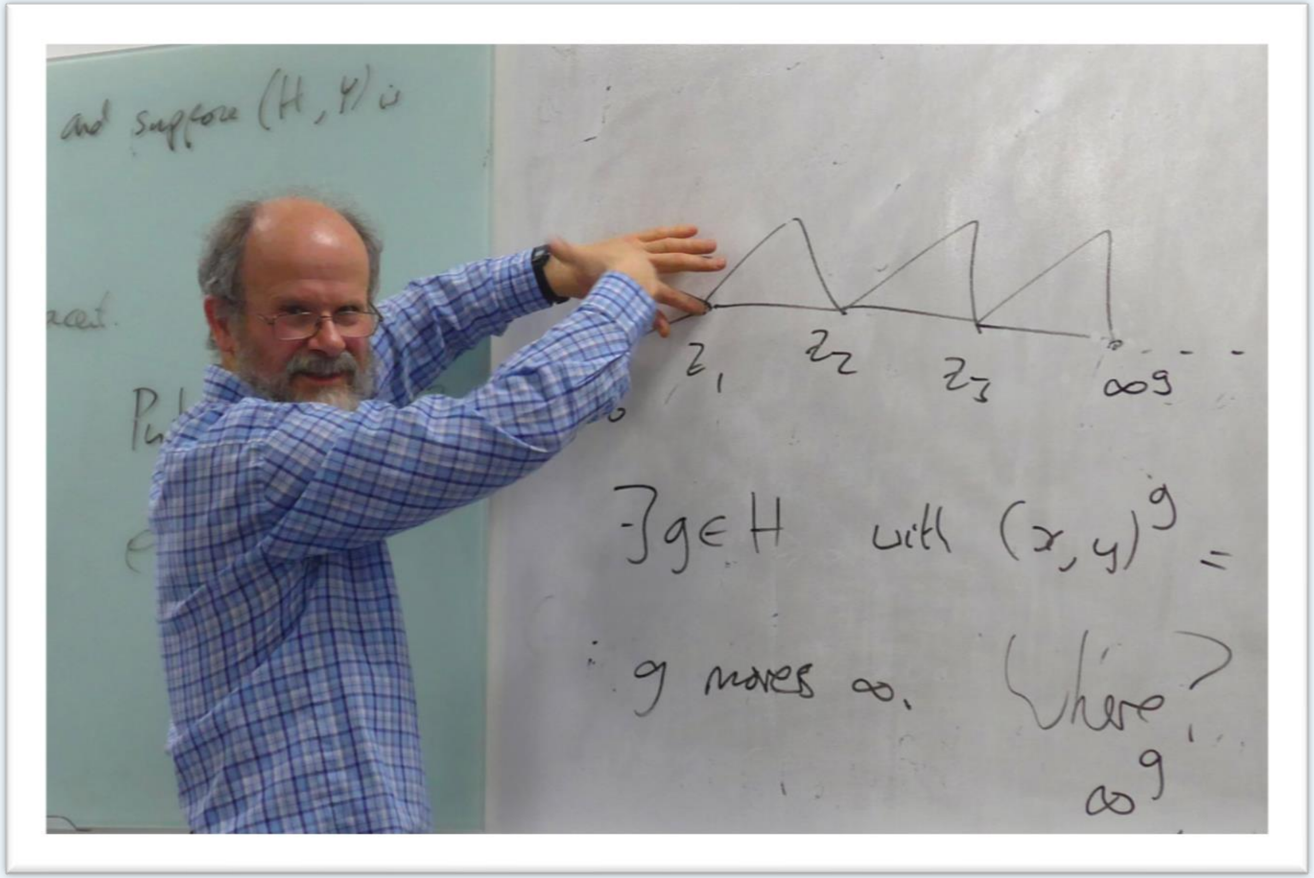
THE UNREASONABLE EFFECTIVENESS OF LOGIC

Richard Elwes

Public Talk as part of: "From permutation groups to model theory: a workshop inspired by the interests of Dugald Macpherson, on the occasion of his 60th birthday."

UNIVERSITY OF LEEDS

Happy Birthday Dugald!



A Foundational Crisis!

Can mathematicians be confident in the validity of their proofs?

Georg Cantor (1845-1918)

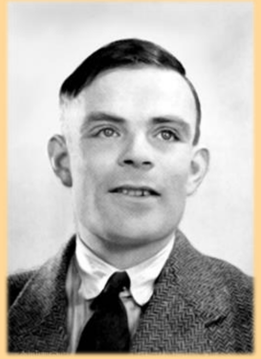
Gottlob Frege (1848-1925)

David Hilbert (1862-1943)

Bertrand Russell (1872-1970)

Alfred North Whitehead (1861-1947)

Kurt Gödel (1906-1978)



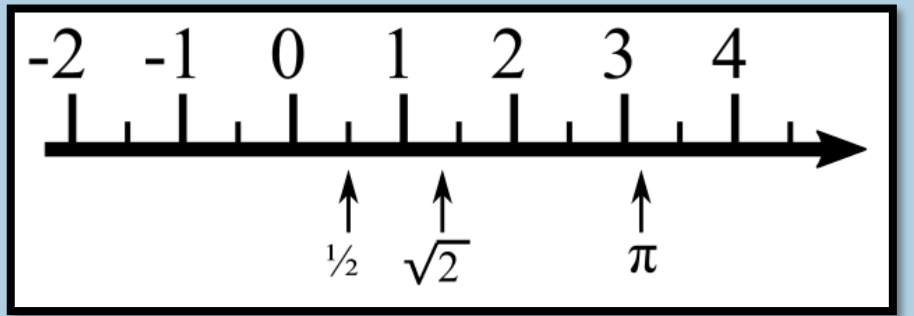
Alan Turing (1912-1954)

Outcome: arithmetic $(\mathbf{Z}, +, \times, -)$ is logically **wild**.

