Low-rank update of preconditioners for sequences of linear systems

Luca Bergamaschi, Department of Civil Environmental and Architectural Engineering
University of Padua, e-mail luca.bergamaschi@unipd.it

Abstract. Fast solution of large and sparse SPD linear systems by Krylov subspace methods is usually prevented by the presence of eigenvalues near zero of the coefficient matrix A. This is particularly true when computing the smallest or interior eigenvalues where, using Lanczos' or the Jacobi-Davidson approach, a system like $(A-\sigma I)x = r$ has to be repeatedly solved, with σ close to the wanted eigenvalue.

We propose and discuss how cost-effective spectral information on the coefficient matrix can be used to construct a spectral preconditioner, i.e. a low-rank modification of a given approximate inverse of A, $P_0 \approx A^{-1}$. The spectral preconditioner usually moves away from zero the smallest eigenvalues of the preconditioned matrices with a consequent, sometimes dramatic, reducing of the condition number and speeding up of the iterative process.

Given a linearly independent set of vectors $v_1, \ldots, v_p$, and defining $V_p = [v_1, \ldots, v_p]$, $\Lambda_p = V_p^T A V_p$, we will investigate the properties of the following low-rank updated preconditioners [5, 6] defined as

$$\begin{aligned} P &= P_0 + V_p \Lambda_p^{-1} V_p^T \\ P &= V_p \Lambda_p^{-1} V_p^T + (I - V_p \Lambda_p^{-1} V_p^T A) P_0 (I - A V_p \Lambda_p^{-1} V_p^T) \\ P &= P_0 - Z \left(Z^T A V_p\right)^{-1} Z^T, \qquad \text{where} \qquad Z = PAV_p - V_p. \end{aligned}$$

We will show that, especially in the case v_i-s are (rough) approximations of the leftmost eigenvectors of A, such preconditioners provide an important acceleration of

- iterative eigensolvers (see e.g. [3, 4])
- PCG for sequences of linear systems in the framework of the interior point method [1]
- very ill-conditioned sequences of linear systems [2] arising from discretizations of PDEs modeling optimal transport problems.

Some hints on parallel implementation of the algorithms (see e.g. [7] for some preliminary results on the eigensolution of large matrices) will be finally given.

References

[1] L. Bergamaschi, V. De Simone, D. di Serafino, and A. Martínez, *BFGS-like updates of constraint preconditioners for sequences of KKT linear systems*, Numer. Lin. Alg. Appl., (2018). Published online on Feb 6, 2018.

[2] L. Bergamaschi, E. Facca, A. Martínez, and M. Putti, *Spectral preconditioners for the efficient numerical solution of a continuous branched transport model*, (2018). Published online on Jan 31, 2018.

[3] L. Bergamaschi and A. Martínez, *Two-stage spectral preconditioners for iterative eigensolvers*, Numer. Lin. Alg. Appl., 24 (2017), pp. 1–14.

[4] ———, *Generalized block tuned preconditioners for SPD eigensolvers*, Springer INdAM Series, (2018). submitted.

[5] B. Carpentieri, I. S. Duff, and L. Giraud, *A class of spectral two-level preconditioners*, SIAM J. Sci. Comput., 25 (2003), pp. 749–765 (electronic).

[6] A. Martínez, *Tuned preconditioners for iterative SPD eigensolvers*, Numer. Lin. Alg. Appl., 23 (2016), pp. 427–443.

[7] A. Martínez and L. Bergamaschi, *Spectral acceleration of parallel iterative eigensolvers for large scale scientific computing*, in ParCo2017, Advances in Parallel Computing, 2018. to appear.