Regularity of minimal and CMC stable hypersurfaces

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May 30, 2018

Lecture plan:

- Some early work on regularity and existence of minimal hypersurfaces in Riemannian manifolds.
- Recent work on regularity of minimal and CMC hypersurfaces.
- An alternstive approach to the existence theory.
- Aspects of the proofs of recent regularity results (time permitting).

Early work: Schoen–Simon–Yau (1975), Schoen–Simon (1981).

A fundamental compactness theory for uniformly mass bounded stable minimal hypersurfaces of any given Riemannian manifold.

Schoen–Simon–Yau established this for low dimensions, and Schoen–Simon for general dimensions.

The methods employed in these two works were very different.

However both theories required the a priori assumption that the interior singular sets (i.e. the set of non-embedded points) of the hypersurfaces M are sufficiently small. In fact Schoen–Simon–Yau required that the singularities are completely absent; the more general Schoen–Simon theory required that

$$\mathcal{H}^{n-2}(\mathrm{sing}\, M) < \infty$$

where $n = \dim M$. Subject to this smallness hypothesis on the singular set, the work established curvature estimates for M and a sharp bound (in general dimensions) on the size of their singular set, namely, that

$$\dim_{\mathcal{H}}(\mathrm{sing}\, M) \leq n - 7.$$

Earlier work (1960-1970): De Giorgi, Federer, Fleming, Almgren, Simons:

If M is locally area minimizing, then $\dim_{\mathcal{H}}(\mathrm{sing}\, M) \leq n - 7$.

Existence theory: Almgren (1960): A deep min-max theory for the area functional, giving existence of a weak minimal k-dimensional submanifold (a stationary integral k-varifold) in any given compact Riemannian manifold N and any $k < \dim N$.

Pitts (1970): an important strengthening of Almgren's theory: allowed regularity of min-max solutions to to be deduced from regularity of local minimizers plus compactness under stability.

Outcome: The above compactness theory for stable hypersurfaces, together with the regularity theory for area minimizing hypersurfaces, implies embeddedness (away from a codim. 7 singular set) of the Almgren–Pitts min-max minimal hypersurfaces.

Marques–Neves–Song: For $2 \leq n \leq 6$ and some metrics (a dense set), there are lots of Almgren–Pitts min-max minimal hypersurfaces.

Recent work on regularity and compactness: In a series of works in the past several years, the Schoen–Simon–Yau and Schoen–Simon theories have been strengthened and extended.

(i) (Wic., 2014) A general regularity and compactness theory for stable hypersurfaces, replacing the Schoen–Simon smallness hypothesis on the singular set with a certain *structural condition*. This has lead to a considerably more efficient PDE alternative to the Almgren–Pitts min-max theory. (Guaraco, Hutchinson–Tonegawa, Tonegawa–Wic.).

(ii) (Bellettini–Wic., 2018) Generalisation of (i) to weakly stable CMC (constant mean curvature) hypersurfaces; weak stability is the natural notion of stability for CMC hypersurfaces. (*Stable CMC integral varifolds of codimension 1: regularity and compactness*, arXiv, 2018);

(iii) (Bellettini–Chodosh–Wic., 2018) Generalisation of the curvature estimates of Schoen–Simon–Yau and Schoen–Simon (for (strongly) stable hypersurfaces) to weakly stable CMC hypersurfaces (*Curvature estimates and sheeting theorems for weakly stable CMC hypersurfaces*, arXiv, 2018).

This work (ii) appears to be the key step in understanding regularity of Morse index controlled hypersurfaces whose (scalar) mean curvature is prescribed by an ambient function.

To describe this work, assume that the ambient space is an open set $U \subset \mathbb{R}^{n+1}$. The local nature of the regularity theorems means that this involves little loss of generality, and indeed the methods do carry over to the general Riemannian setting with only technical modifications.

The work considers a general class of n-dimensional hypersurfaces: codimension 1 integral n-varifolds V whose generalized mean curvature is locally summable to a power $p > n$.

What are these?

First, $V = (M, \theta)$ where $M \subset U$ is n-rectifiable and $\theta : M \to \mathbb{N}^+$ (the multiplicity function) is locally $\mathcal{H}^n$ integrable.

Write $\|V\| = \mathcal{H}^n \lfloor \theta, \quad$ i.e. $\|V\|(A) = \int_{M \cap A} \theta \, d\mathcal{H}^n$. This measure is called the weight (or mass) measure associated with the varifold V.

