Quantum groups, quantum flag manifolds and quantum symmetric spaces

(q-stories in rank 1)

Kenny De Commer
VUB (Brussels, Belgium)

June 11th, 2018
Quantum homogeneous spaces
ICMS, Edinburgh

Disclaimer

We will present a short survey on quantum homogeneous spaces for quantized compact semisimple Lie groups, focusing attention on quantum flag manifolds and quantum symmetric spaces. We will give an overview of various approaches which have been used to clarify their structure, coming from pure algebra, operator algebras, non-commutative geometry, tensor categories and integrable systems. We end with some open problems.

Disclaimer

We will present ~~a~~ ~~short survey~~ ~~on~~ quantum homogeneous spaces for quantized compact semisimple Lie groups, focusing attention on quantum flag manifolds and quantum symmetric spaces. We will give an overview of various approaches which have been used to clarify their structure, coming from pure algebra, ~~operator algebras~~, ~~non-commutative geometry~~, ~~tensor categories~~ ~~and~~ ~~integrable systems~~. We end with some open problems.

- Only rank 1 case
- Low-brow approach

Closed Lie subgroups of $SL(2,\mathbb{C})$

For $\epsilon \in \mathbb{R}$, consider in $SL(2,\mathbb{C})$ the closed Lie subgroup G_ϵ preserving the hermitian form

$$|x|^2 + \epsilon|y|^2.$$

We have

$$\begin{aligned} G_\epsilon &= \{g \in SL(2,\mathbb{C}) \mid g^* \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\} \\ &= \{\begin{pmatrix} \alpha & -\epsilon\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 + \epsilon|\gamma|^2 = 1\}. \end{aligned}$$

In particular

$$G_+ = SU(2), \qquad G_0 = \widetilde{E}(2), \qquad G_- = SU(1,1).$$

Some classical homogeneous spaces

Restricted actions

We have 'dual' actions

$$G_\eta \backslash SL(2,\mathbb{C}) \curvearrowleft G_\epsilon \quad \leftrightarrow \quad G_\eta \curvearrowright SL(2,\mathbb{C})/G_\epsilon.$$

$$\text{Orbits}: \qquad G_\eta g G_\epsilon \curvearrowleft G_\epsilon \quad \leftrightarrow \quad G_\eta \curvearrowright G_\eta g G_\epsilon$$

We have

$$\begin{aligned} G_\eta \backslash SL(2,\mathbb{C}) \quad &\cong \quad P_\eta \quad := \quad \{g^* \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} g \mid g \in SL(2,\mathbb{C})\} \\ &\qquad\qquad\;\; = \quad \left\{ \begin{pmatrix} |a|^2 + \eta |c|^2 & \overline{a}b + \eta \overline{c} d \\ \overline{b}a + \eta c\overline{d} & |b|^2 + \eta |d|^2 \end{pmatrix} \right\}, \end{aligned}$$

$$SL(2,\mathbb{C})/G_\epsilon \quad \cong \quad Q_\epsilon \quad := \quad \{(g^*)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} g^{-1} \mid g \in SL(2,\mathbb{C})\}.$$

Some classical homogeneous spaces

Examples compact group action

P_+ is $SU(2)$-space of positive-definite matrices of determinant 1,

$$\text{Orbit space: } \mathbb{R}_+^\times/\mathrm{Sym}_2, \qquad \text{Orbits: } \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \mathbb{T}\backslash SU(2).$$

P_0 is $SU(2)$-space of positive-semi-definite matrices of rank 1,

$$\text{Orbit space: } \mathbb{R}_+^\times, \qquad \text{Orbits: } \mathbb{T}\backslash SU(2).$$

P_- is $SU(2)$-space of selfadjoint matrices of determinant -1,

$$\text{Orbit space: } \mathbb{R}_+^\times, \qquad \text{Orbits: } \mathbb{T}\backslash SU(2).$$

Some classical homogeneous spaces

Flag spaces and symmetric spaces

$SU(2)$-orbits $O_0(p)$ in P_0 are flag manifolds:

$$O_0(p) \subseteq P_0 \cong \widetilde{E}(2)\backslash SL(2,\mathbb{C}) \twoheadrightarrow B_-\backslash SL(2,\mathbb{C}) \cong \mathbb{C}P^1,$$

where B_- Borel subgroup $\{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^\times, c \in \mathbb{C}\}$.

