Characters of inductive limits of finite alternating groups

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Don't Bury the Lead ...

Definition

- G is an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite alternating groups G_i.
- Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.

Theorem (Thomas 2016)

If G is an $L(\mathrm{Alt})$-group and $G \not\cong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are precisely the associated characters of its ergodic IRSs.

Invariant random subgroups

- Let G be a countable group and let $\mathrm{Sub}_G \subset 2^G$ be the compact space of subgroups $H \leqslant G$.
- Then $G \curvearrowright \mathrm{Sub}_G$ via conjugation: $H \overset{g}{\mapsto} gHg^{-1}$.

Definition (Abért)

A G-invariant Borel probability measure ν on Sub_G is called an invariant random subgroup or IRS.

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A Trivial Example

If $N \trianglelefteq G$, then the Dirac measure δ_N is an IRS of G.

Stabilizer distributions

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- Then the stabilizer distribution $\nu = f_* \mu$ is an IRS of G.

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- If $B \subseteq \mathrm{Sub}_G$, then $\nu(B) = \mu(\{ z \in Z \mid G_z \in B \})$.

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Theorem (Abért-Glasner-Virag 2012)

If ν is an IRS of G, then ν is the stabilizer distribution of a measure-preserving action $G \curvearrowright (Z, \mu)$.

Ergodicity

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A measure-preserving action $G \curvearrowright (Z, \mu)$ *is* ***ergodic*** *if* $\mu(A) = 0, 1$ *for every G-invariant* μ*-measurable subset* $A \subseteq Z$.

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Theorem (Creutz-Peterson 2013)

If ν *is an ergodic IRS of* G*, then* ν *is the stabilizer distribution of an ergodic action* $G \curvearrowright (Z, \mu)$.

A classification problem

Open Problem

Classify the ergodic IRSs of the simple locally finite groups.

Definition

A countably infinite group G is locally finite if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of an increasing chain of finite groups.

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Question

But why focus on locally finite groups?

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Theorem

With the above hypotheses, for μ-a.e. $z \in Z$, for all $g \in G$,

$$\mu(\operatorname{Fix}_Z(g)) = \lim_{n \to \infty} |\operatorname{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|.$$

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Remark

Note that the $|\, \mathrm{Fix}_{\Omega_n(z)}(g)\,| / |\,\Omega_n(z)\,|$ is the probability that an element of $(\,\Omega_n(z), \mu_n\,)$ is fixed by $g \in G_n$, where μ_n is the uniform probability measure on $\Omega_n(z)$

The ergodic IRSs of the $L(\mathrm{Alt})$-groups

Joint work with Tucker-Drob (2016)

- The classification of the nontrivial ergodic IRSs of the $L(\mathrm{Alt})$-groups $G \ncong \mathrm{Alt}(\mathbb{N})$.

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Example

Hall's universal locally finite group has no nontrivial ergodic IRSs.

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Example

Hall's universal locally finite group has no nontrivial ergodic IRSs.

Remark

- Vershik (essentially) classified the ergodic IRSs of $\mathrm{Alt}(\mathbb{N})$.

Characters of countable groups

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Definition

If G is a countable group, then $\chi : G \to \mathbb{C}$ is a character if the following conditions are satisfied:

(i) $\chi(h\,g\,h^{-1}) = \chi(g)$ *for all* $g, h \in G$.

(ii) $\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ *for all* $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ *and* $g_1, \cdots, g_n \in G$.

(iii) $\chi(1_G) = 1$.

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Example

If G is finite and π is a finite-dimensional representation, then $\chi(g) = \mathrm{trace}(\,\pi(g)\,)/\pi(1)$ is a character of G.

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Example

If $G \curvearrowright (Z, \mu)$ is a measure-preserving action on a probability space, then $\chi(g) = \mu(\,\mathrm{Fix}_Z(g)\,)$ is a character.

Characters of finite groups

Notation

- If G is a finite group, then $\widehat{G}$ denotes the set of (unitary equivalence classes) of irreducible representation of G.
- If $\pi \in \widehat{G}$, then $\chi^\pi(g) = \mathrm{trace}(\, \pi(g)\,)$.