Some Good News

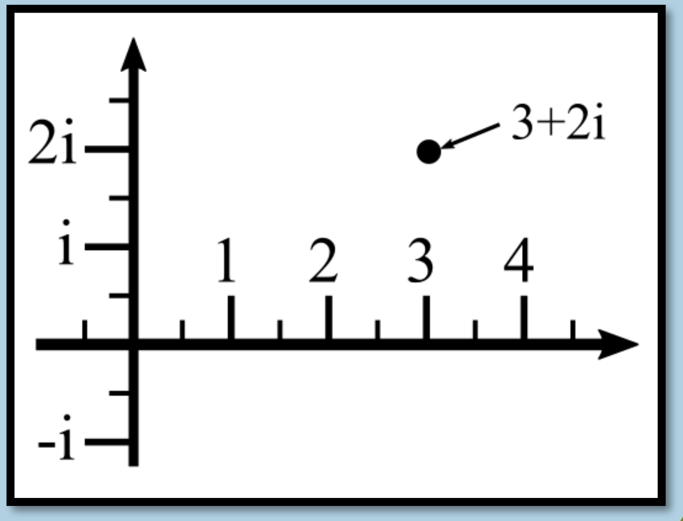
Other important structures are logically **tame**.

- The field of real numbers $(\mathbf{R}, +, -, \times, \div)$



- The field of complex numbers $(\mathbf{C}, +, -, \times, \div)$

Alfred Tarski
(1901-1983)