$SU(2)$-orbit $O_- = O_-(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ in P_- is symmetric space:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O_- = \{\nu(U)^* U \mid U \in SU(2)\} \cong SU(2)^\nu \backslash SU(2),$$

where ν involution

$$\nu(u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{Fix points } SU(2)^\nu = \mathbb{T}.$$

Hopf $*$-algebras

Hopf $*$-algebras

Algebra A: 'quantum complex affine variety X', classically

$$X = \mathrm{Spec}_m(A)$$

$*$-algebra A: 'quantum real affine variety X', classically

$$X = \mathrm{Spec}^*_m(A) \subseteq X_{\mathbb{C}} = \mathrm{Spec}_m(A).$$

Hopf $*$-algebra (A, Δ): coassociative $*$-homomorphism

$$\Delta : A \to A \otimes A$$

with counit and antipode: 'quantum real affine group variety G'

Hopf $*$-algebras

Some examples

Recall: $G_\epsilon = \{\begin{pmatrix} \alpha & -\epsilon\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 + \epsilon|\gamma|^2 = 1\}$.

For $q > 0$ we define $\mathcal{O}_q(G_\epsilon)$ as $*$-algebra generated by α, γ with

$$\alpha\gamma = q\gamma\alpha \qquad \gamma^*\gamma = \gamma\gamma^*, \qquad \alpha\gamma^* = q\gamma^*\alpha,$$

$$\alpha\alpha^* + q^2\epsilon\gamma\gamma^* = 1 = \alpha^*\alpha + \epsilon\gamma^*\gamma$$

It becomes a Hopf $*$-algebra by asking that

$$X_\epsilon = \begin{pmatrix} \alpha & -q\epsilon\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is a corepresentation:

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}.$$

Hopf $*$-algebras

Embedding into quantum $SL(2,\mathbb{C})$

Question: $\exists$ quantum $SL(2,\mathbb{C})$ into which quantum G_ϵ embeds?

Answer: Yes, but for each ϵ separately:

$$\mathcal{O}_q^\epsilon(SL(2,\mathbb{C})) \twoheadrightarrow \mathcal{O}_q(G_\epsilon).$$

Question: $\exists$ left and right actions of quantum G_η and G_ϵ?

Answer: Yes, but for each η,ϵ separately:

$$\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \to \mathcal{O}_q(G_\eta) \otimes \mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})),$$

$$\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \to \mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \otimes \mathcal{O}_q(G_\epsilon).$$

Quantum $SL(2,\mathbb{C})$: holomorphic variables

Quantum $SL(2,\mathbb{C})$ as complex affine group:

Define $\mathcal{O}_q^{\mathbb{C}}(SL(2,\mathbb{C}))$ as algebra generated by a, b, c, d with

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc,$$

$$ad - qcb = 1 = da - q^{-1}cb.$$

Hopf algebra structure with fundamental corepresentation

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note: $R_{12}Y_{13}Y_{23} = Y_{23}Y_{13}R_{12}$ + quantum determinant 1.

Hopf $*$-algebras

Quantum $SL(2,\mathbb{C})$ as a real variety

For quantum $SL(2,\mathbb{C})$ as real affine variety, we need commutations between holomorphic and anti-holomorphic variables. We let them depend on η, ϵ:

Define $\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C}))$ as $*$-algebra generated by $\mathcal{O}_q^{\mathbb{C}}(SL(2,\mathbb{C}))$ and

1. $ad^* = d^*a,\ bc^* = c^*b,\ cc^* = c^*c,$
2. $ac^* = qc^*a,\ cd^* = q^{-1}d^*c,$
3. $aa^* = a^*a + \eta(1-q^2)c^*c,\ ab^* = q^{-1}b^*a + \eta(q^{-1}-q)d^*c,$
4. $dd^* = d^*d - \epsilon(1-q^2)c^*c,\ bd^* = qd^*b - \epsilon q(1-q^2)c^*a,$
5. $bb^* = b^*b + (1-q^2)(\eta d^*d - \epsilon a^*a) - \eta\epsilon(1-q^2)^2c^*c.$

