Strong cosmic censorship in general relativity

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General relativity: Lorentzian geometry

- A *spacetime* is a time-oriented $(3+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$.
- The manifold is equipped with a *Lorentzian metric* g: a symmetric bilinear form on the tangent space with signature $(-,+,+,+)$.
- Basic example: Minkowski spacetime $(\mathbb{R}^{3+1}, m)$,

$$m = -dt^2 + dx^2 + dy^2 + dz^2$$

- A Lorentzian metric g defines a *lightcone* on the tangent space of each point. Accordingly, a vector X can be classified as
  - *spacelike*, if $g(X,X) > 0$
  - *timelike*, if $g(X,X) < 0$
  - *null*, if $g(X,X) = 0$

General relativity: Einstein equation

- The Einstein equations ($T_{\mu\nu}$ describes some matter fields) for $(3+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ is given by

$$Ric_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R = 2 T_{\mu\nu}.$$

- In vacuum ($T_{\mu\nu} = 0$), this reduces to

$$Ric_{\mu\nu}(g) = 0.$$

- Matter fields determine the geometry, and freely-falling observers follow timelike/null geodesics.
- In an appropriate coordinate system, the Einstein vacuum equations form to a system of quasilinear wave equations for the metric components.

Initial value problem in general relativity

Theorem (Choquet-Bruhat–Geroch)

Given

- *a (smooth) three dimensional manifold* Σ
- *with a (smooth) Riemannian metric* $\hat{g}$*,*
- *and a (smooth) symmetric 2-tensor* $\hat{k}$

satisfying the constraint equations, there exists a unique maximal Cauchy development $(\mathcal{M}, g)$ *solving the Einstein vacuum equations such that* $\hat{g}$ *and* $\hat{k}$ *are the induced first and second fundamental forms respectively.*

- Appropriate matter fields can be added in the initial value problem.

PDE questions for the Einstein equations

Given the initial value formulation, we are interested in the following PDE questions:

- Given initial data set, is the maximal Cauchy development global-in-time (geodesically complete)?
- Are special solutions globally nonlinearly stable?
- If the maximal Cauchy development is not global, what happens? Is there a singularity?
- Can a solution be extended further (e.g. beyond the singularity)? Are extensions unique?

Minkowski spacetime

- Recall again that the Minkowski metric is given by

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

  on the manifold $\mathbb{R}^{3+1}$.
- This is the maximal Cauchy development of data $(\bar{g}, \bar{k}) = (\delta, 0)$ on $\mathbb{R}^3$,

$$\delta = dx^2 + dy^2 + dz^2$$

- The maximal Cauchy development is geodesically complete and is inextendible with a C^0 metric.
- (Christodoulou–Klainerman 1993) Minkowski is globally nonlinearly stable.

Future development of a flat ball

- Consider as data $(B(0,1), \delta, 0)$.
- By domain of dependence arguments, the maximal Cauchy development is a strict subset of Minkowski.
- The solution is geodesically incomplete and smoothly extendible, but this is due to incompleteness of data.
- This motivates the study of complete initial data — we will in fact consider only complete asymptotically flat data in this talk.

The Schwarzschild spacetime

- The (maximal) Schwarzschild spacetime $(\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^2, g_{Sch})$
  - solves $Ric(g_{Sch}) = 0$,
  - and in local coordinates

$$g_{Sch} = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2\gamma_{\mathbb{S}^2},$$

  where $M > 0$ and $\gamma_{\mathbb{S}^2}$ is the standard round metric.
- $(\mathcal{M}, g_{Sch})$ has complete 2-ended asymptotically flat initial data and is geodesically incomplete.
- Associated to the incompleteness is a singularity where $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \to \infty$.
- Moreover, $(\mathcal{M}, g_{Sch})$ is inextendible in C^0.

The Schwarzschild spacetime

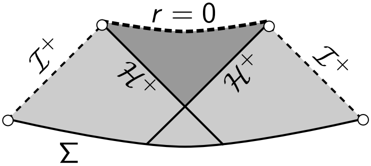


Figure: Maximal future Cauchy development of Schwarzschild

The Kerr spacetime

The Schwarzschild family of solutions turns out to be a subfamily of the larger Kerr family of solutions (to the Einstein vacuum equations). The manifold is again $\mathbb{R}^2 \times \mathbb{S}^2$ and the metric is given in a local coordinate system (with $0 \leq |a| < M$) by

$$g_{Ke} = -(1-\frac{2Mr}{\Sigma})dt^2+\frac{\Sigma}{\Delta}dr^2+\Sigma d\theta^2+R^2\sin^2\theta d\phi^2-\frac{4Mar\sin^2\theta}{\Sigma}d\phi dt,$$

where

$$\Sigma = r^2+a^2\cos^2\theta, \quad R^2 = r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\Sigma}, \quad \Delta = r^2+a^2-2Mr.$$

The Kerr spacetime

- Kerr has complete 2-ended asymptotically flat initial data.
- The maximal Cauchy development is geodesically incomplete.
- However, when $0 < |a| < M$, there is no singularity.
- Instead, there is a smooth Cauchy horizon.
- In fact, it has infinitely many smooth extensions that solve the Einstein vacuum equations!