Generalized mean curvature of V: This is defined variationally:

Let $X : U \to \mathbb{R}^{n+1}$ be a smooth vector field with compact support. We can use X to deform V by moving it by a smooth 1-parameter family of diffeomorphisms $\varphi_t : U \to U$, $t \in (-\epsilon, \epsilon)$, with $\varphi_0 = \mathit{identity}$, $\varphi_t|_{U \setminus \mathrm{spt}\, X} = \mathit{identity}|_{U \setminus \mathrm{spt}\, X}$ and with initial velocity $\frac{d}{dt}\varphi_t(x)\big|_{t=0} = X(x)$.

For any such family φ_t, set $\varphi_{t\,\#}(V) = (\varphi_t(M), \theta \circ \varphi_t^{-1})$. Then a calculation shows that the *first variation*

$$\delta_X(V) \equiv \left.\frac{d}{dt}\right|_{t=0} \|\varphi_{t\,\#}(V)\|(U) = \int_M \mathrm{div}_M X \, d\|V\|.$$

Here $\mathrm{div}_M X(x) = \sum_{j=1}^n < \tau_j, \nabla_{\tau_j} X(x) >$ for any orthonormal basis $\{\tau_1, \tau_2, \ldots, \tau_n\}$ for $T_x M$. If M is a C^2 submanifold with mean curvature H_M and nice boundary and if $\theta = 1$, then along M we can write $X = X^T + X^\perp$, the sum of tangential and normal components. Since $\mathrm{div}_M X^\perp = \sum_{j=1}^n < \tau_j, \nabla_{\tau_j} X^\perp >= - < H_M, X >$, we get by the divergence theorem

$$\int_M \mathrm{div}_M X \, d\mathcal{H}^n = -\int_M < H_M, X > \, d\mathcal{H}^n + \int_{\partial M} < X, \nu > \, d\mathcal{H}^{n-1}.$$

Definition: $V = (M, \theta)$ has generalized mean curvature H_V if $H_V \in L^1_{loc}(\|V\|)$ and

$$\int_M \mathrm{div}_M X \, d\|V\| = -\int_M < H_V, X > \, d\|V\|$$

for every $X \in C^\infty_c(U; \mathbb{R}^{n+1})$.

Allard (1972): If $H_V \in L^p_{loc}(\|V\|)$ for $n < p < \infty$, then $\mathrm{spt}\,\|V\|$ is n-rectifiable, $\mathcal{H}^n\left((\mathrm{spt}\,\|V\| \setminus M) \cup (M \setminus \mathrm{spt}\,\|V\|)\right) = 0$ and the C^1 embedded part $\mathrm{reg}_1\, V$ of $\mathrm{spt}\,\|V\|$ is a relatively open, dense subset of $\mathrm{spt}\,\|V\|$. In fact $\mathrm{reg}_1\, V$ is of class $C^{1,1-\frac{n}{p}}$.

Brakke: a.e. regularity does not follow from $H_V \in L^p_{\mathrm{loc}}$; There is an integral 2-varifold W in $\mathbb{R}^3$ with $H_W \in L^\infty_{\mathrm{loc}}$ such that $\mathrm{sing}\, W$ has positive $\mathcal{H}^2$ measure.

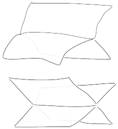
This example uses Catenoidal necks as bulilding blocks, and hence it has a "lot of curvature."

Main new discovery in (Wic. 2014), (Bellettini–Wic., 2018): For codimension 1 integral varifolds V with $H_V \in L^p_{loc}(\|V\|)$ for some $p > n$ and satisfying appropriate variational hypotheses, certain *structural conditions* (described below) are sufficient (and necessary) to give the codimension 7 conclusion on the (genuine) singular set.

These structural conditions are easier to check in principle, and are less stringent than the Schoen–Simon smallness hypothesis on the singular set.

First structural condition: V has no *classical singularities*.

A classical singularity is a point $Y \in \mathrm{spt}\,\|V\|$ near which $\mathrm{spt}\,\|V\|$ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary for some $\alpha \in (0,1)$ that meet only along a common free boundary containing Y.



No-classical-singularities is the only structural condition necessary in the minimal case.

Theorem (Wic., 2014): a stationary (i.e. $H_V = 0$) codimension 1 integral n-varifold with stable regular part and no classical singularities is smoothly embedded away from a closed singular set of dimension $\leq n-7$; moreover, any uniformly area bounded collection of such varifolds is compact.