Note: $Y_{23}^* R_{\eta,12} Y_{13} = Y_{13} R_{\epsilon,12} Y_{23}^*.$

Hopf $*$-algebras

Actions of quantum G_ϵ

With $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$ we get coassociative $*$-homomorphisms

$$\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \to \mathcal{O}_q^{\eta,\theta}(SL(2,\mathbb{C})) \otimes \mathcal{O}_q^{\theta,\epsilon}(SL(2,\mathbb{C})).$$

Moreover, Hopf $*$-algebra surjections

$$\pi_\epsilon : \mathcal{O}_q^{\epsilon,\epsilon}(SL(2,\mathbb{C})) \to \mathcal{O}_q(G_\epsilon), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -q\epsilon\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

Hence indeed

$$\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \to \mathcal{O}_q(G_\eta) \otimes \mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})),$$

$$\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \to \mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \otimes \mathcal{O}_q(G_\epsilon).$$

Remark: Semiclassically Poisson bi-torsor $(SL(2,\mathbb{C}), \{-,-\}_{\eta,\epsilon})$.

Hopf $*$-algebras

Associated Borel and AN-groups

Put

$$\mathcal{O}_q^{\eta,\epsilon}(B^+) = \mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C}))/(c), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ 0 & A^{-1} \end{pmatrix},$$

$$\mathcal{O}_q^{\eta,\epsilon}(AN^+) = \mathcal{O}_q^{\eta,\epsilon}(B^+)/(A - A^*).$$

Then $G_\eta AN^+$-decomposition

$$\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \hookrightarrow \mathcal{O}_q(G_\eta) \otimes \mathcal{O}_q^{\eta,\epsilon}(AN^+).$$

Note: for $q \neq 1$, we can interpret $\mathcal{O}_q^{\eta,\epsilon}(AN^+) = U_q^{\eta,\epsilon}(\mathfrak{su}(2))$:

$$BB^* - B^*B = (1-q^2)(\eta A^{-2} - \epsilon A^2) \quad \leftrightarrow \quad EF - FE = \frac{\eta K^{-2} - \epsilon K^2}{q^{-1} - q}.$$

Hopf $*$-algebras

Quantization of quotient spaces

Put

$$Z = Y^* \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} Y \in M_2(\mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})).$$

Then from $X_\eta^* \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} X_\eta = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$, we find

$$Z \in M_2(\mathcal{O}_q^{\eta,\epsilon}(G_\eta \backslash SL(2,\mathbb{C}))),$$

where

$$\mathcal{O}_q^{\eta,\epsilon}(G_\eta \backslash SL(2,\mathbb{C})) = \{x \in \mathcal{O}_q^{\eta,\epsilon}(SL(2,\mathbb{C})) \mid (\pi_\eta \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}.$$

Hopf $*$-algebras

Relations for Z

For $Z = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in M_2(\mathcal{O}_q^{\eta,\epsilon}(G_\eta \backslash SL(2,\mathbb{C})))$, we have $Z^* = Z$ so

$$x^* = x, \qquad w = y^*, \qquad z = z^*,$$

and with $T = z + q^2\epsilon x$, we have T selfadjoint central and

$$xy = q^2 yx, \qquad xy^* = q^{-2} y^* x,$$

$$y^* y = -q^2\eta + q^2 Tx - q^4 \epsilon x^2, \quad yy^* = -q^2\eta + Tx - \epsilon x^2.$$

Note:

- twisted reflection equation

$$R_{21} Z_{13} R_{\epsilon,12} Z_{23} = Z_{23} R_{\epsilon,21} Z_{13} R_{12}$$

- $\mathcal{O}_q^{\eta,\epsilon}(G_\eta \backslash SL(2,\mathbb{C})) \underset{\text{dense}}{\hookrightarrow} \mathcal{O}_q^{\eta,\epsilon}(AN^+)$ (Cholesky).

Hopf $*$-algebras

Case $\epsilon = +$: action quantum $SU(2)$

We may intepret

$$\mathcal{O}_q^{\eta,+}(G_\eta \backslash SL(2,\mathbb{C})) \cong \mathcal{O}_q(P_\eta)$$

with further 'spectral condition' $Z > 0$ ($\eta = +$) or $Z \geq 0$ ($\eta = 0$). We moreover have right coaction

$$\mathcal{O}_q(P_\eta) \to \mathcal{O}_q(P_\eta) \otimes \mathcal{O}_q(SU(2)), \quad Z \mapsto U_{13}^* Z_{12} U_{13}$$

and

$$\mathcal{O}_q(P_\eta/SU(2)) = \mathscr{Z}(\mathcal{O}_q(P_\eta)) = \mathbb{C}[T].$$

Hence the orbit space is classical!

