Relational Complexity of a Finite Primitive Structure

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Contents

There and back again

Structure	*Permutation Group*
A	G

There and back again

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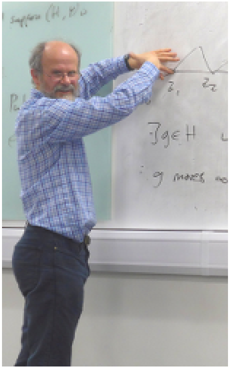
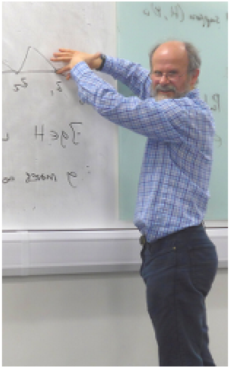
A	$\xrightarrow{\text{Aut}}$	G



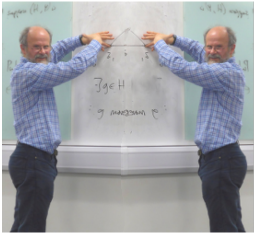
There and back again

Structure		*Permutation Group*
A	$\xleftarrow{A^k/G}$	G



There and back again

Structure		*Permutation Group*
A	$\leftrightarrow$	G



Remark

A is homogeneous in the canonical language. (Orbits are isomorphism types.)

Example

$$\frac{A \qquad\qquad G}{C_n \qquad\qquad D_{2n}}$$

L_2: path metric $\boxed{d(x,y) = i}$

Example

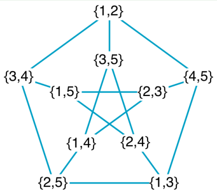
A	G
C_n	D_{2n}

L_2: path metric $\boxed{d(x,y)=i}$

- *k*-closed: $G = \mathrm{Aut}(A{\restriction}L_k)$
- L_k-homogeneous: L_k-isomorphism types determine G-orbits

k-closure and homogeneity

Example (Petersen Graph)



$\mathrm{Aut}(P) = \mathrm{Sym}(5)$ (2-closed).

L_3-homogeneous.

k-closure and homogeneity

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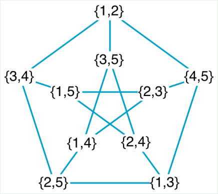

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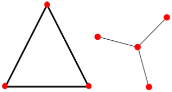
L_3-homogeneous.

Independent triples:

$\{1,2\}$, $\{1,3\}$, $\{2,3\}$ (triangle); $\{1,2\}$, $\{1,3\}$, $\{1,4\}$ (star).



Relational Complexity

$$\rho(G) = \min(r : A{\restriction}L_r \text{ is } G\text{-homogeneous})$$

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rc-spectrum

$\{r \mid \exists (a_1, \ldots, a_r), (a'_1, \ldots, a'_r)$
Not G-conjugate
all proper restrictions G-conjugate$\}$

$$\rho(G) = \sup(\text{rc-spectrum})$$

Model Theoretic Background

Lachlan Homogeneous for a finite relational language
ρ bounded
A^ρ/G bounded.
(Stability theory)

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Generalization: A^4/G bounded.
Kantor-Liebeck-Macpherson Classified in the primitive case.
Classical or semi-classical geometries.
C-H Structure theory based on the primitive classification
(neostability theory)

Questions for the primitive case

- What can we say about A if ρ is bounded?
- What can we say about ρ (and possibly the spectrum) when A is "natural?"
- What is the meaning of gaps in the spectrum?

A few more examples

1. $\mathrm{SL}_n < G \leq \mathrm{GL}_n$: $n+1$ (linear algebra)
   - SL_n: n
   - $\mathrm{ASL}_n < G \leq \mathrm{AGL}_n$: $n+2$ unless $n=1$, $G = D_{2\cdot q}$
2. $O^{\pm}(n,q)$, $q \neq 2$: $\begin{cases} n & \text{isotropic} \\ 2 & \text{anisotropic} \end{cases}$

   (linear algebra or inner products)
3. P^1: 4 (cross ratio)
4. $(\mathcal{P}([n]), \mathrm{Sym}(n))$: $\lfloor \log_2 n \rfloor + 1$

   "$|\alpha(\bar{S})| = i$" α a Boolean atom

Small ρ: $\rho = 2$

Conjecture (Binary Conjecture)

The (finite) primitive binary structures are

- $\vec{C}_p$ *(regular action)*
- $\mathrm{Sym}(n)$ *(theory of equality)*
- $AO(n,q)$ *anisotropic*

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Cherlin, Wiscons: reduced to almost simple case
(Very dependent on the value $\rho = 2$)

Almost Simple Case

Gill, Spiga, Dalla Volta, Hunt, Liebeck

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Theorem (Gill, Spiga)

The Binary Conjecture holds for alternating socle.

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The Binary Conjecture holds for alternating socle.