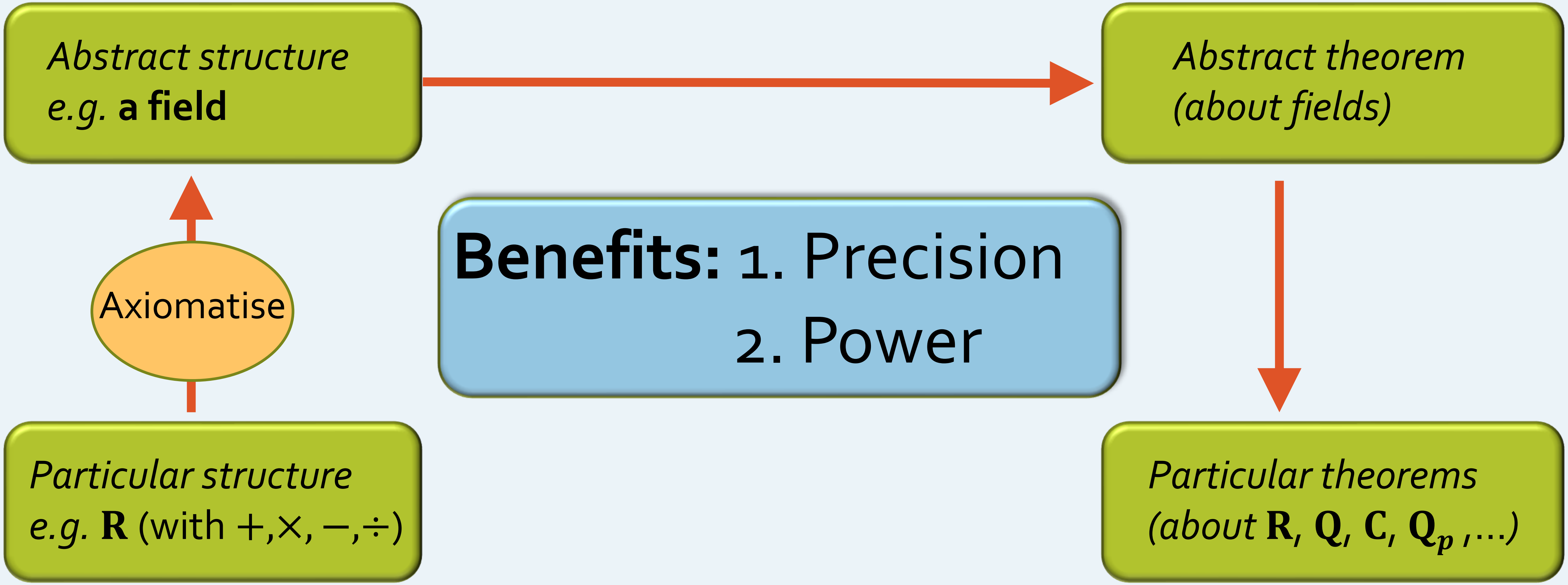
A new branch of logic

Model Theory *is "the Geography of Tame Mathematics"*

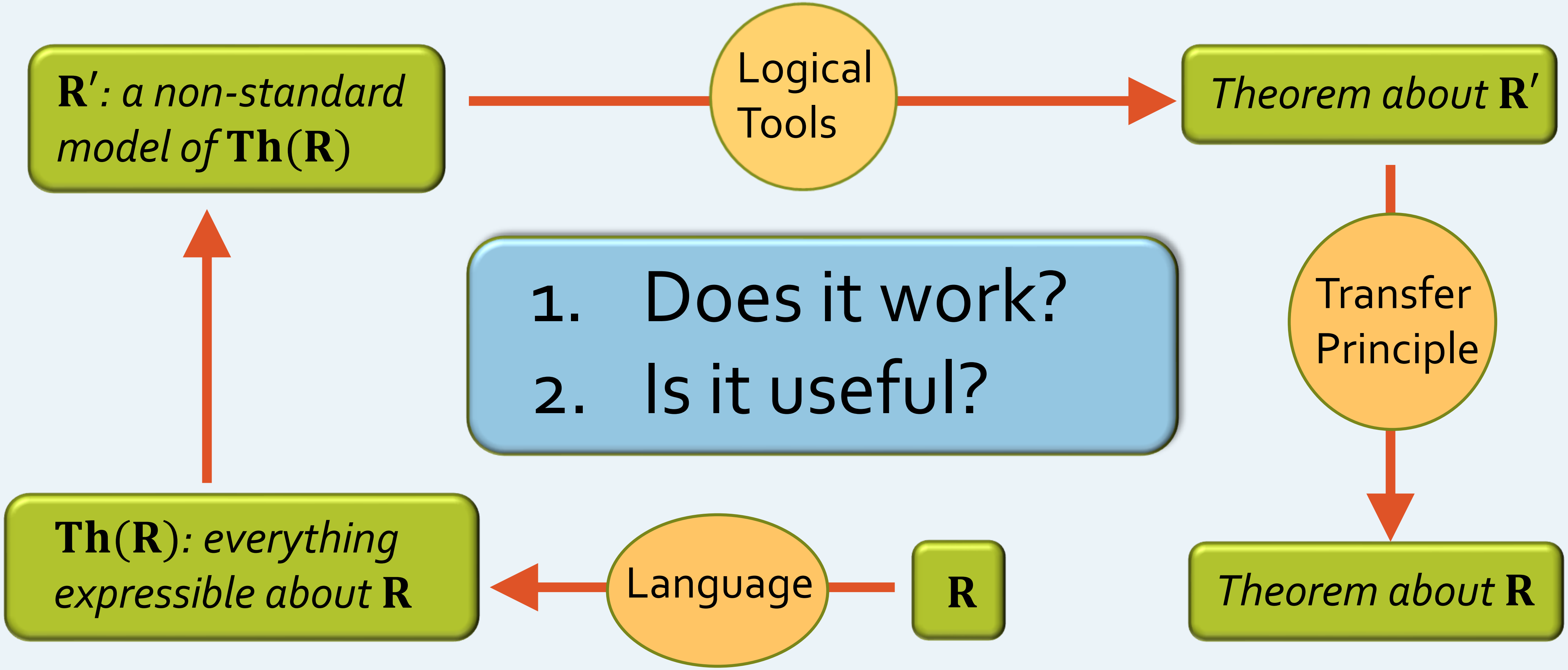
– Ehud Hrushovski

Question: does logical tameness yield useful mathematics?

Abstract Algebra



Model-Theoretic Algebra



1. Does it work?
2. Is it useful?

Example: Hilbert's 17th Problem

A real rational function is non-negative ⟺ it is a sum of squares.

First proof by Artin (1927). Model-theoretic proof by Abraham Robinson (1955).

David Hilbert
(1862-1943)

Emil Artin
(1898-1962)

Abraham Robinson
(1918-1974)

Example: Hilbert's 17th Problem

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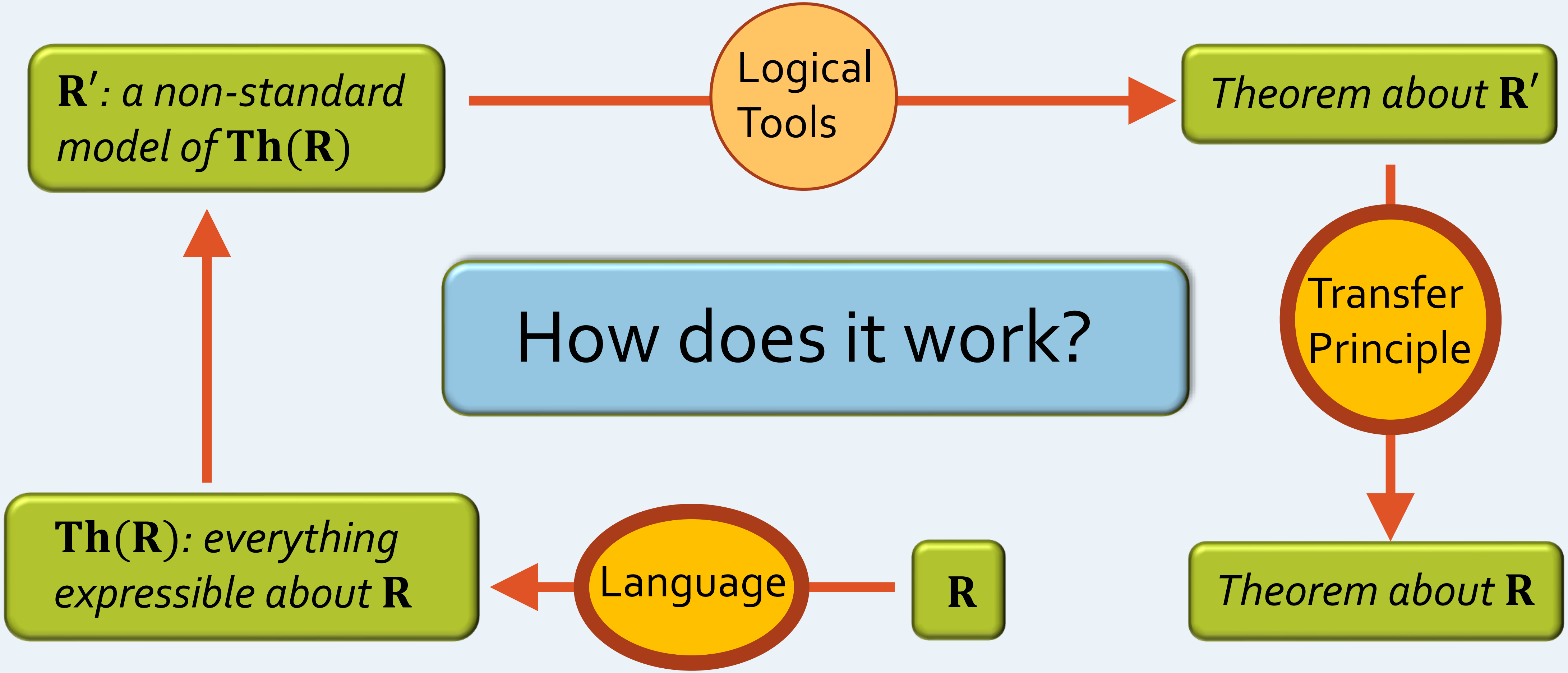
Terminology

- **Polynomial**, e.g. $(x, y) \to x^2 + 3y + 2$.
- **A rational function** is a fraction of polynomials, e.g. $(x, y) \to \frac{x^2y^2}{x^2+1}$.
- **Sum of squares**, e.g. $\frac{x^2y^2}{x^2+1} = \left(\frac{x^2y}{x^2+1}\right)^2 + \left(\frac{xy}{x^2+1}\right)^2$.

Sketch proof: Suppose $f(x, y, \ldots, z)$ is not a sum of squares.

Build a non-standard model $\mathbf{R}' = \mathbf{R}(x, y, \ldots, z)$ in which $f(x, y, \ldots, z) < 0$.

