Martin boundary of the duals of free unitary quantum groups

(joint work with Sergey Neshveyev)

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Plan

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- Classical Martin boundary

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- Martin boundary of a dual discrete quantum group

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- Free unitary quantum groups $A_u(F)$

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- Idea of the proof

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- For any $n \in \mathbb{N}$, define $(P_\mu^n u)(s) = \sum_{t \in \Gamma} p_\mu^{(n)}(s,t) u(t)$, where

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- the **Green kernel** is the operator $G_\mu = \sum_{n=0}^{\infty} P_\mu^n$.

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Assumptions:

(i) *transience*, $G_\mu(s,t) := \sum_{n=0}^{\infty} p_\mu^{(n)}(s,t) < \infty$, for any $s,t \in \Gamma$.

(ii) *irreducibility*, for any $s,t \in \Gamma$, there exists $n \in \mathbb{N}$ such that $p_\mu^{(n)}(s,t) > 0$.

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Fix $e \in \Gamma$. For any $s,t \in \Gamma$ the **Martin kernel** is the operator

$$K_\mu \colon c_c(\Gamma) \to \ell^\infty(\Gamma), \qquad \text{defined by} \qquad K_\mu(s,t) := (K_\mu\,\delta_t)(s) = \frac{G_\mu(s,t)}{G_\mu(e,t)}.$$

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- The **Martin boundary** is the compact space $\partial\Gamma_M = \overline{\Gamma}_M \setminus \Gamma$.

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Main properties used: almost multiplicativity of kernels along geodesics

$$\frac{1}{C_1} G(s,v)G(v,t) \le G(s,t) \le C_1\, G(s,v)G(v,t), \quad C_1 > 0, \quad v \in [s,t].$$

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- $\mathbb{C}[G]^* \simeq \prod_{s \in I} B(H_s)$, $\hat{\Delta}(\omega)(a \otimes b) = \omega(ab)$ for $\omega \in \mathbb{C}[G]^*$ and $a, b \in \mathbb{C}[G]$.

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- *dual discrete quantum group* is $\ell^\infty(\hat{G}) = \ell^\infty\text{-}\bigoplus_{s \in I} B(H_s)$, with $\hat{\Delta}\colon \ell^\infty(\hat{G}) \to \ell^\infty(\hat{G}) \mathbin{\bar{\otimes}} \ell^\infty(\hat{G})$.

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- For $s \in I$, $I_s \in B(H_s)$ minimal central projection.

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- The **Martin boundary** is the quotient C^*-algebra $M_\mu := \bar{M}_\mu / c_0(\hat{G})$.

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- $F \in \mathrm{GL}_n(\mathbb{C})$, $n \geq 2$, such that $\mathrm{Tr}(F^*F) = \mathrm{Tr}((F^*F)^{-1})$;
- $A_u(F)$ is the universal unital C^*-algebra generated by entries of a matrix $U = (u_{ij})_{i,j=1}^n$ such that

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- the comultiplication Δ is defined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj};$$

- $(A_u(F), \Delta)$ is called the **free unitary quantum group**;

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- The set I of irreducible representations of $A_u(F)$ identified with $\mathbb{N} * \mathbb{N}$:

$$\mathbb{1} \rightsquigarrow e, \quad U \rightsquigarrow \alpha, \quad \bar{U} \rightsquigarrow \beta, \qquad\qquad \bar{\beta} = \alpha, \quad \bar{\alpha} = \beta.$$

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- fusion rules

$$x \otimes y \simeq \bigoplus_{\substack{z \in \mathbb{N}*\mathbb{N} \\ x = x_0 z,\ y = \bar{z} y_0}} x_0 y_0.$$

- Example $\alpha\beta \otimes \alpha\beta = \alpha\beta\alpha\beta \oplus \alpha\beta \oplus e$.

