Factorisation of quasi K-matrices for quantum symmetric pairs

Liam Dobson

School of Mathematics, Statistics and Physics
Newcastle University

June 13th 2018
### Outline

- Quantized enveloping algebra $U_q(\mathfrak{g})$
  - Universal $R$-matrix $\mathcal{R}$
    - $\mathcal{R} = R \circ \kappa \circ \text{flip}$
  - Quasi $R$-matrix $R$
    - Factorisation using $R$ for $U_q(\mathfrak{sl}_2)$

- QSP coideal subalgebra $B_{c,s}$
  - Universal $K$-matrix $\mathcal{K}$
    - $\mathcal{K} = \mathfrak{X} \circ \xi \circ T_{w_x}^{-1} T_{w_0}^{-1}$
  - Quasi $K$-matrix $\mathfrak{X}$
    - ?

---

**Liam Dobson**

Factorisation of quasi K-matrices for quantum symmetric pairs
Quantised enveloping algebras

- \( g \) - fin. dim. semisimple Lie algebra over \( \mathbb{C} \);
- \( I \) - indexing set for nodes of Dynkin diagram;
- \( \Pi = \{ \alpha_i \mid i \in I \} \) simple roots; \( W \) - Weyl group of \( g \);
- \( q \) - indeterminate; \( \mathbb{K} \) - field of char 0;
- \( \mathbb{K}(q) \) - field of rational functions in \( q \) with coeffs in \( \mathbb{K} \).
Quantised enveloping algebras

- $\mathfrak{g}$ - fin. dim. semisimple Lie algebra over $\mathbb{C}$;
- $I$ - indexing set for nodes of Dynkin diagram;
- $\Pi = \{\alpha_i \mid i \in I\}$ simple roots; $W$ - Weyl group of $\mathfrak{g}$;
- $q$ - indeterminate; $\mathbb{K}$ - field of char 0;
- $\mathbb{K}(q)$ - field of rational functions in $q$ with coeffs in $\mathbb{K}$.

$$U_q(\mathfrak{g}) = \mathbb{K}(q) \langle E_i, F_i, K_i^{\pm 1} \mid i \in I \rangle / \text{relations}$$

- $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ comultiplication;

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$\Delta(K_i) = K_i \otimes K_i.$$
$U_q(g)$ has the following properties/notation:

- Triangular decomposition $U_q(g) = U^+ \otimes U^0 \otimes U^-$;
- Vector space basis for $U^0$: $K_\lambda = \prod_{i \in I} K_i^{n_i}$ for $\lambda = \sum_{i \in I} n_i \alpha_i \in Q$;
- Algebra gradings $U^+ = \bigoplus_{\mu \in Q^+} U^+_\mu$, $U^- = \bigoplus_{\mu \in Q^+} U^-_{-\mu}$.

For $w = s_{i_1} \cdots s_{i_t}$ reduced, define $T_w = T_{i_1} \cdots T_{i_t}$. 
\( U_q(\mathfrak{g}) \) has the following properties/notation:

- **Triangular decomposition** \( U_q(\mathfrak{g}) = U^+ \otimes U^0 \otimes U^-; \)
- **Vector space basis for \( U^0 \):** \( K_\lambda = \prod_{i \in I} K_i^{n_i} \) for \( \lambda = \sum_{i \in I} n_i \alpha_i \in Q; \)
- **Algebra gradings** \( U^+ = \bigoplus_{\mu \in Q^+} U^+_\mu, \quad U^- = \bigoplus_{\mu \in Q^+} U^-_{-\mu}; \)
- **Action of \( Br(\mathfrak{g}) \) on \( U_q(\mathfrak{g}) \) by algebra automorphisms \( T_i: \)

\[
\underbrace{T_i T_j T_i \cdots}_m = \underbrace{T_j T_i T_j \cdots}_{m}
\]

\( m_{ij} \) factors \( m_{ij} \) factors

- For \( w = s_{i_1} \cdots s_{i_t} \) reduced, define \( T_w = T_{i_1} \cdots T_{i_t}. \)
Quantum symmetric pairs

Let $\theta : g \to g$ be an involution automorphism;

- Classified (up to conjugation) by Satake diagrams.
Quantum symmetric pairs

Let $\theta : g \rightarrow g$ be an involutive automorphism;

- Classified (up to conjugation) by Satake diagrams.