By a transfer principle, $\mathbf{R}$ also contains elements $a, b, \ldots, c$ where $f(a, b, \ldots, c) < 0$. ■


$\mathbf{R}'$: *a non-standard model of* $\mathbf{Th}(\mathbf{R})$
Logical Tools
Theorem about $\mathbf{R}'$
How does it work?
Transfer Principle
$\mathbf{Th}(\mathbf{R})$: *everything expressible about* $\mathbf{R}$
Language
$\mathbf{R}$
Theorem about $\mathbf{R}$

Transfer Principle & Choice of Language

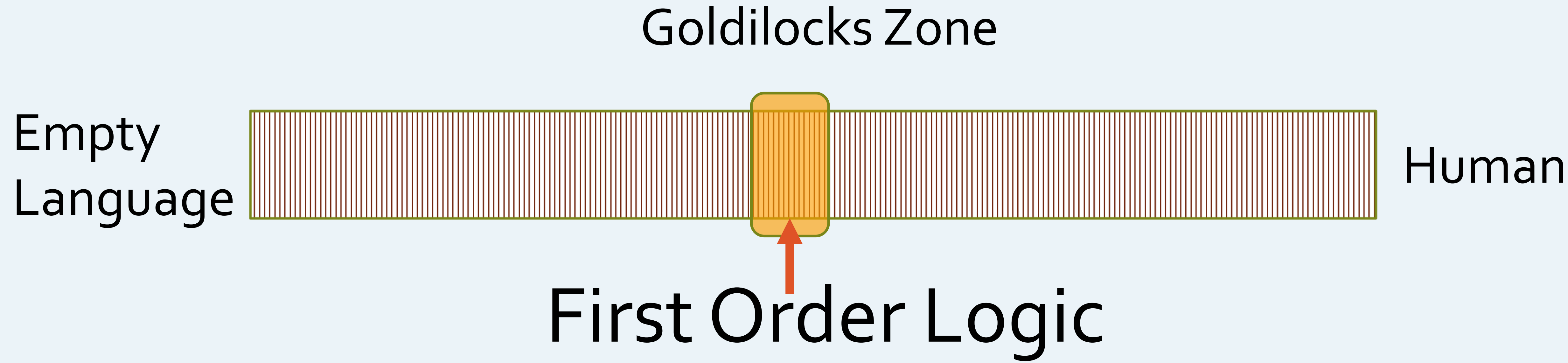
More Expressive Language
Fewer, more similar models
Transference is easier

Empty Language

Human

More Flexible Language
More & more diverse models
Transference is more useful

Transfer Principle & Choice of Language




David Hilbert (1862-1943)


Wilhelm Ackermann (1896-1962)


Thoralf Skolem (1887-1963)


Leopold Löwenheim (1878-1957)


Hermann Weyl (1885-1955)


John von Neumann (1903-1957)

First Order Logic

<u>Restrictions:</u>

1. Specify in advance the mathematical features of interest (e.g. $+, \times, -, \div$).

2. Expressions can't be infinitely long.

3. We can say *"for every element"* but <u>not</u> *"for every subset/ sequence/ function"*. (Similarly for "*there exists*".)

Properties of First Order Logic

Anything Which Can Happen Does Happen ***(somewhere)***

Gödel's Completeness Theorem (1929)

syntax $\equiv$ *semantics*

logical consistency $\Leftrightarrow$ *model existence*

This fails for (e.g.) second order logic.

Kurt Gödel
(1906-1978)

Properties of First Order Logic

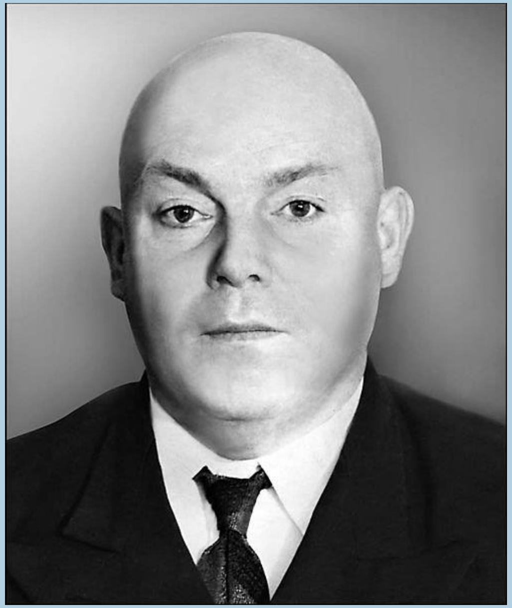
Anything Can Happen *(unless there's a good reason why not)*

Compactness Theorem (Gödel 1930, Maltsev 1936):

consistency $\Leftrightarrow$ *finite consistency*

This also fails for (e.g.) second order logic.

E.g. it is consistent with $\mathrm{Th}(\mathbf{R})$ that there is an infinite element.



Anatoly Maltsev
(1909-1967)

