Commutative-by-finite Hopf Algebras and their Finite Dual

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joint work with Ken Brown & Astrid Jahn

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Overview

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Sweedler's notation: $\Delta(h) = \sum h_1 \otimes h_2$

$H^+ := \ker \epsilon$

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- Convolution product:
  take $f, g \in H^*$ and $h \in H$, write $\Delta(h) = \sum h_1 \otimes h_2$

$$(fg)(h) = \sum f(h_1)g(h_2).$$

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- H° is Hopf algebra: restrict the maps $(\Delta^*, \epsilon^*, m^*, u^*, S^*)$.

Motivation

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- *G acts on $U(\mathrm{Lie}\, G)$ by conjugation.*

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Non-example: Gelaki & Letzter [arXiv:math/0112038].

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$\overline{H} = k\langle \bar{g}, \bar{x} : \bar{g}^n = 1, \bar{x}^n = 0, \bar{x}\bar{g} = q\bar{g}\bar{x} \rangle = T_f(n, t, q)$.

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- In general, these Hopf algebras are *not* regular.

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$\iota^\circ : H^\circ \twoheadrightarrow A^\circ$ *is a surjection of Hopf algebras.*

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$\Pi^\circ : A^\circ \hookrightarrow H^\circ$ is an embedding of right A°-comodules.

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Extend these identifications of kG and $U(\operatorname{Lie} G)$ to H.

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Definition

Let H be a commutative-by-finite Hopf algebra. The *tangential component* of H° is

$$W := \{f \in H^\circ : f((A^+H)^n) = 0, \text{for some } n \in \mathbb{N}\}.$$

Theorem (B.-C.-J.)

Let H be commutative-by-finite. Then,

1. *W is a normal Hopf subalgebra of H°;*
2. *and as algebras*
$$W \cong \overline{H}^* \#_\sigma U(\operatorname{Lie} G).$$

The character component

$A^\circ \cong U(\mathrm{Lie}\, G) * kG$,

- $kG = \{f \in A^\circ : f(\mathbf{m}_{g_1} \cap \ldots \cap \mathbf{m}_{g_r}) = 0, \text{ for some } g_i \in G\}$.

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Extending to H:

- H acts on A by adjoint actions: for every $a \in A, h \in H$

$$ad_l(h)(a) := \sum h_1 a S(h_2) \in A \qquad \text{and} \qquad ad_r(h)(a) := \sum S(h_1) a h_2 \in A.$$

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- H acts on $G = \mathrm{MaxSpec}(A)$.

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Let H be commutative-by-finite.

1. Let $\mathbf{m} \in \mathrm{MaxSpec}(A)$. We define $\mathbf{m}^{\overline{H}} = \{a \in \mathbf{m} : \overline{h} \cdot a \in \mathbf{m}, \forall \overline{h} \in \overline{H}\}$.

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2. The *character component* of H° is

$$\widehat{kG} = \{f \in H^\circ : f(\mathbf{m}_{g_1}^{\overline{H}} H \cap \ldots \cap \mathbf{m}_{g_r}^{\overline{H}} H) = 0, \text{ for some } g_i \in G\}.$$

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Definition

Let H be commutative-by-finite. We say it is *orbitally semisimple* if $A/\mathbf{m}_g^{\overline{H}}$ is semisimple for every $g \in G$.

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Theorem (B.-C.-J.)

Let H be an orbitally semisimple commutative-by-finite Hopf algebra.