**Definition**

Let $X \subset I$ and $\tau : I \rightarrow I$ a diagram automorphism satisfying $\tau(X) = X$. The triple $(I, X, \tau)$ is called a **Satake diagram** if:

1. $\tau^2 = \text{id}_I$.
2. $-w_X(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in X$.
3. If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$.
Quantum symmetric pairs

Let $\theta : g \to g$ be an involutive automorphism;

- Classified (up to conjugation) by Satake diagrams.

**Definition**

Let $X \subset I$ and $\tau : I \to I$ a diagram automorphism satisfying $\tau(X) = X$. The triple $(I, X, \tau)$ is called a **Satake diagram** if:

1. $\tau^2 = \text{id}_I$.
2. $-w_X(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in X$.
3. If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$. 

Diagram:

```
1 ---- 2 ---- 3 ---- n
```

Liam Dobson

Factorisation of quasi K-matrices for quantum symmetric pairs
Let $\theta : g \to g$ be an involutive automorphism;

- Classified (up to conjugation) by Satake diagrams.

**Definition**

Let $X \subset I$ and $\tau : I \to I$ a diagram automorphism satisfying $\tau(X) = X$. The triple $(I, X, \tau)$ is called a **Satake diagram** if:

1. $\tau^2 = \text{id}_I$.
2. $-w_X(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in X$.
3. If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho^\vee_X) \in \mathbb{Z}$.

- **Induced map** $\Theta : \mathfrak{h}^* \to \mathfrak{h}^*$ defined by $\Theta = -w_X \circ \tau$;
- Fixed Lie subalgebra $\mathfrak{k} = \{x \in g \mid \theta(x) = x\}$ - form **quantum analogue** of $U(\mathfrak{k})$;
Quantum symmetric pairs

- $\mathcal{M}_X$ - subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in X\}$;
- $U_0^\Theta$ - subalgebra generated by $\{K_\lambda \mid \lambda \in Q, \Theta(\lambda) = \lambda\}$;
- $\mathbf{c} = (c_i)_{i \in I \setminus X}, \mathbf{s} = (s_i)_{i \in I \setminus X}$ - parameters with added constraints;
- $s : I \rightarrow \mathbb{C}^\times$ - function depending on $X$ and $\tau$. 

Definition (Letzter '99)

Let $(I, X, \tau)$ be a Satake diagram. The quantum symmetric pair coideal subalgebra $B_{\mathbf{c}, \mathbf{s}}$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\mathcal{M}_X, U_0^\Theta$ and elements $B_i = F_i - c_i s(\tau(i)) T \mathcal{W} X (E_{\tau(i)}) K_{-1}^i + s_i K_{-1}^i$ for $i \in I \setminus X$.

Key property: $\Delta(B_{\mathbf{c}, \mathbf{s}}) \subseteq B_{\mathbf{c}, \mathbf{s}} \otimes U_q(\mathfrak{g})$. 

Liam Dobson

Factorisation of quasi K-matrices for quantum symmetric pairs
Quantum symmetric pairs

- $\mathcal{M}_X$ - subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in X\}$;
- $U_\Theta^0$ - subalgebra generated by $\{K_\lambda \mid \lambda \in Q, \Theta(\lambda) = \lambda\}$;
- $c = (c_i)_{i \in I \setminus X}, s = (s_i)_{i \in I \setminus X}$ - parameters with added constraints;
- $s : I \to \mathbb{C}^\times$ - function depending on $X$ and $\tau$.

Definition (Letzter '99)

Let $(I, X, \tau)$ be a Satake diagram. The quantum symmetric pair coideal subalgebra $B_{c,s}$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\mathcal{M}_X$, $U_\Theta^0$ and elements

$$B_i = F_i - c_i s(\tau(i)) T_{wx}(E_{\tau(i)}) K_i^{-1} + s_i K_i^{-1} \quad \text{for } i \in I \setminus X.$$
Quantum symmetric pairs

- $\mathcal{M}_X$ - subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in X\}$;
- $U^0_\Theta$ - subalgebra generated by $\{K_\lambda \mid \lambda \in Q, \Theta(\lambda) = \lambda\}$;
- $c = (c_i)_{i \in I \setminus X}, s = (s_i)_{i \in I \setminus X}$ - parameters with added constraints;
- $s : I \to \mathbb{C}^\times$ - function depending on $X$ and $\tau$.

