A reconstruction notion for monoids and clones

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Outline

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Presenting joint work with...

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What is reconstruction?

Reconstruction results are intended to answer:

Given $\operatorname{Aut}\mathbb{A}$, to what extent can an $\aleph_0$-categorical structure $\mathbb{A}$ be recovered from $\operatorname{Aut}\mathbb{A}$?

$\operatorname{Aut}\mathbb{A}$ can be considered as

- abstract group (alg.)
- topological group (alg. + top.) (Tichonov (=pointw. conv.) top.)
- permutation group ((alg. + top. +) act.)

$[(A, \operatorname{Aut}\mathbb{A})]_{\cong}$ Perm.Group	$[\operatorname{Aut}\mathbb{A}]_{\cong,\text{top}}$ Top.Group	$[\operatorname{Aut}\mathbb{A}]_{\cong}$ Group

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Reconstruction up to bi-interpretability

The case where $\operatorname{Aut}\mathbb{A}$ is viewed as a topological group:

Theorem ((Coquand, 1980s) Ahlbrandt and Ziegler)

Let $\mathbb{A}$ and $\mathbb{B}$ be $\aleph_0$-categorical structures.

$\mathbb{A}$ *and* $\mathbb{B}$ *are bi-interpretable* $\iff$

$\operatorname{Aut}\mathbb{A}$ *and* $\operatorname{Aut}\mathbb{B}$ *are isomorphic as topological groups.*

We focus on the following reconstruction question

Let $\mathbb{A}$ be an $\aleph_0$-categorical structure and $G = \operatorname{Aut} \mathbb{A}$.

If $\mathbb{B}$ is also $\aleph_0$-categorical and $G \cong \operatorname{Aut} \mathbb{B}$ as an abstract group,
is $G \cong \operatorname{Aut} \mathbb{B}$ as a topological group?

Two approaches

- One is via the small index property.
- The second method is Rubin's forall-exists interpretations.

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Proposition (Dugald's Survey)

Let $\mathbb{A}$ and $\mathbb{B}$ be $\aleph_0$-categorical structures, and $\mathbb{A}$ has S.I.P.

If $\operatorname{Aut} \mathbb{A} \cong \operatorname{Aut} \mathbb{B}$ *as abstract groups*,

then $\mathbb{A}$ and $\mathbb{B}$ are bi-interpretable.

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Proposition (folklore)

Let $\mathbb{A}$ and $\mathbb{B}$ be relational structures, and $\mathbb{A}$ has S.I.P. Then any

$$\xi : \operatorname{Aut}\mathbb{A} \xrightarrow{\cong} \operatorname{Aut}\mathbb{B}, \text{ is continuous.}$$

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Corollary (D. Lascar (1991))

Any continuous isomorphism between closed subgroups of Sym (A) is a homeomorphism.

Automatic homeomorphicity

Definition (M.Bodirsky, M.Pinsker, A.Pongrácz)

A **closed** subgroup $G \leq \mathrm{Sym}\,(A)$ has reconstruction $:\Longleftrightarrow$ for every other closed subgroup $G' \leq \mathrm{Sym}\,(B)$, if there exists a group isomorphism between G and G', then there also exists a group isomorphism between G and G' which is a homeomorphism.

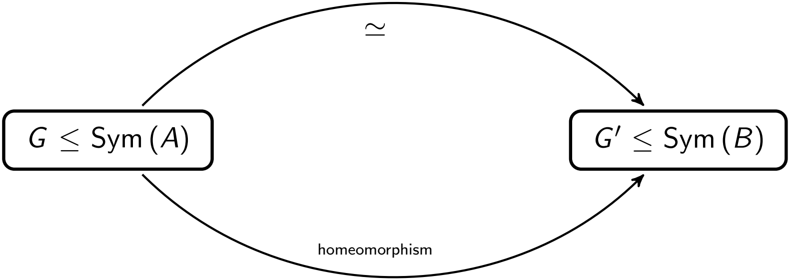
Definition (M.Bodirsky, M.Pinsker, A.Pongrácz)

A **closed** group $G \leq \mathrm{Sym}\,(A)$ has automatic homeomorphicity (A.H.) $:\Longleftrightarrow$ every group isomorphism from G to a closed $G' \leq \mathrm{Sym}\,(B)$ is a homeomorphism.

Reconstruction up-to a homeomorphism

$$G \leq \mathrm{Sym}\,(A) \xrightarrow{\;\simeq\;} G' \leq \mathrm{Sym}\,(B)$$

Reconstruction up-to a homeomorphism



The advantage of looking for A.H.