Theorem

If G is a finite group and $\chi : G \to \mathbb{C}$ is a character, then χ is a convex combination of $\{\, \chi^\pi / \pi(1) \mid \pi \in \widehat{G} \,\}$.

Indecomposable characters

Definition

A character χ is indecomposable *if it is impossible to express*

$$\chi = r\chi_1 + (1 - r)\chi_2,$$

where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters.

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Remark

Indecomposable characters of countable groups give rise (via the Gelfand-Naimark-Siegel construction) to the factor representations of finite type.

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One of Vershik's many insights

The indecomposable characters of the group $\mathrm{Fin}(\mathbb{N})$ of finite permutations of the natural numbers are "closely related" to its ergodic IRSs.

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If G is an $L(\mathrm{Alt})$-group and $G \ncong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of G are precisely the associated characters of its ergodic IRSs.

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- Then we can define a corresponding character χ by

$$\begin{aligned}\chi(g) &= \nu(\{\, H \in \mathrm{Sub}_G \mid gHg^{-1} = H \,\}) \\ &= \nu(\{\, H \in \mathrm{Sub}_G \mid g \in N_G(H) \,\}).\end{aligned}$$

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- But we can also define a second character χ' by

$$\chi'(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \}).$$

Trivial IRSs and characters of simple groups G

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The trivial characters of G are the regular character χ_{reg} and the constant character χ_{con}, where:

- $\chi_{reg}(g) = 0$ *for all* $1 \neq g \in G$*; and*
- $\chi_{con}(g) = 1$ *for all* $g \in G$.

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Definition

If ν is an IRS of the countable group G, then the associated character is defined by $\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \})$.

The indecomposable characters of the $L(\mathrm{Alt})$-groups

Joint work with Tucker-Drob (2016)

- The classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups $G \ncong \mathrm{Alt}(\mathbb{N})$.

Corollary

- *The classification of the indecomposable characters of the $L(\mathrm{Alt})$-groups $G \ncong \mathrm{Alt}(\mathbb{N})$.*

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- *The classification of the indecomposable characters of the $L(\mathrm{Alt})$-groups $G \ncong \mathrm{Alt}(\mathbb{N})$.*

Remark

- Thoma classified the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$.

The Indecomposability Problem

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If $G \curvearrowright (Z, \mu)$ is ergodic and $N_G(G_z) = G_z$ for μ-a.e. $z \in Z$, then $\chi(g) = \mu(\mathrm{Fix}_Z(g))$ is an indecomposable character.

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Theorem (Thomas-Tucker-Drob 2016)

If $G \ncong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-groups and $\nu \neq \delta_1$ is an ergodic IRS of G, then $N_G(H) = H$ for ν-a.e. $H \in Sub_G$.

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Corollary (Thomas-Tucker-Drob 2016)

If $G \ncong \operatorname{Alt}(\mathbb{N})$ is an $L(\operatorname{Alt})$-groups and ν is an ergodic IRS of G, then the associated character χ_ν is indecomposable.

What about Alt($\mathbb{N}$)?

What about $\mathrm{Alt}(\mathbb{N})$?

Theorem (Thoma 1964)

If χ is a character of $\mathrm{Alt}(\mathbb{N})$, then the following are equivalent:

(i) *χ is indecomposable.*

(ii) *If $g, h \in \mathrm{Alt}(\mathbb{N})$ have disjoint supports, then $\chi(gh) = \chi(g)\chi(h)$.*

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- Let $\varphi : 2^{\mathbb{N}} \to \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N}}$ be the $\mathrm{Alt}(\mathbb{N})$-equivariant map defined by $\varphi(\xi) = \mathrm{Alt}(B_0^{\xi}) \times \mathrm{Alt}(B_1^{\xi})$.

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The associated character is

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and hence

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while, on the other hand,

$$\chi_\nu((1\,2)(3\,4)(5\,6)(7\,8)) = \frac{\binom{4}{0} + \binom{4}{2} + \binom{4}{4}}{2^8} = 1/2^5.$$

The Main Theorem Revisited

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A non-realizable indecomposable character

Fact

Thoma's classification of the indecomposable characters of Alt($\mathbb{N}$) *includes*

$$\chi(g) = \prod_{n=2}^{\infty} \left((1/2)^n + (-1)^{n+1}(1/2)^n \right)^{c_n(g)},$$

where $c_n(g)$ is the number of cycles of length n in the cyclic decomposition of the permutation g.