Properties of First Order Logic

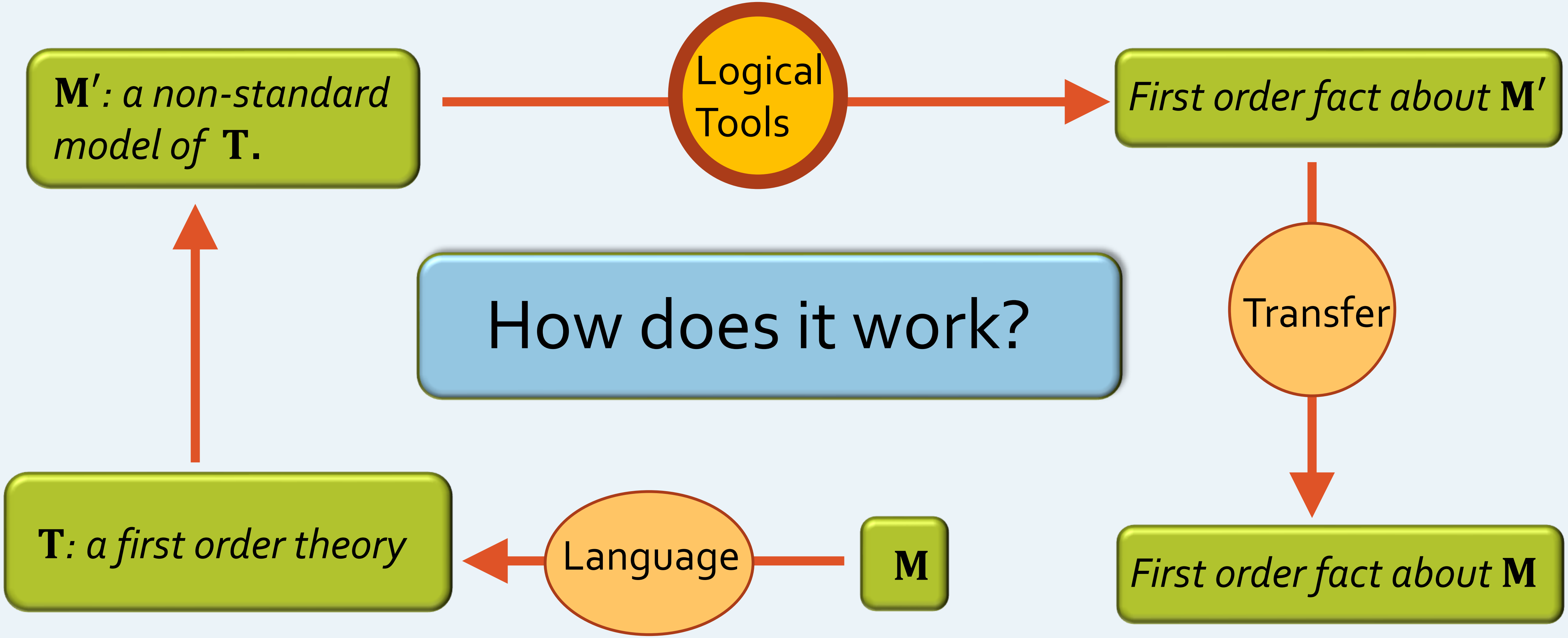
Everything That Can Happen Happens In The Same Place

Saturated models (Morley & Vaught, 1960)

E.g. a saturated model of $\mathrm{Th}(\mathbf{R})$ contains infinite elements.

Michael D. Morley

Robert Vaught
(1926-2002)



$\mathbf{M}'$: *a non-standard model of* $\mathbf{T}$.
Logical Tools
First order fact about $\mathbf{M}'$
How does it work?
Transfer
$\mathbf{T}$: *a first order theory*
Language
$\mathbf{M}$
First order fact about $\mathbf{M}$

Logical Tools: Universal Domains

Example: $x^2 = 2$ is defined over $\mathbf{Q}$, has a solution in $\mathbf{R}$, but not in $\mathbf{Q}$.

Fundamental Theorem of Algebra (Argand, 1806): *a polynomial equation over $\boldsymbol{C}$ which has a solution anywhere, already has one in* $\mathbf{C}$.

Algebraists: "$\mathbf{C}$ is an *algebraically closed field*" ($\mathbf{ACF}$).

Logicians: "$\mathbf{C}$ is *existentially closed* in the language of fields".

Or: "$\mathbf{C}$ is a *universal domain* for polynomial algebra."

Logical Tools: Universal Domains

Universal Domains for enriched fields:

- **RCF (real closed fields):** universal domains for *ordered* fields.
- **DCF (differentially closed fields):** universal domains for differential algebra.
- **ACFA (algebraically closed fields with a generic automorphism)**: universal domains for difference algebra.
- **ACVF (algebraically closed valued fields)**

The Origins Of Stability

***A Logician's Principle:** If a language is ambiguous, there is something special about structures it can pin down (essentially) unambiguously.*

- Morley (1965) analysed such **Uncountably Categorical** structures. Zilber & Morales (2017) call them **Logically Perfect**. E.g.
  - Vector Spaces (Linear Geometry)
  - **ACF** (Algebraic Geometry)

- Shelah (1990) solved a major **Classification Problem** on the number of models a first order theory can have.

"On any reckoning this is one of the major achievements of mathematical logic since Aristotle." – Wilfrid Hodges (1997)

Saharon Shelah

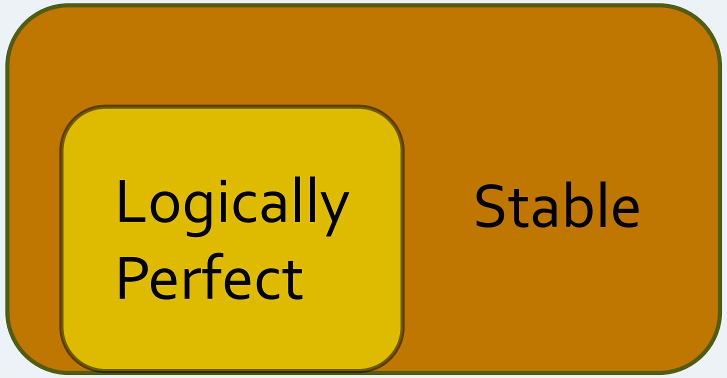
Stability

Key to Shelah's work are notions of **dimension** and **independence**, highly reminiscent of ideas from linear & algebraic geometry.

*'Perhaps the most remarkable feature of model-theoretic classification theory is that it exposes a **geometric** nature of "perfect" structures.'*

– Zilber & Morales (2017)

Structures with these geometric features are called **stable**.



Stability

Unfortunately…

"Not many interesting mathematical structures are stable."

- Enrique Casanovas (2000)

Stability

Or maybe….

"Not many ~~interesting~~ core mathematical structures are stable."