Important remark: No-classical-singularities condition is implied by the following: There is a set $Z \subset \mathrm{spt}\,\|V\|$ with $\mathcal{H}^{n-1}(Z) = 0$ such that no tangent cone at a point $Y \in \mathrm{spt}\,\|V\| \setminus Z$ is the union of three or more half-hyperplanes meeting along a common axis.

CMC varifolds: Next theorems generalise the above result to CMC varifolds. Key new difficulties arise due the the combination of: (1) failure of two-sided maximum principle for CMC hypersurfaces, and (2) lack of any smallness hypothesis on the singular set.

An important feature of the regularity theorems in this more general setting is that they require variational hypotheses only on the regular parts of the varifold. So consider first the classical (i.e. C^2) setting.

Let M be an embedded, oriented, boundaryless C^2 hypersurface of an open subset U of $\mathbb{R}^{n+1}$, with ν a continuous choice of unit normal on M. For open $\mathcal{O} \subset\subset U$, write

$$\mathcal{A}_{\mathcal{O}}(M) = \mathcal{H}^n(M \cap \mathcal{O}),$$

$$vol_{\mathcal{O}}(M) = \frac{1}{n+1}\int_{M\cap\mathcal{O}} x \cdot \nu(x)\, d\mathcal{H}^n(x).$$

$vol_{\mathcal{O}}(M)$ is the volume enclosed by M in case M is the boundary of an open set $\Omega \subset \mathcal{O}$ and ν is the outward pointing unit normal. (Proof: $\mathrm{div}_{\mathbb{R}^{n+1}}\, x = n+1$; integrate this over Ω and use the divergence theorem.)

For fixed $\lambda \in \mathbb{R}$, let $\mathcal{J}_{\mathcal{O}}(M) = \mathcal{A}_{\mathcal{O}}(M) + \lambda\, vol_{\mathcal{O}}(M)$.

Write H_M for the mean curvature vector of M.

Variational formulation of the CMC condition:

It is well known that the following are equivalent.

(a) M is CMC with $H_M \cdot \nu = \lambda$.

(b) $\lambda = \frac{1}{\mathcal{A}_\Omega(M)} \int_{M \cap \Omega} H_M \cdot \nu \, d\mathcal{H}^n$ for some $\Omega \subset\subset U$ and, for each open $\mathcal{O} \subset\subset U$, M is stationary for $\mathcal{A}_\mathcal{O}(\cdot)$ w. r. t. $vol_\mathcal{O}(\cdot)$ preserving deformations.

[i.e. for each compact $K \subset \mathcal{O}$, $\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_\mathcal{O}(\varphi_t(M)) = 0$ for diffeomorphisms $\varphi_t : \mathcal{O} \to \mathcal{O}$, $t \in (-\epsilon, \epsilon)$, with $\varphi_0 = identity$, $\varphi_t|_{\mathcal{O} \setminus K} = identity|_{\mathcal{O} \setminus K}$ $\forall t$ and $vol_\mathcal{O}(\varphi_t(M)) = vol_\mathcal{O}(M)$ $\forall t$.]

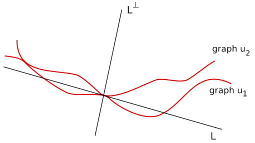
(c) For every open $\mathcal{O} \subset\subset U$, M is stationary for $\mathcal{J}_\mathcal{O}(\cdot)$ for arbitrary deformations (i.e. deformations φ_t as above but not necessarily with $vol_\mathcal{O}(\varphi_t(M)) = vol_\mathcal{O}(M)$ $\forall t$.)

Definition: A CMC hypersurface M of U is weakly stable if for every open $\mathcal{O} \subset\subset U$, $\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{A}_\mathcal{O}(\varphi_t(M)) \geq 0$ for all $vol_\mathcal{O}(\cdot)$ preserving φ_t as in (b) above. ($\implies \int_M |A_M|^2 \zeta^2 \leq \int_M |\nabla \zeta|^2$ for $\zeta \in C^1_c(M)$ with $\int_M \zeta = 0$.)

CMC varifold regularity requires a second structural condition based on the following:

<u>Definition:</u> $Y \in \mathrm{spt}\,\|V\|$ is a **touching singularity** of V if $Y \notin \mathrm{sing}_C\, V \cup \mathrm{reg}_1\, V$ and there is $\sigma > 0$ such that $\mathrm{spt}\,\|V\| \cap B^{n+1}_\sigma(Y)$ is the union of two embedded $C^{1,\alpha}$ hypersurfaces for some $\alpha \in (0,1)$. $\mathrm{sing}_T\, V$ is the set of touching singularities of V.