1. SIP needs index $\Rightarrow$ only makes sense for groups,
2. automatic homeomorphicity: available for (endomorphism) monoids, (polymorphism) clones
3. from structures $\rightsquigarrow$ closed groups, monoids, clones, i.e., from model theory $\rightsquigarrow$ algebra.

$\forall \mathbb{A}, \mathbb{B}$ $\aleph_0$-categorical structures: Aut $\mathbb{A}$ has autom. homeo. $\Rightarrow$

$$\text{Aut}\,\mathbb{A} \cong \text{Aut}\,\mathbb{B} \iff \text{Aut}\,\mathbb{A} \cong_{Top} \text{Aut}\,\mathbb{B}$$
$$\iff \mathbb{A} \text{ and } \mathbb{B} \text{ are bi-interpretable}$$

Towards something stronger. Reconstruction up-to a bijection

M. Rubin's main result [1994] gives a reconstruction criterion for the class of $\aleph_0$-categorical structures without algebraicity.

Let $\mathcal{K}$ be the class of $\aleph_0$-categorical structures without algebraicity. Whenever

- $\mathbb{A}, \mathbb{B} \in \mathcal{K}$
- $\mathbb{A}$ has a so-called weak $\forall\exists$ interpretation,

it is enough to know that $\operatorname{Aut}\mathbb{A} \cong \operatorname{Aut}\mathbb{B}$ as abstract groups in order to conclude that $(A, \operatorname{Aut}\mathbb{A}) \cong (B, \operatorname{Aut}\mathbb{B})$ as permutation groups.

Definition

A closed $G = \operatorname{Aut} \mathbb{A} \leq \operatorname{Sym}(A)$ has **automatic action compatibility** $:\iff$ for every $H = \operatorname{Aut} \mathbb{B} \leq \operatorname{Sym}(B)$, $|B| = |A|$: for every $\varphi : G \cong H$ group isomorphism $\exists\, \theta : A \to B$ bijection that induces φ via conjugation, i.e.

$$\varphi(g) = \theta \circ g \circ \theta^{-1} \text{ for every } g \in G$$

Are there any examples with this property?

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1. Theorem (Rubin, 1994)
   $G = \mathrm{Aut}\,\mathbb{A} \in \mathcal{K}$ for $\mathbb{A}$ with **weak** $\forall\exists$ **interpretation** $\Longrightarrow$ G has **automatic action compatibility** wrt. $\mathcal{K}$, for $\mathcal{K}$ the class of automorphism groups of $\aleph_0$-categorical str without algebraicity.
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2. Theorem (Paolini & Shelah, 2017)
   $G = \mathrm{Aut}\,\mathbb{A} \in \mathcal{K} \implies G$ has **automatic action compatibility** wrt. $\mathcal{K}$,
   for $\mathcal{K}$ the class of automorphism groups of $\aleph_0$-categorical str without algebraicity and strong SIP.

Our contribution

Automatic action compatibility for monoids /clones

- Given closed clone $F \leq \mathrm{O}_A$,
- $M = F^{(1)}$ closed monoid,
- $G \subseteq M$ group of invertible monoid elements,
- How to extend automatic action compatibility:

$$G \rightsquigarrow M \rightsquigarrow F\ ?$$

Lemma from Groups to Monoids

Suppose

- $M \leq A^A$ and $M' \leq B^B$ closed transformation monoids
- with groups of invertibles $G \subseteq M$, $G' \subseteq M'$
- G is dense in M
- $\varphi : M \to M'$ monoid isomorphism
- $\varphi \restriction_G^{G'} : G \to G'$ is induced by bijection $\theta : A \to B$
- $\forall \psi : \; M \cong \overline{G'} : \psi \restriction_G = \varphi \restriction_G \implies \psi = \varphi$.

Then φ is induced by θ: $\quad \forall f \in M : \varphi(f) = \theta \circ f \circ \theta^{-1}$.

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Corollary for $G \subseteq M \subseteq \overline{G} \subseteq A^A$ as above

If $\forall G' \subseteq M' \leq B^B, \ |A| = |B| \ \forall \varphi : \ M \cong M' \ \forall \psi : \ M \cong \overline{G'} :$

$\psi \restriction_G = \varphi \restriction_G \implies \psi = \varphi$, then

G has autom action comp. $\implies$ M has autom action comp.