**Definition (Letzter '99)**

Let $(I, X, \tau)$ be a Satake diagram. The quantum symmetric pair coideal subalgebra $B_{c,s}$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\mathcal{M}_X$, $U^0_\Theta$ and elements

$$B_i = F_i - c_i s(\tau(i)) T_{wx}(E_{\tau(i)}) K_i^{-1} + s_i K_i^{-1} \quad \text{for} \ i \in I \setminus X.$$

- **Key property**: $\Delta(B_{c,s}) \subseteq B_{c,s} \otimes U_q(\mathfrak{g})$. 

Liam Dobson

Factorisation of quasi K-matrices for quantum symmetric pairs
An element $R$ that intertwines two bar involutions on $\Delta(U_q(g))$:

- **Bar involution** $\overline{-}^U : U_q(g) \rightarrow U_q(g)$

  $E_i^U = E_i$, $F_i^U = F_i$, $K_i^U = K_i^{-1}$, $q^U = q^{-1}$;
An element $R$ that intertwines two bar involutions on $\Delta(U_q(\mathfrak{g}))$:

- **Bar involution** $\overline{\cdot}^U : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$

  \[
  \overline{E}^U_i = E_i, \quad \overline{F}^U_i = F_i, \quad \overline{K}^U_i = K_i^{-1}, \quad \overline{q}^U = q^{-1};
  \]

**Theorem (Lusztig)**

There is a uniquely determined element $R = \sum_{\mu \in Q^+} R_{\mu}$ with $R_{\mu} \in U^-_{-\mu} \otimes U^+_{\mu}$ and $R_0 = 1 \otimes 1$ such that

\[
\Delta(\overline{u}^U)R = R\Delta(u)^{U \otimes U} \quad \text{for all } u \in U_q(\mathfrak{g}).
\]
Quasi $R$-matrices

An element $R$ that intertwines two bar involutions on $\Delta(U_q(\mathfrak{g}))$:

- **Bar involution** $\overline{\cdot}^U : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$

  $$
  \overline{E_i}^U = E_i, \quad \overline{F_i}^U = F_i, \quad \overline{K_i}^U = K_i^{-1}, \quad \overline{q}^U = q^{-1};
  $$

**Theorem (Lusztig)**

There is a uniquely determined element $R = \sum_{\mu \in Q^+} R_{\mu}$ with $R_{\mu} \in U^-_{-\mu} \otimes U^+_{\mu}$ and $R_0 = 1 \otimes 1$ such that

$$
\Delta(\overline{u}^U)R = R\Delta(u)^{U \otimes U}
$$

for all $u \in U_q(\mathfrak{g})$.

**Applications:**

- Canonical bases;
- Universal $R$-matrix.
An element $\hat{X}$ that intertwines between two bar involutions on $B_{c,s}$:

- **Bar involution** $\bar{\phantom{B}}^B : B_{c,s} \rightarrow B_{c,s}$.

[Bao, Wang '13], [Ehrig, Stroppel '13], [Balagović, Kolb '16]

\[
\bar{\phantom{B}}^B \big|_{\mathcal{M}_X u^0_\Theta} = \_, \quad \bar{B}^B_i = B_i;
\]
An element $\mathcal{X}$ that intertwines between two bar involutions on $B_{c,s}$:

- **Bar involution** $\overline{\phantom{B}}^B : B_{c,s} \rightarrow B_{c,s}$.