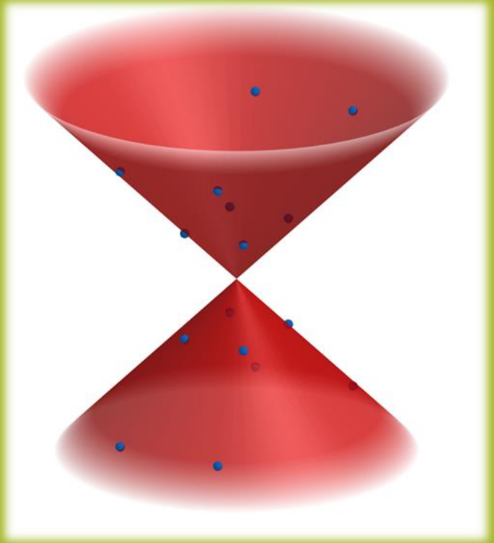
Exceptions: Vector Spaces, $\mathbf{ACF}$…

Plus: Planar Graphs, Modules, Free Groups, Abelian Groups, Algebraic Groups over $\mathbf{ACF}$, Separably Closed Fields,…

Also: certain *auxiliary* structures, notably **Differentially Closed Fields** ($\mathbf{DCF}$).

Application: Diophantine Geometry

Diophantine questions initially look for integer solutions to polynomial equations (e.g. $x^2 + y^2 = z^2$).



"Ehud Hrushovski has achieved a remarkable synthesis… connecting geometric stability theory to diophantine questions… One meeting ground is the subject of differentially closed fields."

- Lou Van Den Dries (2007)

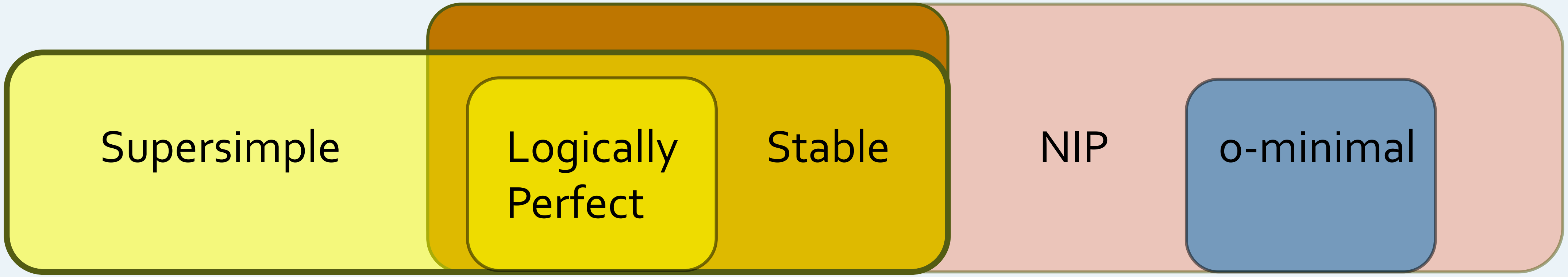
Ehud Hrushovski

Neo-Stability

"Many interesting mathematical structures are neo-stable."

Stability-style techniques now extend far beyond stable structures.

- The Erdős–Rényi graph is *supersimple* [Shelah, Kim, Pillay]
- The p-adic field is *NIP* *(Not the Independence Property)* [Matthews (1993)]
- The real exponential field is *o-minimal* [Wilkie (1996)]
- The ring of all algebraic integers is *real rosy* [van den Dries, Macintyre, Ealy, Onshuus]



Neo-Stability

Neo-Stability also covers many auxiliary structures & universal domains:

- ACFA is *supersimple* [Chatzidakis, Hrushovski (1999)]
- ACVF is *metastable* [Haskell, Hrushovski, Macpherson (2008)]
- Pseudofinite fields are *supersimple* [Chatzidakis, van den Dries, Macintyre (1992)]

Zoe Chatzidakis

Dugald Macpherson (1958-)