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Corollary for $G \subseteq M \subseteq \overline{G} \subseteq A^A$ as above

If M **has automatic homeomorphicity**, then
G has autom action comp. $\implies$ M has autom action comp.

Lemma from Groups to Monoids

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Then φ is induced by θ: $\forall f \in M : \varphi(f) = \theta \circ f \circ \theta^{-1}$.

Corollary for $G \subseteq M \subseteq \overline{G} \subseteq A^A$ as above

If every inj. $\varphi : M \to M$ with $\varphi \restriction_G = id \restriction_G$ is the id_M, then G has autom action comp. $\implies$ M has autom action comp.

Theorem from Monoids to Clones

Suppose for **any** sets A, B

- $F \leq \mathrm{O}_A$ and $F' \leq \mathrm{O}_B$ closed clones
- $\xi : F \to F'$ surjective clone homomorphism
- $F^{(1)}$ weakly directed
  $\left(\forall a_1, a_2 \in A \; \exists a_0 \in A \; \exists f_1, f_2 \in F^{(1)} : f_1(a_0) = a_1 \wedge f_2(a_0) = a_2\right)$
- $\xi \restriction_{F^{(1)}}^{F'^{(1)}} : F^{(1)} \to F'^{(1)}$ is induced by bijection $\theta : A \to B$

Then ξ is induced by θ:

$$\forall f \in F : \quad \xi(f) = \theta \circ f \circ \left(\theta^{-1} \times \cdots \times \theta^{-1}\right).$$

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Corollary for $F \subset \mathrm{O}_A$ with $F^{(1)}$ weakly directed

$F^{(1)}$ has autom action comp. $\Longrightarrow$ F has autom action comp.

Summarizing, making things concrete

Corollary

- $F \leq \mathrm{O}_A$ *closed clone,* $F^{(1)}$ *weakly directed*
- G *dense (group of invertibles) in* M.
- $\mathbb{A}$ $\aleph_0$*-categorical str, no algebraicity, weak* $\forall\exists$ *interpretations*
- $\mathrm{Aut}\,\mathbb{A} = G$
- *Every inj.* $\varphi : M \to M$ *with* $\varphi \restriction_G = id \restriction_G$ *is the* id_M*, or* M *has automatic homeomorphicity*

$\implies$ F *has automatic action compatibility wrt.* $\mathcal{C}$,
$\implies$ M *has automatic action compatibility wrt.* $\mathcal{K}$,
$\mathcal{K}$ *the class of endomorphism monoids of* $\aleph_0$*-categorical str, w/o algebraicity*
$\mathcal{C}$ *the class of polymorphism clones of* $\aleph_0$*-categorical str, w/o algebraicity*

Applying results to concrete examples

Let $\mathbb{A}$ be one of

- $(\mathbb{Q}, <)$
- the random (Rado) graph
- the random directed graph
- the countable universal homogeneous tournament
- the countable universal k-uniform hypergraph for $k \geq 2$
- the countable universal homogeneous $\mathbb{K}_n$-free graph, for $n \geq 3$
- any (countable universal homogeneous) Henson digraph

$\implies$ $M = \operatorname{Emb} \mathbb{A}$ has automatic action compatibility wrt. endomorphism monoids of $\aleph_0$-categorical str, w/o algebraicity

$\implies$ $F = \overline{F} \leq \mathrm{O}_A$ with $F^{(1)} = \operatorname{Emb} \mathbb{A}$, e.g. $\operatorname{Pol} \mathbb{A}^{\mathbf{C}}$ has automatic action compatibility wrt. polymorphism clones of $\aleph_0$-categorical str, w/o algebraicity

$\implies$ $\operatorname{Pol}(\mathbb{Q}, <)$ has automatic action compatibility

Open Problems

Problem 1

Which of $(\mathbb{Q}, \text{ betw })$, $(\mathbb{Q}, \text{ circ })$ and $(\mathbb{Q}, \text{ sep })$ have a weak $\forall\exists$-interpretation?

Problem 2

For which $\mathbb{A}$ among

- random strict poset
- countable universal homogeneous biapartite graph
- countable dense local order $\mathbb{S}_2$
- Cherlin's countable myoptic local order $\mathbb{S}_3$

does $\operatorname{Emb}\mathbb{A}$ have automatic homeomorphicity / satisfy the condition the only injective monoid endomorphism fixing $\operatorname{Aut}\mathbb{A}$ pointwise is $id_{\operatorname{Emb}\mathbb{A}}$?

Thank you :)