[13x128]Bao, Wang '13], [Ehrig, Stroppel '13], [Balagović, Kolb '16]

\[
\overline{\phantom{B}}^B \mid_{\mathcal{M}_X U_0^0} = \overline{\phantom{U}}^U \mid_{\mathcal{M}_X U_0^0}, \quad \overline{B}_i^B = B_i;
\]

**Theorem (Bao, Wang '13; Balagović, Kolb '16)**

There is a uniquely determined element $\mathcal{X} = \sum_{\mu \in \mathbb{Q}^+} \mathcal{X}_\mu$ with $\mathcal{X}_\mu \in U_\mu^+$ and $\mathcal{X}_0 = 1$ such that

\[
\overline{\phantom{B}}^U \mathcal{X} = \mathcal{X} \overline{\phantom{B}}^B \quad \text{for all } b \in B_{c,s}.
\]
An element $\mathcal{X}$ that intertwines between two bar involutions on $B_{c,s}$:

- **Bar involution** $\overline{\cdot}^B : B_{c,s} \to B_{c,s}$.

[Bao, Wang '13], [Ehrig, Stroppel '13], [Balagović, Kolb '16]

$$\overline{\cdot}^B \left|_{\mathcal{M}_X U_0^\Theta} = \overline{\cdot}^U \left|_{\mathcal{M}_X U_0^\Theta}, \quad \overline{B}_i^B = B_i;$$

**Theorem (Bao, Wang '13; Balagović, Kolb '16)**

There is a uniquely determined element $\mathcal{X} = \sum_{\mu \in Q^+} \mathcal{X}_\mu$ with $\mathcal{X}_\mu \in U^+_\mu$ and $\mathcal{X}_0 = 1$ such that

$$\overline{b}^U \mathcal{X} = \mathcal{X} \overline{b}^B \quad \text{for all } b \in B_{c,s}.$$ 

**Applications:**

- Canonical bases [Bao, Wang '13, '16];
- Universal $K$-matrix [Balagović, Kolb '16]
Recursive definition of $\mathcal{X}$

From now on, $s = (0, 0, \ldots, 0)$.

- Linear map $i\tau : U^+ \rightarrow U^+$:

$$
i\tau(E_j) = \delta_{ij}, \quad i\tau(xy) = i\tau(x)y + q^{(\alpha_i, \mu)} x \cdot i\tau(y) \text{ for } x \in U_\mu^+;
$$
Recursive definition of $\mathcal{X}$

From now on, $s = (0, 0, \ldots, 0)$.

- Linear map $i_r : U^+ \to U^+$:
  \[
i_r(E_j) = \delta_{ij}, \quad i_r(xy) = i_r(x)y + q^{(\alpha_i, \mu)}x \cdot i_r(y) \quad \text{for} \ x \in U^+;\]

- $q_i = q^{(\alpha_i, \alpha_i)}/2$;

- Consequence of intertwiner relation $\overline{b}^U \mathcal{X} = \mathcal{X} \overline{b}^B$:
  \[
i_r(\mathcal{X}_\mu) = (q_i - q_i)^{-1} q^{-(\Theta(\alpha_i, \alpha_i))} c_i s(\tau(i)) T_{w_\chi}(E_{\tau(i)}) \mathcal{X}_{\mu + \Theta(\alpha_i) - \alpha_i}.\]
Recursive definition of $X$

From now on, $s = (0, 0, \ldots, 0)$.

- Linear map $i_r : U^+ \rightarrow U^+$:
  \[
  i_r(E_j) = \delta_{ij}, \quad i_r(xy) = i_r(x)y + q^{(\alpha_i, \mu)}x \cdot i_r(y) \quad \text{for} \ x \in U^+_\mu;
  \]

- $q_i = q^{(\alpha_i, \alpha_i)}/2$;

- Consequence of intertwiner relation $b^U X = X b^B$:
  \[
  i_r(X_\mu) = (q_i - q_i)^{-1} q^{-(\Theta(\alpha_i), \alpha_i)} c_i s(\tau(i)) T_{wx}(E_{\tau(i)}) X_{\mu + \Theta(\alpha_i) - \alpha_i}.
  \]

More compact form

\[
  i_r(X) = (q_i - q_i)^{-1} q^{-(\Theta(\alpha_i), \alpha_i)} c_i s(\tau(i)) T_{wx}(E_{\tau(i)}) X.
  \]
Idea: Build $R$ from quasi $R$-matrices for $U_q(sl_2)$.

