Commutative-by-finite Hopf Algebras and their Finite Dual

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joint work with Ken Brown & Astrid Jahn

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Overview

1 Preliminaries and Motivation

2 Commutative-by-finite Hopf Algebras

3 Finite Dual of this Class

4 Subspaces of the dual
   - The tangential component
   - The character component
$k$ a field, $\bar{k} = k$, $\text{char}(k) = 0$
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$\epsilon : H \to k$

$S : H \to H$
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Sweedler’s notation: $\Delta(h) = \sum h_1 \otimes h_2$

$H^+ := \ker \epsilon$
$(H, m, u, \Delta, \epsilon, S)$ finite-dimensional Hopf algebra
Hopf dual

\((H, m, u, \Delta, \epsilon, S)\) finite-dimensional Hopf algebra

\[ H^* = \text{Hom}_k(H, k) \]

\[(H^*, \Delta^*, \epsilon^*, m^*, u^*, S^*)\]
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- Convolution product:
  take $f, g \in H^*$ and $h \in H$, write $\Delta(h) = \sum h_1 \otimes h_2$

  $(fg)(h) = \sum f(h_1)g(h_2)$. 
(H, m, u, Δ, ϵ, S) infinite-dimensional Hopf algebra
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Problem: $m^* : H^* \to (H \otimes H)^*$ and $H^* \otimes H^* \subset (H \otimes H)^*$. 
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\[= \{ f \in H^* : f|_I = 0 \text{ for some ideal } I \triangleleft H \text{ of finite codimension} \} \]
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$H^\circ$ is Hopf algebra: restrict the maps $(\Delta^*, \epsilon^*, m^*, u^*, S^*)$. 

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Com.-by-fin. Hopf algebras and duals  
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Motivation

Theorem (Cartier)

Let $H$ be an affine commutative Hopf algebra. Then, $H$ is reduced, hence

$$H \cong \mathcal{O}(G)$$

for some affine algebraic group $G$. 

Theorem (Cartier-Gabriel-Kostant)

Let $G$ be an affine algebraic group. Then,

$$\mathcal{O}(G) \circ \cong U(\text{Lie } G)^*_{kG},$$

where $G$ identifies with the group of characters of $\mathcal{O}(G)$.

$U(\text{Lie } G)$ is the set of maps that vanish on some power of $\mathcal{O}(G)^+ = \ker \epsilon$.

$G$ acts on $U(\text{Lie } G)$ by conjugation.
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- $G$ acts on $U(\text{Lie } G)$ by conjugation.
An affine Hopf algebra $H$ is *commutative-by-finite* if it is a finitely-generated left (or right) module over some normal commutative Hopf subalgebra $A$. A being normal means $\text{ad}(h)(a) = \sum h_1 a S(h_2) \in A$ and $\text{ad}_r(h)(a) = \sum S(h_1) a h_2 \in A$, for every $a \in A, h \in H$. A normal $\implies A + H$ is a Hopf ideal of $H \implies H := H/A + H$ f.d. quotient Hopf algebra.
Setting

Definition

An affine Hopf algebra $H$ is \textit{commutative-by-finite} if it is a finitely-generated left (or right) module over some normal commutative Hopf subalgebra $A$.

A being \textit{normal} means

$$ad_l(h)(a) := \sum h_1 a S(h_2) \in A \quad \text{and} \quad ad_r(h)(a) := \sum S(h_1) a h_2 \in A,$$

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- Noetherian PI Hopf domains of Gelfand-Kirillov dimension 2. (Goodearl & Zhang [arXiv:0905.0621]).
- Non-example: Gelaki & Letzter [arXiv:math/0112038].
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\[ H = T(n, t, q) = k\langle g, x : g^n = 1, xg = qgx \rangle, \]
\[ \Delta(g) = g \otimes g \quad \epsilon(g) = 1 \quad S(g) = g^{-1} \]
\[ \Delta(x) = x \otimes g^t + 1 \otimes x \quad \epsilon(x) = 0 \quad S(x) = -xg^{-t} \]

where \( q \) is a primitive \( n \)th root of unity and \( 0 < t < n \). Assume \( (n, t) = 1 \).
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\[ A = k[x^n]. \]
\[ \overline{H} = k\langle \bar{g}, \bar{x} : \bar{g}^n = 1, \bar{x}^n = 0, \bar{xg} = q\bar{g}\bar{x} \rangle = T_f(n, t, q). \]
Proposition

Let $H$ be a commutative-by-finite Hopf algebra. Then,

1. $H$ and $A$ are affine and noetherian.

2. $H$ is a faithfully flat and projective $A$-module.

3. If $H$ is pointed, $H$ is a free $A$-module.

4. $H$ is a PI-ring.

5. [Skryabin] The antipode $S$ is bijective.


In general, these Hopf algebras are not regular.
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Idea: decompose $H^\circ$ in terms of $H^*$ and $A^\circ$. 
Finite Dual of this Class

Let $H$ be commutative-by-finite.