If $Y \in \mathrm{sing}_T\, V$, there are a $\sigma > 0$, an affine hyperplane L through p and two $C^{1,\alpha}$ functions $u_1, u_2 : L \to L^\perp$ such that $\mathrm{spt}\,\|V\| \cap B^{n+1}_\sigma(Y) = (\mathrm{graph}\, u_1 \cup \mathrm{graph}\, u_2) \cap B^{n+1}_\sigma(Y)$.



Note: $u_1(Y) = u_2(Y)$, $Du_1(Y) = Du_2(Y)$.

CMC REGULARITY THEOREM (Bellettini-Wic., 2018)

Let V be an integral n-varifold in open $U \subset \mathbb{R}^{n+1}$ with $H_V \in L^p_{loc}(\|V\|)$ for some $p > n$. Suppose:

structural hypotheses:

(a) V has no classical singularities.

(b) For each $y \in \mathrm{sing}_T V$, there is $\rho > 0$ such that
$$\mathcal{H}^n\left(\{z : \Theta(\|V\|, z) = \Theta(\|V\|, y)\} \cap B^{n+1}_\rho(y)\right) = 0;$$

variational hypotheses:

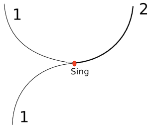
(c) stationarity: whenever $\mathcal{O} \subset U \setminus (\mathrm{spt}\,\|V\| \setminus \mathrm{reg}_1 V)$ is such that $\mathrm{reg}_1 V \cap \mathcal{O}$ is orientable, there is a continuous choice of unit normal ν on $\mathrm{reg}_1 V \cap \mathcal{O}$ such that $V \lfloor \mathcal{O}$ is stationary for $\mathcal{J}_{\mathcal{O}}(V) = \|V\|(\mathcal{O}) + \lambda \int_{\mathrm{reg}_1 V \cap \mathcal{O}} x \cdot \nu \, d\|V\|$;

(d) stability: the C^2 immersed part of $\mathrm{spt}\,\|V\|$ (which contains $\mathrm{reg}_1 V$ by (c)) is stable (as a classical CMC immersion) w.r.t. $vol(\cdot)$ preserving deformations.

Then except on a closed set Σ of dimension $\leq n-7$, $\mathrm{spt}\,\|V\| \cap U$ is quasi-embedded, i.e. is locally either a single smoothly embedded disk or the union of two smoothly embedded disks intersecting tangentially and each lying on one side of the other; moreover, $H_V = \lambda\nu$ on $(\mathrm{spt}\,\|V\| \setminus \Sigma) \cap U$.

Remarks: (1) Hypothesis (a) (no-classical-singularities) cannot be dropped. Consider e.g. a piece of two intersecting unit spheres. In the presence of classical singularities, branch points may develop.

(2) If (b) is dropped, then C^2 regularity can fail.



Important corollary: Caccioppoli sets. A measurable set $E \subset \mathbb{R}^{n+1}$ is a Caccioppoli set (a set of locally finite perimeter) if $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^{n+1})$. So there is a Radon measure μ_E on $\mathbb{R}^{n+1}$ and a vector field ν_E with $|\nu_E| = 1$ μ_E-a.e. such that

$\int_E \operatorname{div} X \, d\mathcal{H}^{n+1} = \int X \cdot \nu_E \, d\mu_E$ for every $X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$.

For open $\mathcal{O} \subset\subset \mathbb{R}^{n+1}$, let $J_{\mathcal{O}}(E) = \mu_E(\mathcal{O}) + \lambda \mathcal{H}^{n+1}(E \cap \mathcal{O})$.

THEOREM (Bellettini-Wic., 2018)

Let E be a Caccioppoli set in $\mathbb{R}^{n+1}$ and let $\lambda \in \mathbb{R}$. Suppose:

(a) no point $p \in \operatorname{spt} \mu_E$ is a classical singularity of $\operatorname{spt} \mu_E$;

(c) For each open $\mathcal{O} \subset\subset \mathbb{R}^{n+1}$, E is stationary with respect to $J_{\mathcal{O}}(\cdot)$ for deformations that fix E outside $\mathcal{O}$ and

(d) the smoothly immersed part of $\operatorname{spt} \mu_E$ is stable (as an immersion) with respect to $J_{\mathcal{O}}(\cdot)$ for deformations that fix E outside $\mathcal{O}$ and preserve $\mathcal{H}^{n+1}(E \cap \mathcal{O})$.