Deirdre Haskell

Angus Macintyre

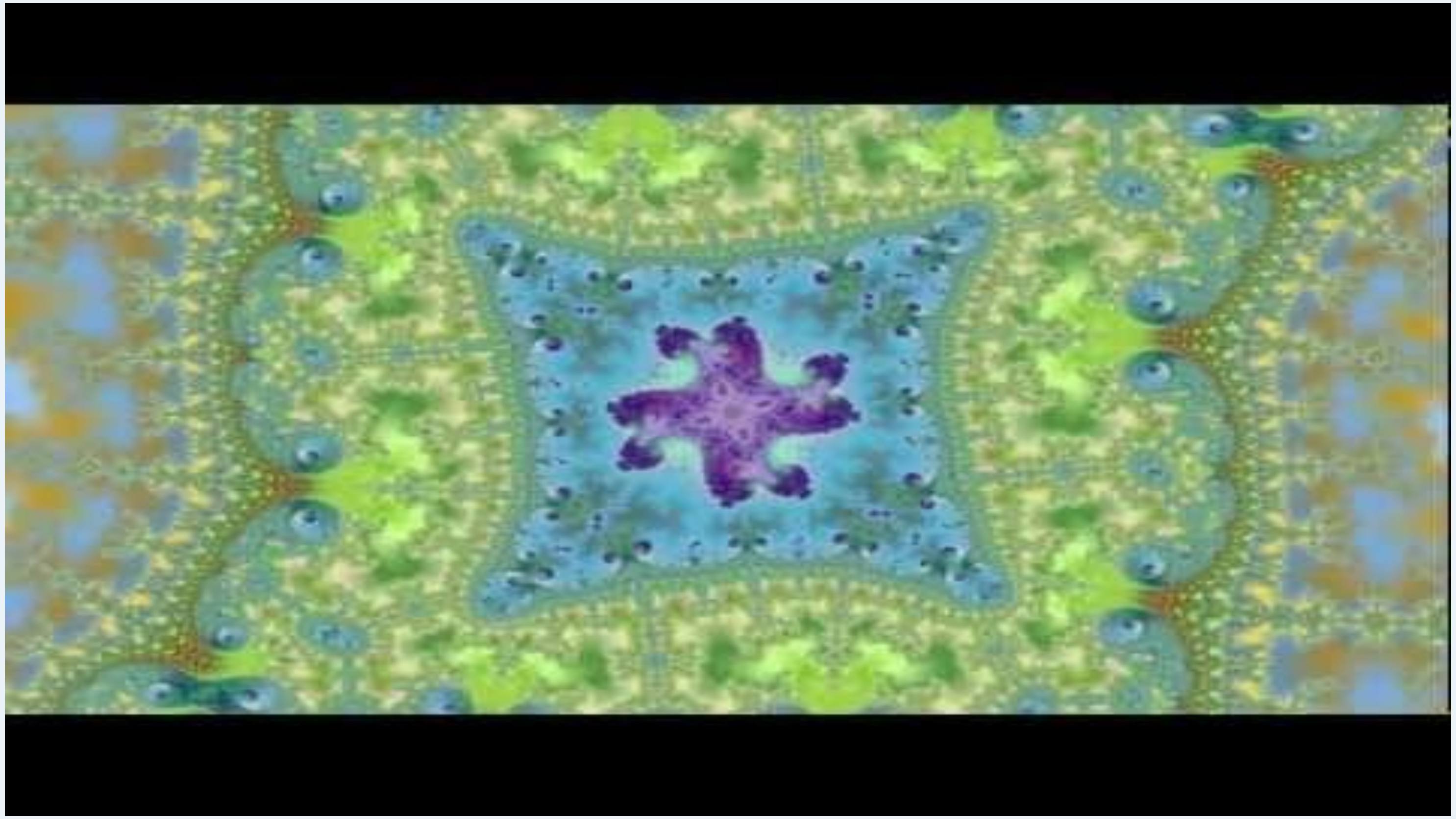
Neo-Stability

Some amazing applications:

- Applications of o-minimality to the André-Oort conjecture
  [Pila (2009)]
- Applications of stable group theory to additive combinatorics
  [Hrushovski (2012)]
- Rationality of p-adic Poincaré series
  [Denef (1984)]
- Pseudo-exponentiation & Schanuel's Conjecture
  [Zilber (2005)]

Algebraic Dynamics

An *algebraic dynamical system* repeatedly applies a polynomial function.
The results can be famously complicated…



The **Julia Set** of

$$z \to z^2 - 0.55 + i0.63$$

Video credit: Youtube @CommandLineCowboy

Algebraic Dynamics

Algebraic Dynamics (Medvedev & Scanlon, 2014)
Given a 'nice' polynomial function $f\colon \mathbf{C}^n \to \mathbf{C}^n$, they precisely characterise the algebraic sets fixed by f.