Hopf $*$-algebras

Realisation as coideal, case $\epsilon = +, \eta = 0$

For $\epsilon = +, \eta = 0$ we have $*$-character

$$\chi : x, y \mapsto 0, \quad T \mapsto 1.$$

Hence embedding as coideal $*$-subalgebra (of subgroup type)

$$Z \mapsto (\mathrm{id} \otimes \chi \otimes \mathrm{id})(U_{13}^* Z_{12} U_{13}) = U^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U = \begin{pmatrix} \gamma^*\gamma & \gamma^*\alpha^* \\ \alpha\gamma & \alpha\alpha^* \end{pmatrix},$$

where range coideal is quantum flag manifold

$$\mathrm{Pol}_q(\mathbb{T}\backslash SU(2)) = \{x \in \mathrm{Pol}_q(SU(2)) \mid (\pi_{\mathbb{T}} \otimes \mathrm{id})\Delta(x) = 1 \otimes x\}$$

for $\pi_{\mathbb{T}} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}$, with $\mathbb{T} = \left\{ \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \right\}$.

Realisation as coideal, case $\epsilon = +, \eta = -$

For $\epsilon = +, \eta = -$ we have $*$-character

$$\chi_- : x \mapsto 0, \quad y \mapsto q, \quad T \mapsto 0.$$

Hence embedding as coideal $*$-subalgebra

$$Z \mapsto U^* \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} U = q \begin{pmatrix} \gamma^*\alpha + \alpha^*\gamma & (\alpha^*)^2 - q(\gamma^*)^2 \\ \alpha^2 - q\gamma^2 & -q\gamma\alpha^* - q\alpha\gamma^* \end{pmatrix},$$

where range coideal is quantum symmetric space

$$\mathrm{Pol}_q(\mathbb{T}' \backslash SU(2))$$

via twisted primitive element of $\mathbb{T}' = \left\{ \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \right\}$, the fixed points of $SU(2)$ for involution $\theta = \mathrm{Ad} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Representation theory

Double the fun!

Classically, for G semisimple linear Lie group:

- Representation theory $\mathcal{O}(G)$ not interesting: points!
- Representation theory G: mostly via $*$-algebra $U(\mathfrak{g})$.
- Full (unitary) representation theory arbitrary G still unknown!

In the quantum world:

- Representation theory $\mathcal{O}_q(G)$ interesting: non-commutative!
- Representation theory $U_q(\mathfrak{g})$ geometrized: $U_q(\mathfrak{g}) \cong \mathcal{O}_q(\hat{G})$.

Challenges:

- Analytical objects $C_{0,q}(G), L_q^\infty(G)$ not defined *a priori*.
- How to describe $C_{0,q}(G), L_q^\infty(G)$?

Representation theory

New opportunities

$*$-representation theory of

$$\mathcal{O}_q^{\eta,\epsilon}(G_\eta \backslash SL(2,\mathbb{C})) \underset{\text{dense}}{\subseteq} \mathcal{O}_q^{\eta,\epsilon}(AN^+) \cong U_q^{\eta,\epsilon}(\mathfrak{su}(2)).$$

For $\epsilon = +$:

$\rightsquigarrow$ 'Big cell' representations allow highest weight description.

$\rightsquigarrow$ Irreps always bounded, but can be infinite-dimensional.

$$\begin{aligned}
\mathrm{Irrep}(\mathcal{O}_q(P_+)) &= \{M_n(\mathbb{C}) \mid T = q(q^{-n} + q^n), n \in \mathbb{Z}_{>0}\}, \\
\mathrm{Irrep}(\mathcal{O}_q(P_0)) &= \{B(l^2(\mathbb{N})) \mid T > 0\}, \\
\mathrm{Irrep}(\mathcal{O}_q(P_-)) &= \{B(l^2(\mathbb{N})) \oplus B(l^2(\mathbb{N})) \mid T = t \in \mathbb{R}\}.
\end{aligned}$$

Problems in general:

- Which representations are positive (on Hilbert spaces)?
- Which central characters factor over coideal?