Let $\{r\}_i = 1 + q_i^2 + \cdots + q_i^{2(r-1)}$.

Quasi $R$-matrix corresponding to $i \in I$ is given by

$$R_i = \sum_{r \geq 0} (-1)^r \frac{(q_i - q_i^{-1})^r}{\{r\}_i!} F_i^r \otimes E_i^r.$$
**Idea:** Build $R$ from quasi $R$-matrices for $U_q(\mathfrak{sl}_2)$.

Let $\{r\}_i = 1 + q_i^2 + \cdots + q_i^{2(r-1)}$.

Quasi $R$-matrix corresponding to $i \in \mathcal{I}$ is given by

$$R_i = \sum_{r \geq 0} (-1)^r \frac{(q_i - q_i^{-1})^r}{\{r\}_i!} F_i^r \otimes E_i^r.$$ 

**Theorem (Levandorskii, Soibelman ’90; Kirillov, Reshetikhin ’90)**

For any reduced expression $w_0 = s_{i_1} \cdots s_{i_t}$ of the longest element $w_0 \in \mathcal{W}$ define

$$R^{[j]} = (T_{i_1} \cdots T_{i_{j-1}} \otimes T_{i_1} \cdots T_{i_{j-1}})(R_{ij}) \quad \text{for } j = 1, \ldots, t.$$ 

The quasi $R$-matrix for $U_q(\mathfrak{g})$ can be written as

$$R = R^{[t]} \cdot R^{[t-1]} \cdots R^{[2]} \cdot R^{[1]}.$$
Subgroup \( \tilde{W} \) generated by \( \tilde{s}_i = w_X w_{\{i, \tau(i)\} \cup X} \) for \( i \in I \setminus X \);
Subgroup $\tilde{W}$ generated by $\tilde{s}_i = w_X w_{\{i, \tau(i)\}} w_X$ for $i \in I \setminus X$;

**Theorem (Lusztig)**

$\tilde{W}$ is a Coxeter group.
Subgroup $\tilde{\mathcal{W}}$ generated by $\tilde{s}_i = w_{X} w_{\{i, \tau(i)\} \cup X}$ for $i \in I \setminus X$;

**Theorem (Lusztig)**

$\tilde{\mathcal{W}}$ is a Coxeter group.

Interpret $\tilde{\mathcal{W}}$ as Weyl group of restricted root system $\Sigma$:
- Reduced expressions in $\tilde{\mathcal{W}}$ are reduced in $\mathcal{W}$;
- Set of simple roots $\tilde{\Pi} = \{\tilde{\alpha}_i \mid i \in I \setminus X\}$ where $\tilde{\alpha}_i = \frac{\alpha_i - \Theta(\alpha_i)}{2}$. 
Idea: Take subdiagrams of $(I, X, \tau)$ corresponding to subsets of $\tau$-orbits.
Idea: Take subdiagrams of \((I, X, \tau)\) corresponding to subsets of \(\tau\)-orbits.

- rank of \((I, X, \tau)\) = number of distinct \(\tau\)-orbits of white nodes.
**Idea:** Take subdiagrams of \((I, X, \tau)\) corresponding to subsets of \(\tau\)-orbits.

- **rank** of \((I, X, \tau)\) = number of distinct \(\tau\)-orbits of white nodes.

Let \(\mathcal{X}_i\) denote quasi \(K\)-matrix for rank one Satake diagram corresponding to \(\tau\)-orbit \(\{i, \tau(i)\}\).
### Rank one Satake diagrams

<table>
<thead>
<tr>
<th>Type</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AI_1$</td>
<td><img src="AI1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$AII_3$</td>
<td><img src="AII3.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$AIII_{11}$</td>
<td><img src="AI11.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$AIV$, $n \geq 2$</td>
<td><img src="AIV.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BII$, $n \geq 2$</td>
<td><img src="BII.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$CII$, $n \geq 3$</td>
<td><img src="CII.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$DII$, $n \geq 4$</td>
<td><img src="DII.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$FII$</td>
<td><img src="FII.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