Then except on a closed set of dimension $\leq n-7$, $\operatorname{spt} \mu_E$ is locally either a single smoothly embedded disk or the union of precisely two smoothly embedded disks intersecting tangentially.

Proof: Follows from the preceding theorem since by De Giorgi's structure theory for Caccioppoli sets, the structural condition (b) is automatic.

Advantage of the structural conditions: they involve only the parts of the varifold consisting of regular pieces coming together in a regular fashion. You are allowed to assume regularity in checking them!

In fact in the above results, surprisingly, *all* hypotheses except for $H_V \in L^p_{loc}(\|V\|)$ concern only the regular parts of the varifold, making them easy to check in principle.

Proofs of the above theorems use the Schoen–Simon estimates (for stable hypersurfaces with small singular sets), the powerful machinery developed by L. Simon in his work on asymptotics for minimal submanifolds in multiplicity 1 classes, as well as ideas and results from a number of other fundamental works in GMT due to: De Giorgi, Allard, Almgren, Federer, Simons and Hardt–Simon.

CMC COMPACTNESS THEOREM (Bellettini-Wic., 2018)

Let (V_j) be a sequence integral n-varifolds in open $U \subset \mathbb{R}^{n+1}$ satisfying $H_{V_j} \in L^p_{loc}(\|V_j\|)$ for some $p > n$ and (a)-(d) as in the above theorem with $V = V_j$. If $\limsup_{j\to\infty} \|V_j\|(K) < \infty$ for each compact $K \subset U$, then there is an integral n-varifold V in U satisfying (a)-(d), and a subsequence $\{j'\}$ such that $V_{j'} \to V$ as varifolds in U.

Remark: $\mathcal{H}^{n-1}(\text{sing}\, V) = 0 \implies$ both structural conditions (a) & (b).

Proof of regularity under this assumption requires not much more than minor modification to the argument in the minimal case.

But this hypothesis is undesirable; compactness fails, and results not directly applicable to Caccioppoli sets.

Effective versions of compactness: curvature estimates

THEOREM (Bellettini-Chodosh-Wic., 2018)

If $M \subset B_1(0) \subset \mathbb{R}^{n+1}$ is a quasi-embedded, weakly stable CMC hypersurface with $\mathcal{H}^n(M) \leq \Lambda$ and $|H_M| \leq H_0$ then
(a) if $2 \leq n \leq 6$ and $\mathrm{sing}\, M \cap B_1(0) = \emptyset$ then

$$\sup_{x \in M \cap B_{1/2}(0)} |A_M|(x) \leq C,$$

where $C = C(H_0, \Lambda)$ and A_M is the second fundamental form of M.

(b) For general n, there is $\delta_0 = \delta_0(n, H_0, \Lambda) \in (0,1)$ such that if additionally $\dim_{\mathcal{H}}(\mathrm{sing}\, M) \leq n-7$ and $M \subset \{|x^{n+1}| \leq \delta_0\}$, then $\overline{M} \cap (B^n_{1/2}(0) \times \mathbb{R})$ separates into the union of graphs of functions $u_1 \leq \cdots \leq u_k$ defined on $B^n_{1/2}(0)$ satisfying

$$\sup_{B^n_{1/2}(0)} \left(|Du_i| + |D^2 u_i|\right) \leq \delta_0$$

for $i = 1, \ldots, k$; moreover, each u_i is separately a smooth CMC graph.

A remark on the proof of the curvature estimates:

The methods used by Schoen–Simon–Yau and Schoen–Simon for strongly stable hypersurfaces involve the use of *positive* test functions ϕ in the stability inequality, and since these never integrate to zero, it is not clear how to carry over these methods to the setting of weak stability.

The strategy employed here is different: we take a geometric approach, combining the results of Schoen–Simon–Yau and Schoen–Simon for strongly stable hypersurfaces with the fact that complete weakly stable minimal hypersurfaces have only one end, a result due to Cheng–Cheung–Zhou and generalized here to allow the hypersurfaces to have a small singular set. (This generalization is only necessary for part (b), the sheeting theorem).

The proof uses a blow-up procedure relying on the above compactness theorem for weakly stable CMC hypersurfaces.

