class of $F^3$ is central and primitive.

It is also “safe” to add the relation $F^3 = 0$.

The resulting quotient $A := \tilde{A}/\langle a^3 - 1, F^3 \rangle$ is an iterated Ore extension and a finitely generated free $B$–module.
Note that $B$ is the subalgebra of $A$ generated by $x$ and $y$. Let $C$ be the subalgebra generated by $a$, $b$ and $F$. 
Note that \( B \) is the subalgebra of \( A \) generated by \( x \) and \( y \). Let \( C \) be the subalgebra generated by \( a, b \) and \( F \).

**Theorem**

- \( B \subseteq A \) is a quantum homogeneous space.
- \( C \cong A/AB^+ \) as coalgebras.
- Multiplication in \( A \) defines an isomorphism \( C \otimes B \cong A \) as left \( C \)-comodules and right \( B \)-modules, so \( A \) is a **cleft extension** of \( B \).
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**Theorem**

- $B \subseteq A$ is a quantum homogeneous space.
- $C \cong A/AB^+$ as coalgebras.
- Multiplication in $A$ defines an isomorphism $C \otimes B \cong A$ as left $C$-comodules and right $B$-modules, so $A$ is a cleft extension of $B$.

The decomposition $A \cong C \otimes B$ follows explicitly from

$$\left\{ a^i b^j F^k x^m y^n \mid i, k \in \{0, 1, 2\}, m \in \mathbb{N}, j, n \in \{0, 1\} \right\}.$$ 

being a basis of $A$ as a vector space. Since $B^+ = \langle x, y \rangle_B$, the projection $\pi : A \to A/AB^+$ restricts to an isomorphism $C \cong A/AB^+$. 

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Relation to $U_r(\mathfrak{sl}_2)$ and the small quantum group $u_r(\mathfrak{sl}_2)$

One can also define elements

$$E := xa - rax + \frac{1-r}{3}(a - a^2), \quad K := a^2$$

which together with $F$ satisfy the defining relations of $U_r(\mathfrak{sl}_2)$:

$$KE = r^2EK, \quad KF = rFK, \quad [E,F] = \frac{K - K^2}{r - r^2}, \quad KK^{-1} = K^{-1}K = 1.$$
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Remark: Monomials in $a, x$ and $(xa)$ can be replaced by monomials in $E, F$ and $K$ (PBW-like basis), while $b$ and $y$ have the coproduct of the generators of Sweedler’s infinite dimensional Hopf algebra $H$.

Indeed, $A$ can be seen as a quotient of $U_r(\mathfrak{sl}_2) \otimes H$ by the relations

$$F^3 = 0, \quad K^3 = 1, \quad y^2 = \frac{1}{27}E^3, \quad a^3 = b^2, \quad b^2 = 1, \quad yb = -by$$
Relation to $U_r(\mathfrak{sl}_2)$ and the small quantum group $u_r(\mathfrak{sl}_2)$

The **small quantum group** $u_r(\mathfrak{sl}_2)$ is obtained by truncating $U_r(\mathfrak{sl}_2)$, i.e., imposing the relations

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The last two of these relations are already present in \( A \), but the relation \( E^3 = 0 \) is equivalent to \( y^2 = x^2 + x^3 = 0 \). In this case, we obtain \( u_r(\mathfrak{sl}_2) \otimes H_4 \) as a quotient of \( A \), where \( H_4 \) is Sweedler’s 4-dimensional Hopf algebra.
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We have the Casimir element in $A$

$$\Omega := EF + \frac{r^2K + rK^2}{(r - r^2)^2} = (xa)^2 - a^2x - a^2x^2 + \frac{1}{3}$$

which is central.
Ring theoretic properties of $A$

**Proposition**

The algebra $A$ is Noetherian of Gelfand-Kirillov dimension 1, but neither regular nor semiprime.
Proposition

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Noetherianity and GK dimension (resp. non-regularity) follow from $B$ having the mentioned properties and $A$ being a f. g. (resp. free) $B$-module.

The Casimir element $\Omega$ has as minimal polynomial:

$$t^3 - \frac{1}{3}t^2 - \frac{2}{27}t - \frac{1}{3}.$$

From where it follows that $A$ is not semiprime.
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The Casimir element $\Omega$ has as minimal polynomial:

$$t^3 - \frac{1}{3}t + \frac{2}{27} = \left(t - \frac{1}{3}\right)^2 \left(t + \frac{2}{3}\right),$$

from where it follows that $A$ is not semiprime.
Main theorem - second half

Theorem

If $\text{char } k \neq 2$ and $0 \neq I \subseteq A$ is a Hopf ideal, then $B \cap I \neq 0$. In other words, $A$ is a minimal Hopf algebra containing $B$ as a quantum homogeneous space.

Motivation: classically, find a smallest subgroup that still acts transitively.
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Proof: The group of group-likes of $A$ is $\mathbb{Z}_3 \times \mathbb{Z}_2$. The presentation of $A$ in terms of $U_r(\mathfrak{sl}_2)$ and $H$ gives a complete characterization of the Yetter-Drinfel’d module of twisted primitives of $A$:

$$(1, 1): \langle x^2 + x^3 \rangle_k, \quad (1, a): \langle x, axa^2 \rangle_k, \quad (1, b): \langle y \rangle_k.$$
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A Hopf algebra map $A \rightarrow A/I$ such that $B \cap I = 0$ induces a injective map on the level of group-like elements and subsequently, on the level of the twisted primitives of $A$. This implies that $A \rightarrow A/I$ is injective, so $I = 0 \not\leq$. 

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Besides the Casimir, $y^2 = x^2 + x^3$ is also an element in the center. Is the whole center generated by these two?

What is the nilradical of $A$?

For other curves (as in Angela’s work), are the corresponding extensions cleft as well?

Can the $K_0$-group of $B$ be obtained from $A$ in some other way, such as a generalisation of a Mayer-Vietoris sequence to Hopf algebras or Hopf Galois extensions?