Factorisation of quasi K-matrices for quantum symmetric pairs

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Outline

Quantized enveloping algebra $U_q(\mathfrak{g})$	QSP coideal subalgebra $B_{\mathbf{c},\mathbf{s}}$
Universal R-matrix $\mathcal{R}$	Universal K-matrix $\mathcal{K}$
$\mathcal{R} = R \circ \kappa \circ \mathrm{flip}$	$\mathcal{K} = \mathfrak{X} \circ \xi \circ T_{w_X}^{-1} T_{w_0}^{-1}$
Quasi R-matrix R	Quasi K-matrix $\mathfrak{X}$
Factorisation using R for $U_q(\mathfrak{sl}_2)$	?

Quantised enveloping algebras

- $\mathfrak{g}$ - fin. dim. semisimple Lie algebra over $\mathbb{C}$;
- I - indexing set for nodes of Dynkin diagram;
- $\Pi = \{\alpha_i \mid i \in I\}$ simple roots; W - Weyl group of $\mathfrak{g}$;
- q - indeterminate; $\mathbb{K}$ - field of char 0;
- $\mathbb{K}(q)$ - field of rational functions in q with coeffs in $\mathbb{K}$.

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$$U_q(\mathfrak{g}) = {}^{\mathbb{K}(q)\langle E_i, F_i, K_i^{\pm 1} \mid i \in I\rangle}\big/_{\text{relations}}$$

- $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ comultiplication;

$$\begin{aligned}
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i,\\
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i,\\
\Delta(K_i) &= K_i \otimes K_i.
\end{aligned}$$

Properties

$U_q(\mathfrak{g})$ has the following properties/notation:

- Triangular decomposition $U_q(\mathfrak{g}) = U^+ \otimes U^0 \otimes U^-$;
- Vector space basis for U^0: $K_\lambda = \prod_{i \in I} K_i^{n_i}$ for $\lambda = \sum_{i \in I} n_i \alpha_i \in Q$;
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- Algebra gradings $U^+ = \bigoplus_{\mu \in Q^+} U^+_\mu$, $U^- = \bigoplus_{\mu\in Q^+} U^-_{-\mu}$;
- Action of $Br(\mathfrak{g})$ on $U_q(\mathfrak{g})$ by algebra automorphisms T_i:

$$\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}$$

- For $w = s_{i_1} \cdots s_{i_t}$ reduced, define $T_w = T_{i_1} \cdots T_{i_t}$.

Quantum symmetric pairs

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism;

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Definition

Let $X \subset I$ and $\tau : I \to I$ a diagram automorphism satisfying $\tau(X) = X$. The triple (I, X, τ) is called a **Satake diagram** if:

1. $\tau^2 = \mathrm{id}_I$.
2. $-w_X(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in X$.
3. If $j \in I \setminus X$ and $\tau(j) = j$, then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$.

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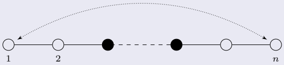

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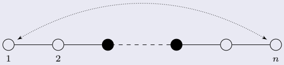

- **Induced map** $\Theta : \mathfrak{h}^* \to \mathfrak{h}^*$ defined by $\Theta = -w_X \circ \tau$;
- Fixed Lie subalgebra $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$ - form **quantum analogue** of $U(\mathfrak{k})$;

Quantum symmetric pairs

- $\mathcal{M}_X$ - subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in X\}$;
- U_Θ^0 - subalgebra generated by $\{K_\lambda \mid \lambda \in Q, \Theta(\lambda) = \lambda\}$;
- $\mathbf{c} = (c_i)_{i \in I \setminus X}, \mathbf{s} = (s_i)_{i \in I \setminus X}$ - parameters with added constraints;
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Definition (Letzter '99)

Let (I, X, τ) be a Satake diagram. The **quantum symmetric pair coideal subalgebra** $B_{\mathbf{c},\mathbf{s}}$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $\mathcal{M}_X$, U_Θ^0 and elements

$$B_i = F_i - c_i s(\tau(i)) T_{w_X}(E_{\tau(i)}) K_i^{-1} + s_i K_i^{-1} \quad \text{for } i \in I \setminus X.$$

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- **Key property**: $\Delta(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}} \otimes U_q(\mathfrak{g})$.

Quasi R-matrices

An element R that intertwines two bar involutions on $\Delta(U_q(\mathfrak{g}))$:

- **Bar involution** $\overline{}^{U} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$

$$\overline{E_i}^U = E_i, \quad \overline{F_i}^U = F_i, \quad \overline{K_i}^U = K_i^{-1}, \quad \overline{q}^U = q^{-1};$$

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Theorem (Lusztig)

There is a uniquely determined element $R = \sum_{\mu \in Q^+} R_\mu$ with $R_\mu \in U^-_{-\mu} \otimes U^+_\mu$ and $R_0 = 1 \otimes 1$ such that

$$\Delta(\overline{u}^U)R = R\overline{\Delta(u)}^{U \otimes U} \quad \text{for all } u \in U_q(\mathfrak{g}).$$

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Applications:

- Canonical bases;
- Universal R-matrix.