1. *$\widehat{kG}$ is a Hopf subalgebra of H°.*

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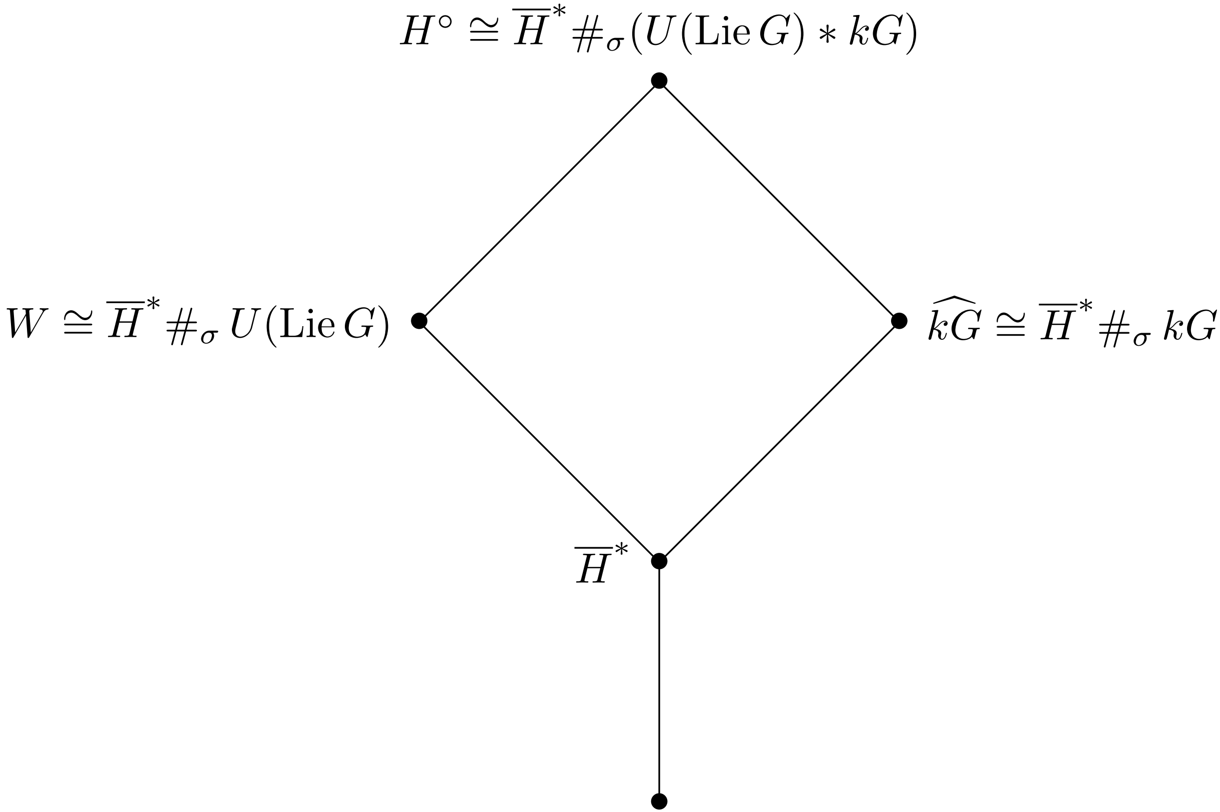
Theorem (B.-C.-J.)

Let H be an orbitally semisimple commutative-by-finite Hopf algebra.

1. *$\widehat{kG}$ is a Hopf subalgebra of H°.*
2. *If H° decomposes as $\overline{H}^* \#_\sigma A^\circ$, then*

$$\widehat{kG} \cong \overline{H}^* \#_\sigma kG.$$

Under certain hypotheses,



Examples

Example 1

$H = kD = k\langle a, b : a^2 = 1, aba = b^{-1}\rangle$

- $A = k\langle b\rangle$, $\overline{H} = kC_2$
- $H^\circ \cong \overline{H}^* \# A^\circ = kC_2 \otimes k[x] \otimes k(k^*, \cdot)$.

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$H = T(n, t, q) = k\langle g, x : g^n = 1, xg = qgx\rangle$

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References

- K.R. Goodearl & R.B. Warfield Jr., *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society, Student Texts **61**, 2004.
- A. Jahn's Doctor of Philosophy thesis, *The finite dual of crossed products*, University of Glasgow, December 2014.
- J.C. McConnell & J.C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, 1987.
- S. Montgomery, *Hopf Algebras and Their Actions on Rings*, American Mathematical Society, 1992.
- D. E. Radford, *Hopf Algebras*, Series on Knots and Everything, Vol. 49, World Scientific Publishing, 2012.
- J.T. Stafford & J.J. Zhang, *Homological Properties of (Graded) Noetherian PI rings*, Journal of Algebra, Vol. 168, pp. 988-1026, 1994.

Thank you.