---

**Liam Dobson**

Factorisation of quasi K-matrices for quantum symmetric pairs
$x = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n E_1^{2n}.$
Rank one examples

\[ x = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n E_1^{2n}. \]

\[ x = \sum_{n \geq 0} \frac{(qc_2)^n}{\{n\}!} [E_2, T_{13}(E_2)]_q^{n-2}. \]

\[ [a, b]_c = ab - cba. \]
\[ \mathcal{X} = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} c_1^n (E_1 E_2)^n. \]
\[ x = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{n\}!} c_1^n (E_1 E_2)^n. \]
Partial quasi $K$-matrices

- Representation of $Br(\tilde{W})$ on $U_q(\mathfrak{g})$ given by automorphisms
  
  $\tilde{T}_i := T_{\tilde{s}_i}$

Main point: $\Psi \circ \tilde{T}_i \circ \Psi^{-1}$:
Partial quasi $K$-matrices

- Representation of $Br(\tilde{W})$ on $U_q(\mathfrak{g})$ given by automorphisms
  \[ \tilde{T}_i := T_{\tilde{s}_i}; \]

- $\tilde{U}^+ = \bigoplus_{\mu \in Q^+(2\Sigma)} U^+_{\mu}; \quad \tilde{U}^+[w] = U^+[w] \cap \tilde{U}^+; \]

- $\tilde{c}_i^2 = c_i c_{\tau(i)} s(i) s(\tau(i))$ for $i \in I \setminus X$;

- Algebra automorphism $\Psi : \tilde{U}^+ \to \tilde{U}^+$,
  \[ \Psi(E) = q^{(\bar{\alpha}_i, \bar{\alpha}_i)} \tilde{c}_i E \quad \text{for} \quad E \in U^+_{2\bar{\alpha}_i}. \]
Partial quasi $K$-matrices

- Representation of $Br(\tilde{W})$ on $U_q(\mathfrak{g})$ given by automorphisms
  \[ \tilde{T}_i := T_{\tilde{s}_i}; \]

- $\tilde{U}^+ = \bigoplus_{\mu \in Q^+(2\Sigma)} U^+_{\mu}; \quad \tilde{U}^+[w] = U^+[w] \cap \tilde{U}^+$;

- $\tilde{c}_i^2 = c_i c_{\tau(i)} s(i) s(\tau(i))$ for $i \in I \setminus X$;

- Algebra automorphism $\Psi : \tilde{U}^+ \to \tilde{U}^+$,
  \[ \Psi(E) = q^{(\alpha_i, \alpha_i)} \tilde{c}_i E \quad \text{for } E \in U^+_{2\alpha_i}. \]

Main point: $\Psi \circ \tilde{T}_i \circ \Psi^{-1} : \tilde{U}^+ [\tilde{s}_i w_0] \to \tilde{U}^+$. 
Partial quasi $K$-matrices

- Representation of $Br(\tilde{W})$ on $U_q(\mathfrak{g})$ given by automorphisms
  \[ \tilde{T}_i := T_{\tilde{s}_i}; \]

- $\tilde{U}^+ = \bigoplus_{\mu \in Q^+(2\Sigma)} U^+_\mu$; \[ \tilde{U}^+[w] = U^+[w] \cap \tilde{U}^+; \]

- $\tilde{c}_i^{-2} = c_i c_{\tau(i)} s(i) s(\tau(i))$ for $i \in I \setminus X$;

- Algebra automorphism $\Psi : \tilde{U}^+ \rightarrow \tilde{U}^+$,
  \[ \Psi(E) = q^{(\tilde{\alpha}_i, \tilde{\alpha}_i)} \tilde{c}_i E \quad \text{for } E \in U^+_{2\tilde{\alpha}_i}. \]

Main point: $\Psi \circ \tilde{T}_i \circ \Psi^{-1} : \tilde{U}^+[\tilde{s}_i w_0] \rightarrow \tilde{U}^+$.