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  [Bao, Wang '13], [Ehrig, Stroppel '13], [Balagović, Kolb '16]

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- Canonical bases [Bao, Wang '13, '16];
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Recursive definition of $\mathfrak{X}$

From now on, $\mathbf{s} = (0, 0, \ldots, 0)$.

- Linear map ${}_ir : U^+ \to U^+$:

$$ {}_ir(E_j) = \delta_{ij}, \quad {}_ir(xy) = {}_ir(x)y + q^{(\alpha_i,\mu)}x\, {}_ir(y) \text{ for } x \in U^+_\mu; $$

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- $q_i = q^{(\alpha_i,\alpha_i)/2}$;
- Consequence of intertwiner relation $\overline{b}^U \mathfrak{X} = \mathfrak{X}\overline{b}^B$:

$$ {}_ir(\mathfrak{X}_\mu) = (q_i - q_i)^{-1} q^{-(\Theta(\alpha_i),\alpha_i)} c_i s(\tau(i)) T_{w_X}(E_{\tau(i)})\mathfrak{X}_{\mu+\Theta(\alpha_i)-\alpha_i}. $$

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More compact form

$$ {}_ir(\mathfrak{X}) = (q_i - q_i)^{-1} q^{-(\Theta(\alpha_i),\alpha_i)} c_i s(\tau(i)) T_{w_X}(E_{\tau(i)}) \mathfrak{X}. $$

Factorisation of quasi R-matrices

Idea: Build R from quasi R-matrices for $U_q(\mathfrak{sl}_2)$.
Let $\{r\}_i = 1 + q_i^2 + \cdots + q_i^{2(r-1)}$.
Quasi R-matrix corresponding to $i \in I$ is given by

$$R_i = \sum_{r \geq 0} (-1)^r \frac{(q_i - q_i^{-1})^r}{\{r\}_i!} F_i^r \otimes E_i^r.$$

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Theorem (Levandorskii, Soibelman '90; Kirillov, Reshetikhin '90)

For any reduced expression $w_0 = s_{i_1} \cdots s_{i_t}$ of the longest element $w_0 \in W$ define

$$R^{[j]} = (T_{i_1} \cdots T_{i_{j-1}} \otimes T_{i_1} \cdots T_{i_{j-1}})(R_{i_j}) \quad \text{for } j = 1, \ldots, t.$$

The quasi R-matrix for $U_q(\mathfrak{g})$ can be written as

$$R = R^{[t]} \cdot R^{[t-1]} \ldots R^{[2]} \cdot R^{[1]}.$$

Weyl group for symmetric Lie algebra $(\mathfrak{g}, \theta)$

Subgroup $\widetilde{W}$ generated by $\widetilde{s}_i = w_X w_{\{i,\tau(i)\}\cup X}$ for $i \in I \setminus X$;

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Theorem (Lusztig)

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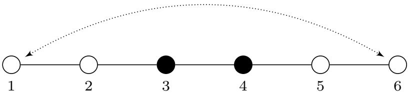
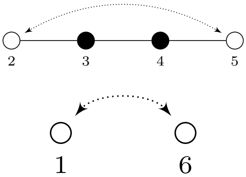
Interpret $\widetilde{W}$ as Weyl group of restricted root system Σ:

- Reduced expressions in $\widetilde{W}$ are reduced in W;
- Set of simple roots $\widetilde{\Pi} = \{\widetilde{\alpha}_i \mid i \in I \setminus X\}$ where $\widetilde{\alpha}_i = \dfrac{\alpha_i - \Theta(\alpha_i)}{2}$.

Satake subdiagrams

Idea: Take subdiagrams of (I, X, τ) corresponding to subsets of τ-orbits.





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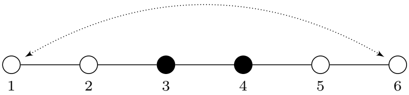

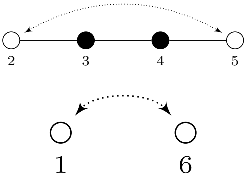

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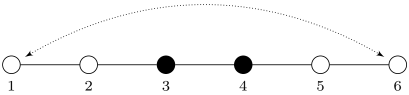

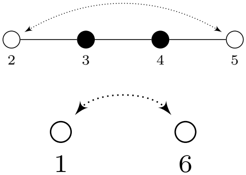

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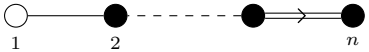
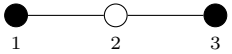
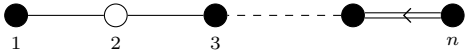
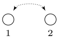
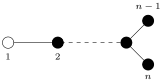
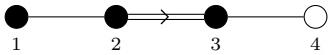
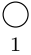
Let $\mathfrak{X}_i$ denote quasi K-matrix for rank one Satake diagram corresponding to τ-orbit $\{i, \tau(i)\}$.