The easy cases:

- $\mathrm{Sym}(n)$ on k-sets: $\lfloor \log_2 k \rfloor + 2$
  (bounded family, but not usually 2)
- $\mathrm{Sym}(n = n_1 n_2)$ on partitions of shape $n_1 \times n_2$: At least

$$\max(n_1, \lfloor \log_2 2(n_2 - 1) \rfloor)$$

Alternating Socle: Primitive Point Stabilizer

The hard case
Primitive point stabilizer $M = G_*$

Key device: Elements of M have few fixed points on $[n]$

If M has an element of order 4 with a fixed point

	$M\backslash G$	$[n]$	
α	?	$(0)(1243)\cdots$	$\in M$
β	?	(01234)	$\notin M$

$\alpha = (0)(1243)\cdots \in M$. $\beta = (01234)$ not in M

$H = \langle \alpha, \beta \rangle \simeq \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, acting naturally on $\{0, 1, 2, 3, 4\}$.

If M has an element of order 4 with a fixed point

$$\begin{array}{cccc} & M\backslash G & [n] & \\ \alpha & (\tilde{0})(\tilde{1},\tilde{2},\tilde{4},\tilde{3})\cdots & (0)(1243)\cdots & \in M \\ \beta & (\tilde{0},\tilde{1},\tilde{2},\tilde{3},\tilde{4})\cdots & (01234) & \notin M \end{array}$$

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Binarity: G induces $\mathrm{Sym}(\tilde{O})$ on $\tilde{O}$.
In particular β has a conjugate β' such that $\beta\beta'$ is nontrivial and fixes $\tilde{0}$.
Return to $[n]$: Many fixed points, in M: contradiction!
(or n is not very large).

If M has an element of order 4 with no fixed point

Then many orbits of length 4 (α^2 has few fixed points).
Take 5 such orbits and make the regular representation of $H = \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, with β having exactly 4 orbits of length 5.

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Finish as before, working mostly in $M\backslash G$.

M has no element of order 4?

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Wander through the various possibilities for M, coming back to M almost simple by the same method.
Then use the classification of finite simple groups (or rather an early result in that direction).

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Then use the classification of finite simple groups (or rather an early result in that direction).
Exceptions occur:
E.g., $\mathrm{Sym}(p)$ on $\mathsf{AGL}(1, p)$
(and its restriction to $\mathrm{Alt}(p)$).

$\mathrm{Sym}(p)$ with stabilizer $\mathrm{AGL}(1, p)$

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(Similarly for $\mathrm{AGL}(1, p) \cap \mathrm{Alt}(p)$ once $p > 5$.)

Sporadic socle

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Theorem

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Most actions are explicitly known. Computation will reach a certain distance (and rather far if supported by a rich range of theoretical tests).
Again, the "small stabilizer" case arises, and the fact that one just needs to understand *one M-orbit* can be very helpful.
Notably, $M = \mathrm{Alt}_4 \times \mathrm{Sym}_5$ in Co_3, $(5:4) \times \mathrm{Alt}_5$ in Ru, where one finds $M \cap M^g =$ 2-Sylow for some g.

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Observation There are relatively few transitive binary actions as well, apparently and this can be remarkably useful in exploiting knowledge about the point stabilizer.

k-sets

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Why?

Relational spectrum

Spectrum: $\mathrm{Sym}(20)$ on 4-tuples: (2–4)
Spectrum: $\mathrm{Alt}(20)$ on 4-tuples: (2–4,8–17). *Both pieces derived from the action of* $\mathrm{Sym}(20)$

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Above $\rho^+ = \rho(k\text{-sets}, \mathrm{Sym}(n))$ the relational spectrum for $\mathrm{Alt}(n)$ on k-sets comes from sequences of k-sets which *just separate points* in $[n]$.
Namely $(X_1, \ldots, X_r)$ and its image under an odd permutation.

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Namely $(X_1, \ldots, X_r)$ and its image under an odd permutation.

Question

What is the longest sequence of k-sets which just separates points in $[n]$?

Just separating sequences

Proposition

Suppose there is a sequence of k-sets of length r which just separates points in $[n]$. Then there is a numerical partition of n into a sum of $n-r$ terms $n = \sum n_i$ with the following splitting property: if $n_i \geq 2$ and n_i is replaced by $(1, n_i - 1)$ then some subsum involving exactly one of these two terms sums to k.

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Application: Look for the shortest sum with the splitting property:

$k=1$	$n=n$	Length 1
$k=2$	$n=(n-1)+1$	Length 2
$n=2(k+1)$	$n=(k+1)+(k+1)$	Length 2
Else	$n=(k-1)+(k-1)+(\cdots)$	Length 3

Then reverse the analysis.

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The analysis: If we omit X_i, there is a pair (a_i, b_i) no longer separated.

This makes an acyclic graph with r edges, so $n-r$ components.