Alice Medvedev

Thomas Scanlon

Algebraic Dynamics

Algebraic Dynamics (Medvedev & Scanlon, 2014)
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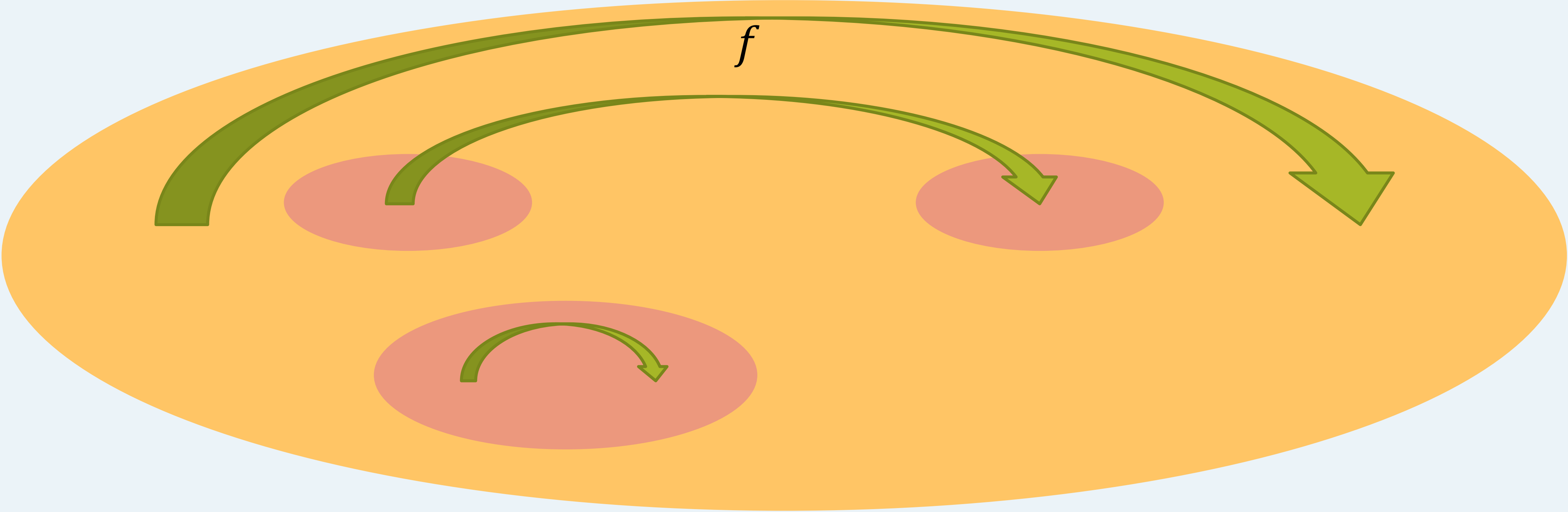
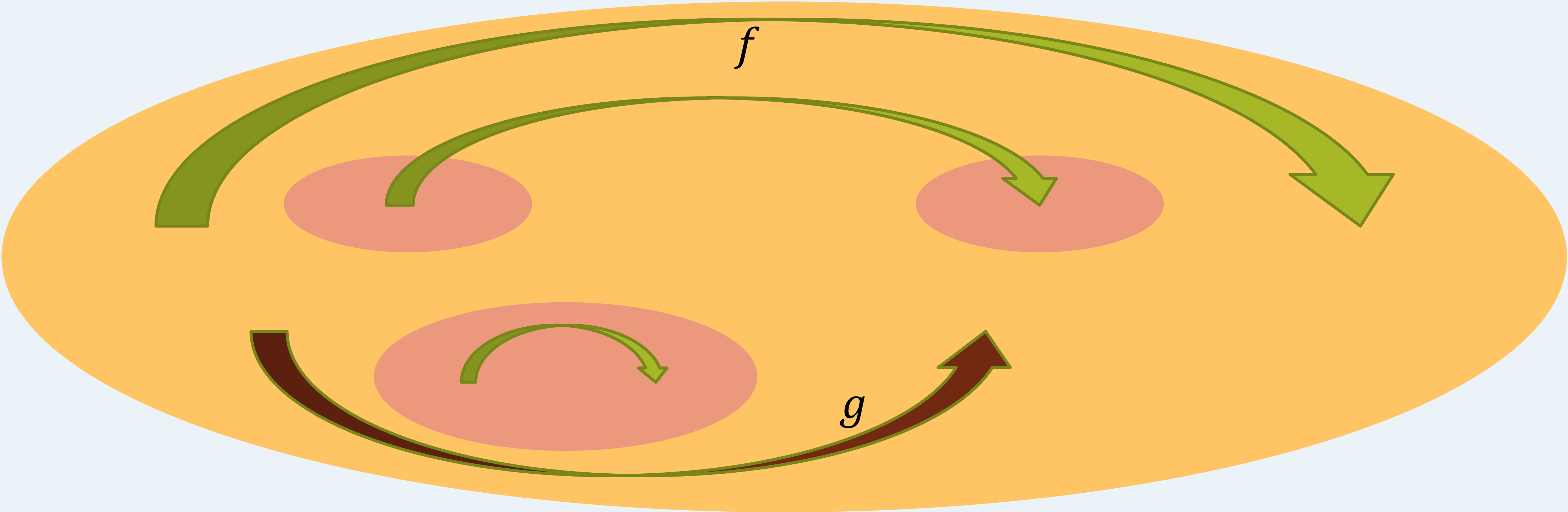
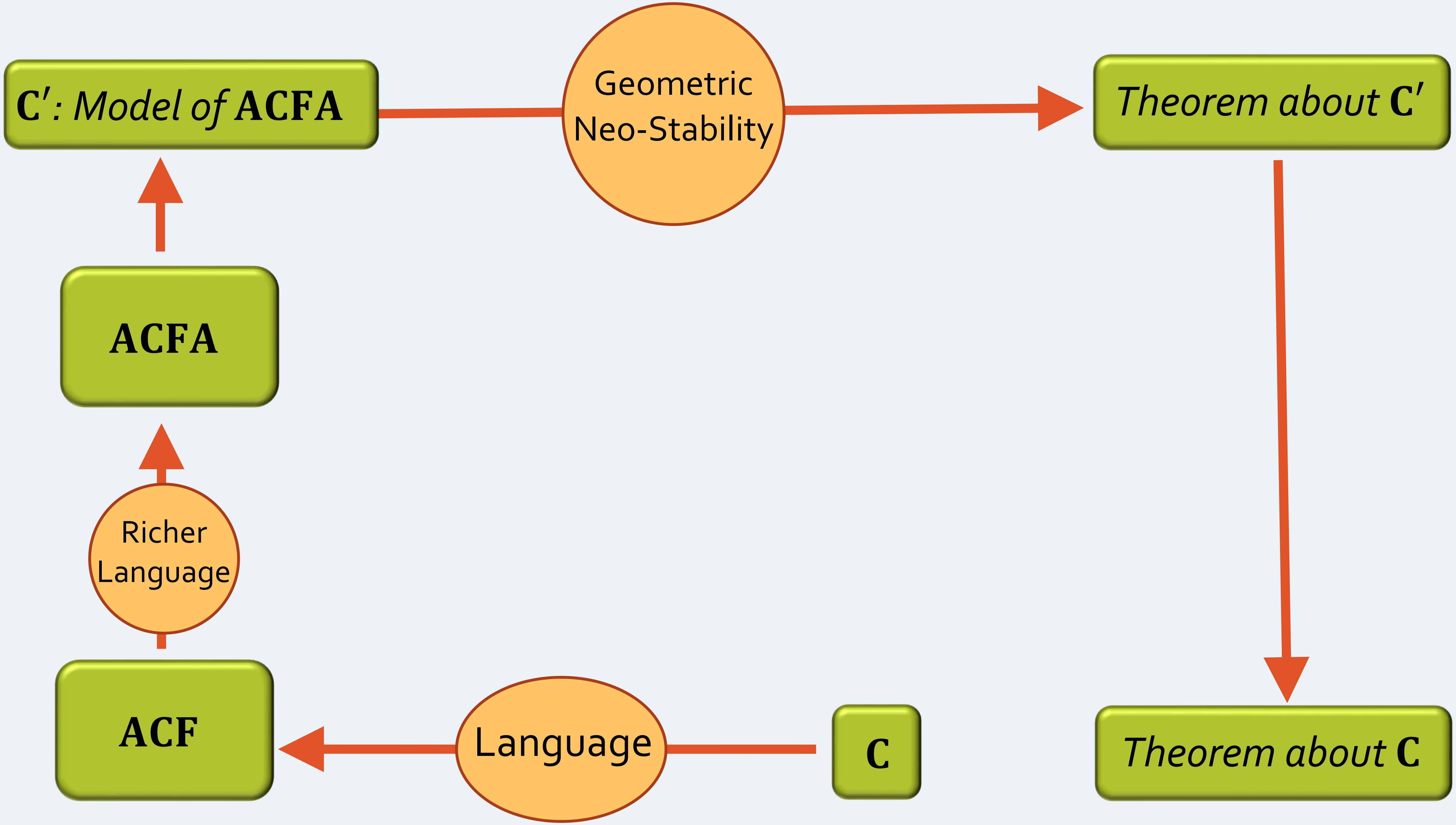


Algebraic Dynamics

Algebraic Dynamics (Medvedev & Scanlon, 2014)
The proof introduces a new function g to get an $\mathbf{ACFA}$.




$\mathbf{C}'$: *Model of* **ACFA**
Geometric
Neo-Stability
Theorem about $\mathbf{C}'$
ACFA
Richer
Language
ACF
Language
$\mathbf{C}$
Theorem about $\mathbf{C}$

Final Thoughts

Prediction: In the next decade, a widening range of mathematical areas will see major progress with the incorporation of a widening range of logical and model-theoretic ideas…

…cross-pollinated with native methods.

Opinion: This is not the only reason to study logic!

THANK YOU!

Richard Elwes, 2018

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