Rank one Satake diagrams

AI_1		BII, $n \geq 2$	
AII_3		CII, $n \geq 3$	
$AIII_{11}$		DII, $n \geq 4$	
AIV, $n \geq 2$		FII	

Rank one examples



$$\mathfrak{X} = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{\{2n\}!!} (q^2 c_1)^n E_1^{2n}.$$

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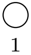

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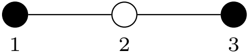
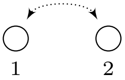


$$\mathfrak{X} = \sum_{n \geq 0} \frac{(qc_2)^n}{\{n\}!} [E_2, T_{13}(E_2)]_{q^{-2}}^n.$$

- $[a, b]_c = ab - cba$.

Rank one examples



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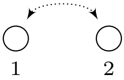

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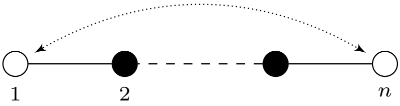

$$\mathfrak{X} = \left(\sum_{k\geq 0} \frac{(c_1 s(n))^k}{\{k\}!} T_1 T_{w_X}(E_n)^k\right) \left(\sum_{k\geq 0} \frac{(c_n s(1))^k}{\{k\}!} T_n T_{w_X}(E_1)^k\right).$$

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Take $\widetilde{w} = \widetilde{s}_{i_1} \cdots \widetilde{s}_{i_t}$ reduced. For $k = 1, \ldots t$ define

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Definition

Let $\widetilde{w} \in \widetilde{W}$ and $\widetilde{w} = \widetilde{s_{i_1}}\widetilde{s_{i_2}} \cdots \widetilde{s_{i_t}}$ be a reduced expression. The **partial quasi K-matrix** $\mathfrak{X}_{\widetilde{w}}$ associated to $\widetilde{w}$ and the given reduced expression is defined by

$$\mathfrak{X}_{\widetilde{w}} = \mathfrak{X}_{\widetilde{w}}^{[t]}\mathfrak{X}_{\widetilde{w}}^{[t-1]} \ldots \mathfrak{X}_{\widetilde{w}}^{[2]}\mathfrak{X}_{\widetilde{w}}^{[1]}.$$

Main result

Theorem (D, Kolb '17)

Let (I, X, τ) be a Satake diagram such that $\mathfrak{g} = \mathfrak{sl}_n$ or $X = \emptyset$. Then

1. $\mathfrak{X}_{\widetilde{w}}$ depends only on $\widetilde{w}$ and not on the chosen reduced expression.
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Conjecture (D, Kolb '17)

The above Theorem holds for any Satake diagram (I, X, τ).

Outline of proof

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- Fix $i \in I \setminus X$.
- Choose a 'nice' reduced expression for $\widetilde{w_0}$.
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Conjecture that this holds for all rank two.

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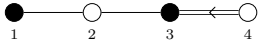
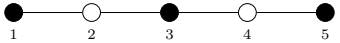
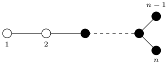
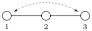
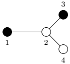
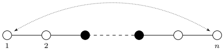
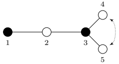
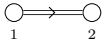
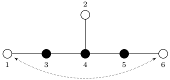
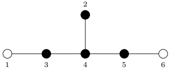
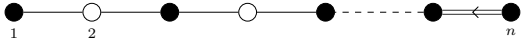
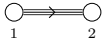
Rank two Satake diagrams

AI_2		CII_4	
AII_5		DI_n, $n \geq 5$	
$AIII_3$		$DIII_4$	
$AIII_n$, $n \geq 4$		$DIII_5$	
$(BC)_2$		$EIII$	
BI_n, $n \geq 3$		EIV	
CII_n, $n \geq 5$		G	

General parameters

Let $\mathfrak{X}_{\mathbf{c},\mathbf{s}}$ denote quasi K-matrix with dependence on parameters $\mathbf{c}, \mathbf{s}$.
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- Algebra isomorphism $\phi_{\mathbf{s}} : B_{\mathbf{c},\mathbf{0}} \to B_{\mathbf{c},\mathbf{s}}$;

$$\phi_{\mathbf{s}}(B_i) = B_i, \quad \phi_{\mathbf{s}}|_{\mathcal{M}_X U^0_\Theta} = \mathrm{id}|_{\mathcal{M}_X U^0_\Theta}.$$

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We have $\mathfrak{X}_{\mathbf{c},\mathbf{s}} = (\chi_{\mathbf{s}} \otimes \mathrm{id})(R^\theta_{\mathbf{c},\mathbf{0}})$.