Actions of automorphism groups of omega-categorical structures on compact spaces.

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THEMES:

- Automorphism groups of countable ω-categorical structures acting on compact Hausdorff spaces.
- Connection with structural Ramsey theory (Kechris - Pestov - Todorčević Correspondence)
- Sparse graphs constructed using Hrushovski amalgamations exhibit interesting new phenomena.

PART 1: Describe the theory due to Kechris, Pestov, Todorčević and Nguyen Van Thé which, under the right conditions, makes use of structural Ramsey theory to give a good description of such actions.

PART 2: Show that the theory breaks down for the ω-categorical structures produced by Hrushovski constructions. (Joint work with Jan Hubička and Jaroslav Nešetřil .)

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Amalgamation classes and Fraïssé limits.

L a 1st-order relational language and $\mathbb{M}$ a countable L-structure.
$\mathrm{Age}(\mathbb{M})$: class of isomorphism types of finite substructures.

$\mathbb{M}$ is *homogeneous* if all isomorphisms between finite substructures of $\mathbb{M}$ extend to automorphisms of $\mathbb{M}$. In this case $\mathcal{C} = \mathrm{Age}(\mathbb{M})$ has the Amalgamation Property.

Conversely: if $\mathcal{C}$ is a countable class of isomorphism types of finite L-structures which is closed under taking substructures, has the joint embedding property and $\mathcal{C}$ has AP,
then there is a countable, homogeneous structure $\mathbb{F}(\mathcal{C})$ with $\mathrm{Age}(\mathbb{F}(\mathcal{C})) = \mathcal{C}$. It is unique up to isomorphism.

$\mathcal{C}$ is an *amalgamation class* and $\mathbb{F}(\mathcal{C})$ is its *Fraïssé limit*.

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Automorphism groups.

M infinite set (usually countable); $\mathrm{Sym}(M)$ symmetric group.

$G \leq \mathrm{Sym}(M) \subseteq M^M$ pointwise convergence topology.

Basic open sets: $\{g \in G : g|A = \gamma\}$, $A \subseteq M$ finite and $\gamma : A \to M$.

G is a topological group.
$\mathrm{Sym}(M)$ complete metrizable if M is countable.

Lemma

$G \leq \mathrm{Sym}(M)$ is closed iff $G = \mathrm{Aut}(\mathbb{M})$ for some 1st order structure $\mathbb{M}$ with domain M.

INTERESTING EXAMPLES: $\mathbb{M}$ countable homogeneous, or ω-categorical.

REMARK: If $G \leq \mathrm{Sym}(M)$ is closed there is a *homogeneous* relational structure $\mathbb{M}$ with $\mathrm{Aut}(\mathbb{M}) = G$ (but the language may have to be infinite).

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Topological Dynamics

G a topological group.

G-flow: compact, Hausdorff, non-empty space X with a continuous G-action.

A G-flow X is *minimal* if every G-orbit on X is dense.

FACT: (Ellis) There is a unique *universal* minimal G-flow, $M(G)$.

[If X is any G-flow, there is a cts G-map $M(G) \to X$.]

QUESTION: Can we describe $M(G)$?

REMARKS: (1) If G is locally compact but not compact, the answer is 'not in a meaningful way';
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G-flows

$G = \mathrm{Aut}(\mathbb{M})$. Some G-flows:

1. Consider $Y = \{0,1\}^{M^n}$ as a G-flow.
   Also consider G-invariant, closed subspaces X of Y.
2. G-invariant, closed subspaces of $St(\mathbb{M})$, Stone space over M.

EXAMPLES: (1) $G = \mathrm{Sym}(M)$. We have a G-flow:

$$LO(M) = \{R \subseteq M^2 : R \text{ is a linear order on } M\}.$$

This is minimal.

(2) Let $\mathbb{P}$ be the Fraïssé limit of the class of all finite partial orders. As above, $LO(P)$ is an $\mathrm{Aut}(\mathbb{P})$-flow. But it is not minimal - the linear orderings which extend the ordering on $\mathbb{P}$ form a subflow.

(3) $\mathbb{M}$ is the dense local order $S(2)$; $St(\mathbb{M})$ is an $\mathrm{Aut}(\mathbb{M})$-flow.