Idea: decompose $H^\circ$ in terms of $\overline{H}^*$ and $A^\circ$.

$$
\pi : H \rightarrow \overline{H} \quad \iota : A \hookrightarrow H \quad \Pi : H \twoheadrightarrow A
$$
\[ \pi : H \to \overline{H} := H/A^+H \]
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**Lemma (B.-C.-J.)**

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**Proposition (B.-C.-J.)**

\[ \iota^\circ : H^\circ \to A^\circ \text{ is a surjection of Hopf algebras.} \]
Lemma (Radford)

\( H \) decomposes into \( A \oplus X \), as a left (or right) \( A \)-module.

\[ \Pi : H \rightarrow A \]

is a left \( A \)-module projection map.

Proposition (B.-C.-J.)

\[ \Pi \circ A \hookrightarrow H \] is an embedding of right \( A \)-comodules.
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Proposition (B.-C.-J.)

\[ \Pi^\circ : A^\circ \hookrightarrow H^\circ \text{ is an embedding of right } A^\circ\text{-comodules.} \]
Theorem (B.-C.-J.)

Let $H$ be commutative-by-finite. We write $H = A \oplus X$ as $A$-modules.

1. If $X$ is a coideal of $H$, then as algebras $H \cong H \ast \# A$.

2. If $X$ is an ideal of $H$, then as algebras $H \cong H \ast \# \sigma A$. 
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Decomposes the Hopf dual of:

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- Noetherian PI Hopf domains of Gelfand-Kirillov dimension 2.
- Quantum groups $U_\epsilon(\mathfrak{sl}_2(k))$ and $U_\epsilon(\mathfrak{sl}_3(k))$.  


Examples

Example 1
\[ H = kD = k\langle a,b : a^2 = 1, aba = b^{-1} \rangle \]
\[ A = k\langle b \rangle = O(k), \quad H = kC \]
\[ H \circ \sim = H \ast \# A \circ = kC \otimes k[x] \otimes k(k^*, \cdot) \]

Example 2
\[ H = T(n,t,q) = k\langle g,x : g^n = 1, xg = qgx \rangle \]
\[ A = k[x^n] = O(k), \quad H = T_f(n,t,q) \]
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\[ H^* \cong \overline{H}^* \# A^* = T_f(n, t, q) \otimes k[z] \otimes k(k, +). \]
A little more about the dual

Recall:

\( A \) is affine commutative \( \Rightarrow A = O(G) \) for some affine algebraic group \( G \).

\( A \circ \sim = U(\text{Lie} G)^* kG \), where \( A^+ = \ker \epsilon \).

\( kG = \{ f \in A \circ : f(\bigcap m g_1 \cap \ldots \cap m g_r) = 0 \}, \) for some \( g_i \in G \).

Under certain hypotheses, \( H \circ \sim = H^* \# \sigma(U(\text{Lie} G)^* kG) \).

Extend these identifications of \( kG \) and \( U(\text{Lie} G)^* \) to \( H \).
Recall: $A$ is affine commutative $\Rightarrow A = \mathcal{O}(G)$ for some affine algebraic group $G$

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The tangential component

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**Definition**

Let \( H \) be a commutative-by-finite Hopf algebra. The *tangential component* of \( H^\circ \) is

\[ W := \{ f \in H^\circ : f((A^+H)^n) = 0, \text{ for some } n \in \mathbb{N} \}. \]
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Let \( H \) be commutative-by-finite. Then,

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Extending to \( H \):

- \( H \) acts on \( A \) by adjoint actions: for every \( a \in A, h \in H \)

\[
ad_l(h)(a) := \sum h_1 aS(h_2) \in A \quad \text{and} \quad ad_r(h)(a) := \sum S(h_1)ah_2 \in A.
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\[
\widehat{kG} = \{ f \in H^\circ : f(m_{g_1}^{\overline{H}} H \cap \ldots \cap m_{g_r}^{\overline{H}} H) = 0, \text{ for some } g_i \in G \}.
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**Definition**

Let \( H \) be commutative-by-finite. We say it is *orbitally semisimple* if \( A/\mathfrak{m}_g^H \) is semisimple for every \( g \in G \).
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Let \( H \) be an orbitally semisimple commutative-by-finite Hopf algebra.

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**Definition**

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Let \( H \) be an orbitally semisimple commutative-by-finite Hopf algebra.

1. \( \hat{kG} \) is a Hopf subalgebra of \( H^\circ \).
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\[
\hat{kG} \cong H^* \#_\sigma kG.
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Under certain hypotheses,

\[ \hat{kG} \cong \overline{H}^* \#_\sigma kG \]

\[ \overline{H}^* \cong \overline{H}^* \#_\sigma \sigma(U(Lie G)) \]

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\[ H^o \cong \overline{H}^* \#_\sigma (U(Lie G) \ast kG) \]
Example 1

\[ H = kD = k\langle a, b : a^2 = 1, aba = b^{-1} \rangle \]

- \( A = k\langle b \rangle, \overline{H} = kC_2 \)
- \( H^\circ \cong \overline{H}^* \# A^\circ = kC_2 \otimes k[x] \otimes k(k^*, \cdot) \).
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References


Thank you.