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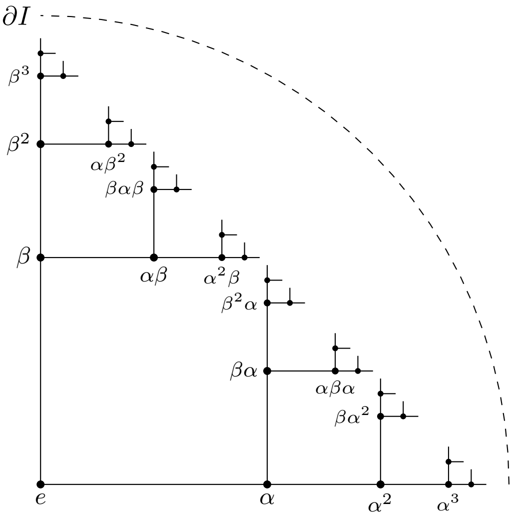
- $q \in (0,1]$ such that $\dim_q(\alpha) = \mathrm{Tr}(F^*F) = q + q^{-1}$.
- F not a unitary 2-by-2 matrix, so $q < 1$.

Cayley graph associated to $A_u(F)$



End compactification

- Boundary $\partial I = \bar{I} \setminus I$ set of infinite words on the left.
- $C(\bar{I})$ identified with the algebra of functions $f \in \ell^\infty(I)$ such that

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- $\mathcal{B}_\infty = \mathcal{B}/c_0(\hat{G})$.

Martin boundary of $A_u(F)$

Theorem (M., Neshveyev)

- *Free unitary quantum group $G = A_u(F)$, with F not a unitary 2-by-2 matrix;*
- *generating finitely supported probability measure μ on I.*

Then the Martin compactification $\bar{M}_\mu$ coincides with the compactification $\mathcal{B}$. It follows that the Martin boundary M_μ coincides with $\mathcal{B}_\infty$.

Actions on $\mathcal{B}$ and on M_μ

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- The Martin compactification $\bar{M}_\mu$ and by restriction the Martin boundary M_μ are equipped with actions

$$\begin{aligned} \alpha_{\hat{G}} \colon M_\mu &\to M(M_\mu \otimes c_0(\hat{G})), \qquad & x &\mapsto \hat{\Delta}(x), \\ \alpha_G \colon M_\mu &\to C(G) \otimes M_\mu, & x &\mapsto \alpha_G(x), \end{aligned}$$

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- The same holds for $\mathcal{B}$ and $\mathcal{B}_\infty$.

Duality Theorem

[De Commer, Yamashita – 2013, Neshveyev – 2014]

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Theorem (Duality for G-C^*-algebras)

Let G be a reduced compact quantum group. Then the categories

(i) *the category of unital G-C^*-algebras with unital G-equivariant $*$-homomorphisms as morphisms;*

(ii) *the category of pairs $(\mathcal{D}, M)$, where $\mathcal{D}$ is a right $\operatorname{Rep} G$-module C^*-category and M is a generating object in $\mathcal{D}$, with equivalence classes of unitary $\operatorname{Rep} G$-module functors respecting the generating objects as morphisms.*

are equivalent. The actions in (i) *are ergodic if and only if the $\operatorname{Rep} G$-module C^*-categories in* (ii) *are semisimple, indecomposable and with simple generating objects.*

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$$\begin{aligned}\mathcal{D}_{\hat{G}}(U,V) &:= \ell^\infty\text{-}\bigoplus_{s\in I} \operatorname{Mor}(U_s \otimes U, U_s \otimes V)\\ &\subset \ell^\infty\text{-}\bigoplus_{s\in I} B(H_s \otimes H_U, H_s \otimes H_V) = \ell^\infty(\hat{G}) \otimes B(H_U, H_V).\end{aligned}$$

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Duality for adjoint actions on discrete quantum groups

- $\ell^\infty(\hat{G})$ equipped with adjoint action α_G.
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- Denote by $\mathcal{D}_{\mathcal{B}}$ the category associated to $\mathcal{B}$ and by $\mathcal{D}_{\mathcal{M}}$ the category associated to $\bar{M}_\mu$.