Recent new proof of existence of minimal hypersurfaces:

Let N be compact Riemannian manifold of dimension $n+1 \geq 3$.

For $\epsilon \in (0,1)$, consider the Allen–Cahn functionals
$E_\epsilon(u) = \int_N \frac{\epsilon|\nabla u|^2}{2} + \frac{W(u)}{\epsilon}\, d\mathcal{H}^{n+1}, \quad u \in W^{1,2}(N).$
$W : \mathbb{R} \to \mathbb{R}$ is a fixed smooth non-negative double-well potential with precisely two non-degenerate minima at ± 1 with $W(\pm 1) = 0$.

Critical points u_ϵ of E_ϵ have a close connection to minimal hypersurfaces: as $\epsilon \to 0$, the level sets $u_\epsilon^{-1}(t)$ for $|t| < 1$ concentrate near a minimal hypersurface. (Modica, Sternberg in the 1980's for minimizers.)

Hutchinson–Tonegawa (1999): If $\epsilon_j \to 0^+$, $u_j \in W^{1,2}(N)$ is a critical point of E_{ϵ_j} (so $-\epsilon_j \Delta u_j + \epsilon_j^{-1} W'(u_j) = 0$) and $E_{\epsilon_j}(u_j) + \sup_N |u_j| \leq c$ for fixed c, then $\sigma^{-1}\epsilon_{j'}|\nabla u_{j'}|^2\, d\mathcal{H}^{n+1} \to \theta\, d\mathcal{H}^n$ where $\sigma = \sigma(W)$ is an explicit constant and $\theta : N \to \mathbb{N} \cup \{0\}$. If $M \equiv \{x \in N : \theta(x) > 0\}$ then M is n-rectifiable and $V_{ac} = (M, \theta)$ is a stationary integral varifold in N.

This is the elliptic version of the pioneering work of Ilmanen (1993): Convergence of the parabolic Allen-Cahn equation to Brakke flow.

Tonegawa (2005): If additionally u_j are stable, i.e. if

$$\frac{d^2}{dt^2}\bigg|_{t=0} E_{\epsilon_j}(u_j + t\varphi) = \int_N \epsilon_j |\nabla\varphi|^2 + \epsilon_j^{-1} W''(u_j)\varphi^2 \geq 0 \quad \forall\, \varphi \in C^1(N),$$

then V_{ac} is stable on its regular part.

Applying the regularity theory of (Wic, 2014) to V_{ac} gives the following:

Tonegawa–Wic., (2012): If u_j are stable then V_{ac} is an embedded stable minimal hypersurface of N with $\dim(\text{sing}\, V_{ac}) \leq n - 7$.

Full strength of (Wic., 2014) is used because:

(1) There is no a priori control of the size of the set T of singularities of V_{ac} where one tangent cone is planar. Convergence of level sets is not known to be strong enough to give any information about T.

(2) The second fundamental form of the level sets could concentrate near some set $Z \subset \mathrm{spt}\,\|V_{ac}\|$. Easy to show that $\dim_{\mathcal{H}}(Z) \leq n-2$, but no more control on Z is available.

The regularity theory (Wic, 2014) requires no control of T, and it only requires ruling out classical tangent cones away from Z, which is not difficult to do.

Recent work of Guaraco completes the existence proof:

Guaraco (2018): For each $\epsilon \in (0,1)$, there is a min-max critical point u_ϵ of E_ϵ such that

(i) Morse index of $u_\epsilon \leq 1$;

(ii) $\sup_N |u_\epsilon| \leq 1$ and

(iii) $0 < \liminf_{\epsilon \to 0^+} E_\epsilon(u_\epsilon) \leq \limsup_{\epsilon \to 0^+} E_\epsilon(u_\epsilon) < \infty$.

Elementary observation: Because $\mathrm{Index}\,(u_\epsilon) \leq 1$, for any ambient ball B, u_ϵ is stable either in B or in $N \setminus \mathrm{clos}\, B$.

Hence by applying Tonegawa–Wic., Guaraco deduces:

Corollary: If u_ϵ are as in the construction above, any sequence $\epsilon_j \to 0^+$ has a subsequence $\epsilon_{j'}$ such that $\sigma^{-1}\epsilon_{j'}|\nabla\, u_{\epsilon_{j'}}|^2\, d\mathcal{H}^{n+1} \to \|V_{ac}\|$ for some non-trivial stationary integral varifold V_{ac} with $\dim_{\mathcal{H}}(\mathrm{sing}\, V_{ac}) \leq n-7$.