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Example

Thus $\chi(\,(\,a\,b\,c\,)\,) = 1/4$ and $\chi(\,(\,a\,b\,)(\,c\,d\,)\,) = 0$.

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- Suppose that ν is an ergodic IRS such that

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- Since $\chi((a\,b\,c)) = 1/4$, there exist $n \neq m$ such that

$$\nu(\{ H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \mid (1\,2\,n), (1\,2\,m) \in H \}) > 0.$$

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- But $(1\,2\,n)(1\,2\,m) = (1\,n)(2\,m)$ and

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The Asymptotic Approach to Characters

Theorem (Vershik-Kerov 1985)

If $G = \bigcup_{i \in \mathbb{N}} G_i$ is the increasing union of the finite subgroups G_i and χ is an indecomposable character of G, then there exist characters θ_i of irreducible representations π_i of G_i such that

$$\chi(g) = \lim_{i \to \infty} \frac{\theta_i(g)}{\theta_i(1)}.$$

Remark

Here $\theta_i(g) = \mathrm{trace}(\, \pi_i(g)\,)$ is the "usual" character associated with an irreducible representation π_i of the finite group G_i.

Irreducible Characters of Finite Symmetric Groups

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Notation

Let $\lambda = (\ell_1, \ell_2, \cdots, \ell_r)$ be a partition of n; i.e. a sequence of integers such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r > 0$ and $\ell_1 + \ell_2 + \cdots + \ell_r = n$.

- φ_λ denotes the corresponding irreducible character of $\mathrm{Sym}(n)$.

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- λ^* denotes the corresponding dual partition.

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Let $\lambda = (\ell_1, \ell_2, \cdots, \ell_r)$ be a partition of n; i.e. a sequence of integers such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r > 0$ and $\ell_1 + \ell_2 + \cdots + \ell_r = n$.

- φ_λ denotes the corresponding irreducible character of $\mathrm{Sym}(n)$.
- The depth of λ is $d(\lambda) = \ell_2 + \cdots + \ell_r$.
- λ^* denotes the corresponding dual partition.
- $\leq$ denotes the lexicographic ordering on the partitions of n.

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Conclusion

The irreducible characters are "almost" parameterized by the partitions λ such that $\lambda \geq \lambda^*$.

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There exist constants $b > 0$ and $0 < q < 1$ such that for sufficiently large n, for every partition $\lambda \geq \lambda^*$ of n and every $g \in \mathrm{Alt}(n)$,

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Full Limits

Definition

Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$.

- The embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is full if $\mathrm{Alt}(\Delta_i)$ has no trivial orbits on Δ_{i+1}.
- G is a full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ if every embedding $\mathrm{Alt}(\Delta_i) \hookrightarrow \mathrm{Alt}(\Delta_{i+1})$ is full.

Lemma (Thomas-Tucker-Drob 2016)

If $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, then $\liminf |\mathrm{supp}_{\Delta_i}(g)|/|\Delta_i| > 0$ for all $1 \neq g \in G$.

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Remark

If $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-group, then we can express $G = \bigcup_{\ell \in \mathbb{N}} G(\ell)$ as an increasing union of full limits $G(\ell)$.

Sketch Proof for Full Limits

- Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and that $\chi \neq \chi_{\mathrm{reg}}, \chi_{\mathrm{con}}$ is a nontrivial indecomposable character of G.

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- Hence there exists an infinite $I \subseteq \mathbb{N}$ and a fixed $(\, \ell_2, \cdots, \ell_r \,)$ such that $\lambda_i = (\, n_i - d(\lambda_i), \ell_2, \cdots, \ell_r \,)$ for all $i \in I$.

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- Hence if $\Omega_i = \Delta_i^d$ and $G_i \curvearrowright \Omega_i$ is the product action, then

$$\chi(g) = \lim_{i \in I} \frac{|\operatorname{Fix}_{\Omega_i}(g)|}{|\Omega_i|}.$$

The Loeb Measure Construction

- For each $i \in I$, let μ_i be the uniform probability measure on Ω_i.
- Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $I \in \mathcal{U}$ and let $(X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ be the corresponding Loeb probablity space.

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