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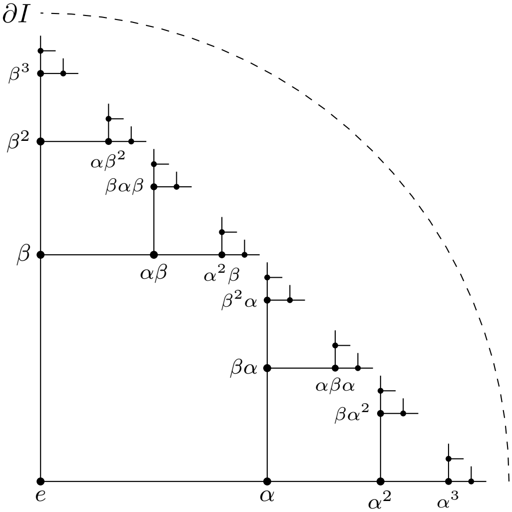
- For a particular element $T \in \bigoplus_{s \in I} \mathrm{Mor}(s, s \otimes y)$,

$$\mathcal{D}_{\mathcal{B}}(e, y) = C(\bar{\Omega}_y) T.$$

Branch $\bar{\Omega}_y$ of the tree



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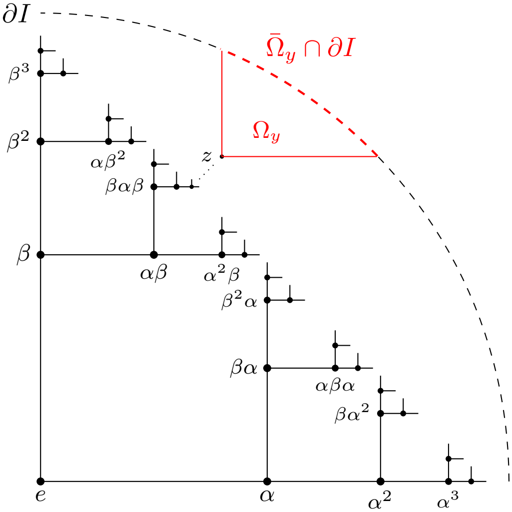

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- Goal: understand asymptotic behaviour of $K_Q(s,\cdot)$.

Random walk on the centre

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Properties of P_μ-random walk:

- There exists a constant $C_1 > 0$ such that

$$\frac{1}{C_1} G_{P,\Delta_x}(s,v) G_{P,\Delta_x}(v,t) \leq G_{P,\Delta_x}(s,t) \leq C_1 \, G_{P,\Delta_x}(s,v) G_{P,\Delta_x}(v,t)$$

  for all $x \in I$, $s,t \in \Delta_x$ and $v \in [s,t]$.

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$$\Longrightarrow \quad K_P(s,\cdot) \in C(\bar{I}).$$

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Estimates that we obtained:

- C constant depending on q, $|y|$ and the support of μ such that

$$|q_\mu(s,t) - p_\mu(s,t)| \leq Cq^{|s|}, \qquad \forall s,t \in \Omega_y.$$

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- Let $\Delta_x = \{ux \,|\, u \in I\}$. There is a constant C_2 such that

$$|G_{Q,\Delta_x}(s,t) - G_{P,\Delta_x}(s,t)| \le C_2 q^{|x|} G_{P,\Delta_x}(s,t)$$

  for all $x \in \Omega_y$ and $s,t \in \Delta_x$.

Proposition

- $s \in \Omega_y$, *the function* $K_Q(s,\cdot) = \dfrac{G_Q(s,\cdot)}{G_P(e,\cdot)}$ *on* Ω_y *extends to a continuous function on* $\bar{\Omega}_y \subset \bar{I}$,

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This concludes the proof.

Thank you for your attention!