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Extreme amenability

Definition

Suppose G is a topological group.

1. G is *amenable* if every G-flow X supports a G-invariant Borel probability measure.
2. G is *extremely amenable* if every G-flow has a fixed point.

EXAMPLE: Suppose $H \leq \mathrm{Sym}(M)$ is e.a. Then H fixes a linear ordering on M.

THEOREM: (1) (Pestov, 1998) $\mathrm{Aut}((\mathbb{Q}; \leq))$ is e.a.
(2) (Glasner, Weiss, 2002) The universal minimal flow of $\mathrm{Sym}(M)$ is $LO(M)$.

Note that as a corollary to (2) we can see that $\mathrm{Sym}(M)$ is amenable.

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The Kechris - Pestov - Todorčević Correspondence

Theorem (KPT, 2005)

Suppose $\mathbb{M}$ is a countable, homogeneous, linearly ordered relational structures with age $\mathcal{A}$. TFAE:

1. $\mathrm{Aut}(\mathbb{M})$ is extremely amenable.
2. $\mathcal{A}$ is a Ramsey class.

So Ramsey classes correspond to homogeneous structures with e.a. automorphism groups.

Ramsey classes

$L^{\leq}$: relational language with $\leq$.
$\mathcal{A}$: a class of finite $L^{\leq}$-structures closed under substrs and satisfying JEP and where $\leq$ is a linear ordering.

DEFINITION: Say that $\mathcal{A}$ is a Ramsey class if whenever $A \subseteq B \in \mathcal{A}$, there is $B \subseteq C \in \mathcal{A}$ such that if

$$\gamma : \binom{C}{A} \to \{0, 1\}$$

is a 2-colouring of the copies of A in C, there is $B' \in \binom{C}{B}$ (a copy of B in C) such that γ is constant on $\binom{B'}{A}$.

EXAMPLES: (1) $L = \{\leq\}$. Take $\mathcal{A} =$ finite linear orders.
(2) (Nešetřil - Rödl) The class $\mathcal{G}^{\leq}$ of linearly ordered finite graphs.

THEOREM: (Nešetřil) If $\mathcal{A}$ is a Ramsey class, then $\mathcal{A}$ has the amalgamation property.

COR: The automorphism group of $\mathbb{F}(\mathcal{G}^{\leq})$ is e.a.

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Computing $M(G)$ (KPT, 2005; Nguyen Van Thé 2013)

- L relational;
- $\mathbb{M}$ a countable homogeneous L-structure;
- $G = \mathrm{Aut}(\mathbb{M})$;
- $\mathcal{A}$ the age of $\mathbb{M}$.

DEF: Say closed $H \leq G$ is *coprecompact* (in G) if for every G-orbit $\Delta \subseteq M^n$, H has finitely many orbits on Δ.

ASSUME: G has a closed coprecompact extremely amenable subgroup H.

Think of H as $\mathrm{Aut}(\mathbb{M}^*)$ for some homogeneous L^*-structure $\mathbb{M}^*$, where $L^* \supseteq L$ is relational. Let $\mathcal{A}^*$ be the age of $\mathbb{M}^*$.

NOTE: Coprecompactness of H in G means that each $C \in \mathcal{A}$ has finitely many expansions in $\mathcal{A}^*$.

Computing $M(G)$ ....

DEF: Let $X(\mathcal{A}^*)$ be the set of expansions of $\mathbb{M}$ to L^*-structures in which the induced structure on each finite subset is in $\mathcal{A}^*$.
Topology: basic open set - all expansions agreeing on a particular finite subset.

– This is a G-flow. Moreover:

1. Any minimal subflow of $X(\mathcal{A}^*)$ is isomorphic to $M(G)$;
2. The G-orbit containing $\mathbb{M}^*$ is comeagre in $X(\mathcal{A}^*)$;
3. H may be chosen so that $X(\mathcal{A}^*)$ is minimal;
4. Every minimal G-flow has a comeagre orbit.
5. $M(G)$ is metrizable.

EXAMPLE: If $\mathbb{M}$ is the random graph, then $M(G) = LO(\mathbb{M})$.