Take $\tilde{w} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_t}$ reduced. For $k = 1, \ldots, t$ define

\[ x_w^{[k]} = \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} \circ \Psi^{-1}(x_{i_k}). \]
Take $\tilde{w} = s_{i_1} \cdots s_{i_t}$ reduced. For $k = 1, \ldots t$ define

$$\mathcal{X}^{[k]}_{\tilde{w}} = \Psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} \circ \Psi^{-1}(\mathcal{X}_{i_k}).$$
Partial quasi $K$-matrices

Take $\tilde{w} = s_{i_1} \cdots s_{i_t}$ reduced. For $k = 1, \ldots, t$ define

$$
\mathcal{X}_\tilde{w}^{[k]} = \psi \circ \tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}} \circ \psi^{-1}(\mathcal{X}_{i_k}).
$$

**Definition**

Let $\tilde{w} \in \tilde{W}$ and $\tilde{w} = s_{i_1} s_{i_2} \cdots s_{i_t}$ be a reduced expression. The **partial quasi $K$-matrix** $\mathcal{X}_\tilde{w}$ associated to $\tilde{w}$ and the given reduced expression is defined by

$$
\mathcal{X}_\tilde{w} = \mathcal{X}_\tilde{w}^{[t]} \mathcal{X}_\tilde{w}^{[t-1]} \cdots \mathcal{X}_\tilde{w}^{[2]} \mathcal{X}_\tilde{w}^{[1]}.
$$
Main result

**Theorem (D, Kolb ’17)**

Let \((I, X, \tau)\) be a Satake diagram such that \(g = \mathfrak{sl}_n\) or \(X = \emptyset\). Then

1. \(\mathfrak{X}_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}\).
Main result

Theorem (D, Kolb '17)

Let \((I, X, \tau)\) be a Satake diagram such that \(g = sl_n\) or \(X = \emptyset\). Then

1. \(X_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(X_{\overline{w_0}} = X\).

Conjecture (D, Kolb '17)

The above Theorem holds for any Satake diagram \((I, X, \tau)\).
Outline of proof

Theorem (D, Kolb ’17)

Let \((I, X, \tau)\) be a Satake diagram such that \(g = \mathfrak{sl}_n\) or \(X = \emptyset\). Then

\begin{enumerate}
\item \(\mathfrak{X}_{\widetilde{w}}\) depends only on \(\widetilde{w}\) and not on the chosen reduced expression.
\item \(\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}\).
\end{enumerate}
Outline of proof

**Theorem (D, Kolb ’17)**

Let \((I, X, \tau)\) be a Satake diagram such that \(g = \mathfrak{sl}_n\) or \(X = \emptyset\). Then

1. \(X_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(X_{\tilde{w}_0} = \mathcal{X}\).

(1) Calculate \(\mathcal{X}\) for Satake diagrams of rank one.
Theorem (D, Kolb ’17)

Let \((I, X, \tau)\) be a Satake diagram such that \(g = \mathfrak{sl}_n\) or \(X = \emptyset\). Then

1. \(\mathcal{X}_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(\mathcal{X}_{\tilde{w}_0} = \mathcal{X}\).

(1) Calculate \(\mathcal{X}\) for Satake diagrams of rank one.
(2) Verify Theorem in rank two.
Theorem (D, Kolb '17)

Let \((I, X, \tau)\) be a Satake diagram such that \(g = sl_n\) or \(X = \emptyset\). Then

1. \(\mathcal{X}_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(\mathcal{X}_{\tilde{w}_0} = \mathcal{X}\).

(1) Calculate \(\mathcal{X}\) for Satake diagrams of rank one.
(2) Verify Theorem in rank two.
(3) Use braid relations for \(\tilde{T}_i\) to show 1.
Theorem (D, Kolb ’17)

Let \((I, X, \tau)\) be a Satake diagram such that \(g = sl_n\) or \(X = \emptyset\). Then

1. \(\tilde{X}_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(\tilde{X}_{\tilde{w}_0} = X\).

(1) Calculate \(X\) for Satake diagrams of rank one.
(2) Verify Theorem in rank two.
(3) Use braid relations for \(\tilde{T}_i\) to show 1.
(4) Show that 
\[
ir(\chi_{\tilde{w}_0}) = (q - q^{-1})q^{-(\alpha_i, \Theta(\alpha_i))}c_i s(\tau(i))T_{w_\chi}(E_{\tau(i)})X_{\chi_{\tilde{w}_0}}
\]
for all \(i \in I \setminus X\). This implies 2.