Remark: Chodosh–Mantoulidis: In dimension $n=2$, strong convergence of level sets of stable Allen–Cahn solutions hold. Hence get existence of minimal surfaces in compact 3 manifolds by entirely PDE methods, without the need of any GMT regularity theory.

Riviére, Pigati–Riviére: A viscocity approach, also for $n=2$, giving branched minimal immersions in arbitrary codimension.

A brief outline of the proof of the CMC regularity and compactness theorems:

First reduce to the case where "strong stability" holds. i.e.

$$\int_{\text{gen-reg } V} |A|^2 \zeta^2 \leq \int_{\text{gen-reg } V} |\nabla \zeta|^2 \text{ for every } \zeta \in C^1_c(\text{gen-reg } V)$$

where gen-reg $V = C^2$ immersed part of $\mathrm{spt}\,\|V\|$.

To describe the proofs, fix $H_0 > 0$ and $p > n$, and let $\mathcal{S}_{H_0}$ be the set of integral n-varifolds V on the open ball $B_2^{n+1}(0) \subset \mathbb{R}^{n+1}$ satisfying $\left(\int |H_V|^p \, d\|V\|\right)^{1/p} \leq H_0$ and the hypotheses:

(a) V has no classical singularities;

(b) $\mathcal{H}^n\left(\{y : \Theta\left(\|V\|, y\right) = \Theta\left(\|V\|, z\right)\} \cap B_\rho^{n+1}(z)\right) = 0$ for each $z \in \mathrm{sing}_T\, V$ and some $\rho > 0$;

(c) V is stationary with respect to the functional J away from $\mathrm{spt}\,\|V\| \setminus \mathrm{reg}_1\, V$; and

(d) V is strongly stable on gen-reg V.

The regularity and compactness theorems are proved by establishing the following three theorems, all proved simultaneously by induction on multiplicity/density.

Sheeting Theorem

Let q be a positive integer. There exists $\epsilon = \epsilon(n, q, H_0) \in (0, 1)$ such that if $V \in \mathcal{S}_{H_0}$, $q - 1/2 \leq \omega_n^{-1}\|V\|\,(B_1^n \times \mathbb{R}) < q + 1/2$ and $E \equiv \int_{B_1^n(0)\times\mathbb{R}} |x^{n+1}|^2\, d\|V\|(X) + \frac{1}{\epsilon}\left(\int_{B_1^n(0)\times\mathbb{R}} |H_V|^p\, d\|V\|(X)\right)^{\frac{1}{p}} < \epsilon$ then

$$V \lfloor \left(B_{1/2}^n(0) \times \mathbb{R}\right) = \sum_{j=1}^{q} |\mathrm{graph}\, u_j|$$

where $u_j \in C^{1,\alpha}\left(B_{1/2}^n(0); \mathbb{R}\right)$, $u_1 \leq \ldots \leq u_q$ and $\|u_j\|^2_{C^{1,\alpha}(B_{1/2}^n(0))} \leq CE$ for some fixed $\alpha = \alpha(n, q, H_0) \in (0, 1/2)$, $C = C(n, q, H_0) \in (0, \infty)$.

Minimum Distance Theorem

Let $\mathbf{C}$ be a stationary cone in $\mathbb{R}^{n+1}$ such that $\mathrm{spt}\,\|\mathbf{C}\|$ consists of three or more n-dimensional half-hyperplanes meeting along a common $(n-1)$-dimensional subspace. There exists $\epsilon = \epsilon(\mathbf{C}, H_0) \in (0, 1)$ such that if $V \in \mathcal{S}_{H_0}$ and $(\omega_n 2^n)^{-1}\|V\|(B_2^{n+1}(0)) < \Theta\left(\|\mathbf{C}\|, 0\right) + 1/2$ then

$$\mathrm{dist}_{\mathcal{H}}\left(\mathrm{spt}\,\|V\| \cap B_1^{n+1}(0), \mathrm{spt}\,\|\mathbf{C}\| \cap B_1^{n+1}(0)\right) > \epsilon.$$