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Aside: The converse

Work of Zucker, Melleray - Nguyen Van Thé - Tsankov; Ben Yaacov - Melleray - Tsankov (2014-15) shows there is a converse:

If $G = \mathrm{Aut}(\mathbb{M})$ has a metrizable universal minimal flow, then G has a coprecompact closed extremely amenable subgroup.

Question

- Question asked (around 2011) by: Bodirsky, Pinsker, Tsankov; Nešetřil; Nguyen Van Thé:
  - If $\mathbb{M}$ is countable ω-categorical, is there an ω-categorical expansion $\mathbb{M}^*$ of $\mathbb{M}$ with $\mathrm{Aut}(\mathbb{M}^*)$ extremely amenable? Equivalently, is there a coprecompact e.a. closed subgroup of $\mathrm{Aut}(\mathbb{M})$?
- Second part of talk: An example where this does not happen.
- Particularly interesting case: $\mathbb{M}$ homogeneous in a finite relational language - still open.
- Why ask the question?
  - Ubiquity of ω-categorical structures with e.a. automorphism groups
  - Ubiquity of Ramsey classes
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Sparse graphs.

DEF: Suppose $k \in \mathbb{N}$. A graph $M = (M; E)$ is *k-sparse* if for all finite $A \subseteq M$ we have $|E[A]| \leq k|A|$.

FACT: If the graph $M = (M; E)$ is k-sparse, then it is *k-orientable*: the edges of M can be directed so that each vertex has at most k directed edges coming out.

DEF: If M is k-sparse, let

$$X(M) = \{D \subseteq M^2 : (M; D) \text{ is a } k\text{-orientation of } M\} \subseteq \{0,1\}^{M^2}.$$

Note that this is an $\mathrm{Aut}(M)$-flow.

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FACT: (Hrushovski) There is an ω-categorical 2-sparse graph M_F with all vertices of infinite valency.

Theorem A (DE, Jan Hubička and Jaroslav Nešetřil)

Suppose M is a countable, k-sparse graph of infinite valency. If $H \leq \mathrm{Aut}(M)$ is amenable, then H has infinitely many orbits on M^2.

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Proof of Thm A: Step 1

- Suppose M is a graph with all vertices of infinite valency and $H \leq \mathrm{Aut}(M)$ has finitely many orbits on M^2.
- If $c \in M$ let H_c denote the stabilizer of c in H.
- For $c \in M$ let $\mathrm{cl}(c)$ be the union of the finite H_c-orbits on M.
- There is $n \in \mathbb{N}$ s.t. $|\mathrm{cl}(c)| \leq n$ for all $c \in M$.
- If $b \in \mathrm{cl}(c)$ then $\mathrm{cl}(b) \subseteq \mathrm{cl}(c)$.
- STEP 1: There are adjacent $a, b \in M$ such that b is in an infinite H_a-orbit and a is in an infinite H_b-orbit.

PROOF: Suppose there do not exist such a, b. Then for every edge a, b in M either $a \in \mathrm{cl}(b)$ or $b \in \mathrm{cl}(a)$. Take b with $\mathrm{cl}(b)$ of maximal size. There is $a \notin \mathrm{cl}(b)$ adjacent to b. By assumption, $\mathrm{cl}(a) \supset \mathrm{cl}(b)$: contradiction.

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Proof of Thm A: step 2

- GIVEN: M is a k-sparse graph, $H \leq \mathrm{Aut}(M)$, and $a, b \in M$ are adjacent and such that a in an infinite H_b-orbit and b is in an infinite H_a-orbit.
- Show H is not amenable.
- Suppose there is an H-invariant probability measure μ on $X(M)$.
- Let $S(ab) = \{D \in X(M) : (a, b) \in D\}$. May assume $p = \mu(S(ab)) > 0$.
- Let $b_1, \ldots, b_n$ be in the same H_a-orbit as b and s_i the characteristic function of $S(ab_i)$. Note $\mu(S(ab_i)) = p$.
- For $D \in X(M)$,

$$\sum_{i \leq n} s_i(D) \leq k \text{ so } \int_{D \in X(M)} \sum_{i \leq n} s_i(D) d\mu(D) \leq k.$$

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QUESTION: (Bodirsky, ...) If $\mathbb{M}$ is a structure homogeneous for a finite relational language, is there a coprecompact e.a. subgroup $H \leq \mathrm{Aut}(\mathbb{M})$?

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