- Fix \(i \in I \setminus X\).
- Choose a 'nice' reduced expression for \(\tilde{w}_0\).
- \(ir\) only acts on the first factor.
Outline of proof

Theorem (D, Kolb ’17)

Let \((I, X, \tau)\) be a Satake diagram such that \(g = \mathfrak{sl}_n\) or \(X = \emptyset\). Then

1. \(\mathfrak{X}_{\tilde{w}}\) depends only on \(\tilde{w}\) and not on the chosen reduced expression.
2. \(\mathfrak{X}_{\tilde{w}_0} = \mathfrak{X}\).

1. Calculate \(\mathfrak{X}\) for Satake diagrams of rank one.
2. Verify Theorem in rank two.
   **Conjecture that this holds for all rank two.**
3. Use braid relations for \(\tilde{T}_i\) to show 1.
4. Show that \(i_r(\mathfrak{X}_{\tilde{w}_0}) = (q - q^{-1})q^{-(\alpha_i, \Theta(\alpha_i))}c_is(\tau(i)) T_{w_X}(E_{\tau(i)}) \mathfrak{X}_{\tilde{w}_0}\) for all \(i \in I \setminus X\). This implies 2.
   - Fix \(i \in I \setminus X\).
   - Choose a 'nice' reduced expression for \(\tilde{w}_0\).
   - \(i_r\) only acts on the first factor.
### Rank two Satake diagrams

<table>
<thead>
<tr>
<th>AI₂</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>CII₄</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AII₅</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>DIₙ, n ≥ 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AIII₃</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>DIII₄</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AIIIₙ, n ≥ 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>DIII₅</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(BC)₂</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>EIII</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BIₙ, n ≥ 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>EIV</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CIIₙ, n ≥ 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>G</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Liam Dobson  | Factorisation of quasi K-matrices for quantum symmetric pairs
Let $X_{c,s}$ denote quasi $K$-matrix with dependence on parameters $c, s$.

**Idea:** Use result for $X_{c,0}$ to find $X_{c,s}$. 
Let $\mathcal{X}_{c,s}$ denote quasi $K$-matrix with dependence on parameters $c, s$.

**Idea:** Use result for $\mathcal{X}_{c,0}$ to find $\mathcal{X}_{c,s}$.

- **Algebra isomorphism** $\phi_s : B_{c,0} \to B_{c,s}$:
  
  $$
  \phi_s(B_i) = B_i, \quad \phi_s|_{\mathcal{M}_X U_0} = \text{id}|_{\mathcal{M}_X U_0}.
  $$

- **One dim representation** $\chi_s = \epsilon \circ \phi_s : B_{c,0} \to \mathbb{K}(q)$;

- **Quasi R-matrix** $R^\theta_{c,s}$ for $B_{c,s}$ [Bao, Wang ’13], [Kolb ’17]:
  
  $$
  R^\theta_{c,s} = \Delta(\mathcal{X}_{c,s}) \cdot R \cdot (\mathcal{X}_{c,s}^{-1} \otimes 1)
  $$
Let $\mathcal{X}_{c,s}$ denote quasi $K$-matrix with dependence on parameters $c, s$.

**Idea:** Use result for $\mathcal{X}_{c,0}$ to find $\mathcal{X}_{c,s}$.

- Algebra isomorphism $\phi_s : B_{c,0} \to B_{c,s}$:
  
  $\phi_s(B_i) = B_i$, $\phi_s|_{\mathcal{M}_x U^0} = \text{id}|_{\mathcal{M}_x U^0}$.

- One dim representation $\chi_s = \epsilon \circ \phi_s : B_{c,0} \to \mathbb{K}(q)$;

- Quasi $R$-matrix $R^\theta_{c,s}$ for $B_{c,s}$ [Bao, Wang '13], [Kolb '17]:
  
  $$R^\theta_{c,s} = \Delta(\mathcal{X}_{c,s}) \cdot R \cdot (\mathcal{X}_{c,s}^{-1} \otimes 1)$$

**Theorem (D, Kolb '17)**

We have $\mathcal{X}_{c,s} = (\chi_s \otimes \text{id})(R^\theta_{c,0})$. 