Higher Regularity Theorem

Assume $q \geq 2$ and that $V \in \mathcal{S}_{H_0}$ is such that

$$V \lfloor \left(B^n_{1/2}(0) \times \mathbb{R} \right) = \sum_{j=1}^{q} |\mathrm{graph}\, u_j|$$

with $u_j \in C^{1,\alpha}\left(B^n_{1/2}(0); \mathbb{R}\right)$ and $u_1 \leq u_2 \leq \ldots \leq u_q$, with $\alpha \in (0, \frac{1}{2})$. Then

$$\mathrm{spt}\|V\| \cap \left(B^n_{1/2}(0) \times \mathbb{R} \right) = \cup_{j=1}^{\tilde{q}} \mathrm{graph}\, \tilde{u}_j$$

for some $\tilde{q} \leq q$; $\tilde{u}^j \not\equiv \tilde{u}^{j+1}$ for each $j \in \{1, ..., \tilde{q} - 1\}$ and moreover

(i) the $\tilde{q}$ graphs $\tilde{u}_j$ can touch at most in pairs, i.e. if there exist $x \in B^n_{1/2}(0)$ and $i \in \{1, 2, ..., \tilde{q} - 1\}$ such that $\tilde{u}_i(x) = \tilde{u}_{i+1}(x)$ then $D\tilde{u}_i(x) = D\tilde{u}_{i+1}(x)$ and $\tilde{u}_j(x) \neq \tilde{u}_i(x)$ for all $j \in \{1, 2, ..., \tilde{q}\} \setminus \{i, i+1\}$.

(ii) each $\tilde{u}_j$ is of class C^2 and solves individually the CMC equation (and hence it is in C^∞).

Note that the higher regularity assertion is only for $\mathrm{spt}\,\|V\|$; the functions u_j as in the conclusion of the Sheeting Theorem need not be C^2.

In the minimal hypersurface case, the Higher Regularity Theorem is a trivial consequence of the Hopf boundary point lemma, and yields strict inequality $\widetilde{u}_i < \widetilde{u}_{i+1}$ for each i.

In the CMC setting too The Higher Regularity Theorem is easy to prove if we assume that $\mathcal{H}^{n-1}(sing\ V) = 0$.

In the above generality however, considerable difficulties arise due to the combination of the failure of two-sided strong maximum principle and the fact that we only have hypothesis (b) concerning the touching singularities p, i.e. $\mathcal{H}^n\left(\{y\,:\,\Theta\left(\|V\|, y\right) = \Theta\left(\|V\|, p\right)\} \cap B^{n+1}_\rho(p)\right) = 0$ for some $\rho > 0$.

The induction scheme for the three theorems is the following: Let $q \geq 2$ be an integer.

INDUCTION HYPOTHESES:

(H1) Sheeting Theorem holds with any $q' \in \{1, \ldots, q-1\}$ in place of q.

(H2) Minimum Distance theorem holds whenever $\Theta(\|\mathbf{C}\|, 0) \in \{3/2, \ldots, q - 1/2, q\}$.

(H3) Higher Regularity Theorem holds with any $q' \in \{1, \ldots, q-1\}$ in place of q.

Completion of induction is achieved by carrying out, assuming (H1), (H2), (H3), the following three steps in the order they are listed:

(i) prove the Sheeting Theorem (for the case of q sheets);

(ii) prove the Minimum Distance Theorem in case $\Theta(\|\mathbf{C}\|, 0) = q + 1/2$;

(iii) prove the Minimum Distance Theorem in case $\Theta(\|\mathbf{C}\|, 0) = q + 1$;

(iv) prove the Higher Regularity Theorem (for the case of q sheets).

Final Remark: Although the Sheeting Theorem and the Minimum Distance Theorem are meaningful independently of each other, it appears to be impossible to prove them one after the other, inductively or otherwise. The reason is the following:

For the Sheeting Theorem, it is possible to make good use of the monotonicity formula whenever there are singular points with density $\geq q$. In regions where all singularities have density $< q$ on the other hand, one has to find a way to use the stability hypothesis to rule out the Catenoid-like behavior. However, in the presence of a potentially large singular set, it is impossible to get information from the stability inequality. This demands controlling the size of the singular set in regions of density $< q$, and to do this (by applying dimension reduction), one needs the Minimum Distance Theorem.

For the Minimum Distance Theorem, the argument is by contradiction, and involves showing that if there is a sequence of varifolds $V_j \in \mathcal{S}_{H_0}$ converging to $\mathbf{C}$ (a cone supported on three or more half-hyperplanes meeting along an axis), then there has to be a classical singularity somewhere in V_j for sufficiently large j. In order to do this, asymptotic decay estimates have to be proved, which demands as a very first step showing that V_j are "well-behaved" away from the axis of $\mathbf{C}$. This demands something like a Sheeting Theorem.