The sizes of the components are the n_i.

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To reverse, use stars and make the k-sets correspondingly (and check).

Cohorts?

This is a mechanism whereby low complexity for one group in a cohort may result in high complexity for smaller groups. But low complexity is not that common.

We will look at a more delicate case.

$\mathrm{Sym}(2n)$ and $\mathrm{Alt}(2n)$ on partitions: shape $n \times 2$

(2017-18, with Wiscons)

$$\rho^+(n \times 2) : n$$

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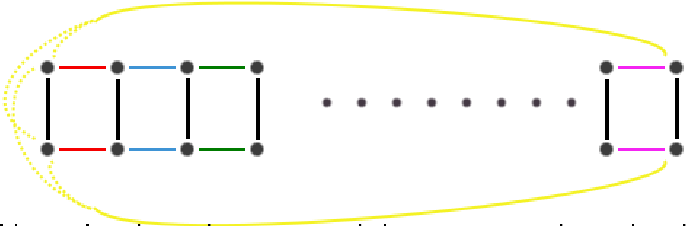
Möbius Band



Edge-colored graph: connected, but any two edge colors have small components.

$\mathrm{Sym}(2n)$ and $\mathrm{Alt}(2n)$ on partitions: shape $n \times 2$

(2017-18, with Wiscons)

$$\rho^+(n \times 2) : n$$

$$\rho^-(n \times 2) : \begin{cases} n+1 & n = 3 \\ n & n = 2, 4;\text{ or odd; or a multiple of } 6 \\ n-1 & n > 6 \text{ even, not a multiple of } 6 \end{cases}$$

(or so it seems)

$\mathrm{Sym}(2n)$ and $\mathrm{Alt}(2n)$ on partitions: shape $n \times 2$

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Some of this follows by direct inheritance from $\mathrm{Sym}(n)$:

- Inheritance for n odd: $\rho^- \geq \rho^+$ because when $n = n_1 + n_2$, one of the parts is odd (Möbius band)
- Sequences of partitions just separating points: $n - 1$ if $n > 2$.

Independent partitions of shape $n \times k$

Maximum sequences of partitions of shape $n \times k$ which just separate points.

$$\begin{cases} n(k-1) & \text{if } m = n = 2 \\ n(k-1) - 1 & \text{if } \min(n,k) = 2 \text{ and } \max(n,k) > 2 \\ n(k-1) - 2 & \text{if } n, k > 2 \end{cases}$$

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$nk = \sum n_i$. The splitting condition:

If $n_i \geq 2$ then the sum with n_i split to $1 + (n_i - 1)$ can be rearranged into n sums equal to k (with $1, (n_i - 1)$ separated).

Examples (Optimal)

$k^{n-2}(k-1)^2 1^2$	$k^{n-1}1^k$	$(k-1)^n 2 1^{n-2}$	$(k+1)1^{(n-1)k-1}$
$n+2$	$n+k-1$	$2n-1$	$(n-1)k$

General shapes

Conjecture

The relational complexity of $\mathrm{Alt}(nk)$ *on shape* $n \times k$ *is well approximated by* $n(k-1)-2$ *(and should always be at least that).*
The relational complexity of $\mathrm{Sym}(nk)$ *on shape* $n \times k$ *is typically much less (but not for* $k=2$*).*

Shape $2 \times k$

For $\mathrm{Alt}(2k)$ we expect $2k-3$.

Examples: $2 \times k$

k	$2k-3$	ρ^-
3	3	4
4	5	5
5	7	7
6	9	≥ 9

Shape $2 \times k$

Examples: $2 \times k$

k	$2k-3$	ρ^-	ρ^+ (L.B.)
3	3	4	*3 (2)*
4	5	5	5 (4)
5	7	7	*4 (4)*
6	9	≥ 9	6 (4)
7			*≥ 5 (4)*

For $\mathrm{Sym}(2k)$ there is a lower bound applying to the point stabilizer, namely

$$2\lfloor \log_2 k \rfloor$$

This may possibly be the true value for the point stabilizer when k is odd.

Problems I

Problem

Show that the relational complexity of $\mathrm{Sym}(nk)$ *acting on cosets of* $\mathrm{Sym}(k) \wr \mathrm{Sym}(n)$ *has relational complexity going to infinity with* n*.*

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Problem

Let $\rho_0(G) = \min(\rho(X, G) \mid \text{primitive})$.
Is this uniformly bounded for G simple?
If so, what is the minimum bound holding for almost all such G?

Problems II

Problem

Show that

$$\lim_{n \to \infty} \rho^+(n \times k)/n = c_k$$

for some explicit constant c_k ($<< k$?).

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$$\lim_{n\to\infty} \rho^{+}(n \times k)/n = c_k$$

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Problem

Determine the relational complexity of $\begin{bmatrix} n \\ k \end{bmatrix}^d$

($k = 1$: Saracino.)