The Kerr spacetime

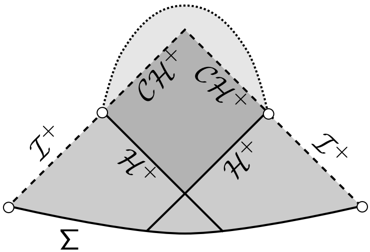


Figure: Maximal future Cauchy development of Kerr and a non-unique extension

Strong cosmic censorship conjecture

The above discussion motivates the following

Conjecture (Strong cosmic censorship, Penrose)

Maximal future Cauchy developments to generic asymptotically flat initial data are future inextendible as suitably regular Lorentzian manifold.

In particular, the smooth Cauchy horizon of Kerr is expected to be unstable!

C^0 formulation of strong cosmic censorship

The original expectation of strong cosmic censorship was modeled on the Schwarzschild solution. One possible way to formulate it is the following:

Conjecture (C^0 formulation of strong cosmic censorship)

Maximal future Cauchy developments to generic asymptotically flat initial data are future inextendible as a Lorentzian manifold with a C^0 metric.

This would be consistent with (though not equivalent to) blow up of Jacobi fields, which has the interpretation that "tidal deformations are infinite in the interior of black holes".

Blue shift instability

- Strong cosmic censorship conjecture has been extensively studied in the physics literature, (Penrose, Penrose–Simpson, McNamara, Gürsel–Sandberg–Novikov–Starobinsky, Chandrasekhar–Hartle, Hiscock, Poisson–Israel, Ori, . . .)
- The original conjecture of Penrose was based in part of the blue shift instability of the Cauchy horizon.

The Reissner–Nordström spacetime

- To simply the problem, we first impose spherical symmetry — but Kerr is not spherically symmetric!
- The Reissner-Nordström spacetime is a 2-parameter family of solutions to the Einstein–Maxwell system. The metric in local coordinates take the form

$$g = -(1 - \frac{2M}{r} + \frac{e^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{e^2}{r^2})^{-1}dr^2 + r^2 d\sigma_{\mathbb{S}^2}.$$

- For $0 < |e| < M$, the Reissner–Nordström spacetime has the same global structure of Kerr spacetime. In particular, it has a smooth Cauchy horizon and can be extended smoothly and non-uniquely as solutions to the Einstein–Maxwell system.

The spherically symmetric Einstein–Maxwell–scalar field system

Consider the Einstein–Maxwell–scalar field system with spherically symmetric initial data.

$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2(T^{(sf)}_{\mu\nu} + T^{(em)}_{\mu\nu}), \\ T^{(sf)}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \\ T^{(em)}_{\mu\nu} = (g^{-1})^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} (g^{-1})^{\alpha\beta} (g^{-1})^{\gamma\sigma} F_{\alpha\gamma} F_{\beta\sigma}, \end{cases}$$

where F is a 2-form and ϕ is a real-valued function satisfying

$$\Box_g \phi = 0, \quad dF = 0, \quad (g^{-1})^{\alpha\mu} \nabla_\alpha F_{\mu\nu} = 0.$$

- Without a scalar field, Minkowski, Schwarzschild and Reissner–Nordström are the only solutions.

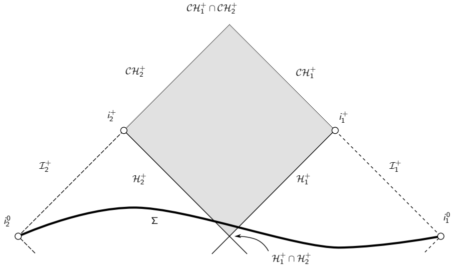
C^0-formulation of strong cosmic censorship is false!

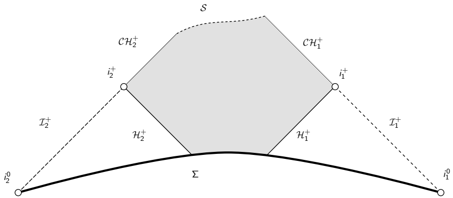
The C^0 formulation of strong cosmic censorship conjecture is false **within the class of spherically symmetric spacetimes**:

Theorem (Dafermos, Dafermos–Rodnianski, 2005)

Consider spherically symmetric, 2-ended asymptotically flat admissible initial data to the Einstein–Maxwell–scalar field system with non-trivial Maxwell field. Then the maximal future Cauchy development is C^0-future-extendible.

Solutions arising from spherically symmetric data





C^2 formulation of strong cosmic censorship in spherical symmetry

Theorem (L.-Oh, 2017)

There exists a generic (open in C^1, dense in C^∞) set $\mathcal{G}$ of spherically symmetric two-ended asymptotically flat admissible smooth initial data to the Einstein–Maxwell–scalar field system with compactly supported initial scalar field such that the maximal future Cauchy development is C^2-future-inextendible.

Comments on the spherically symemtric model

- The C^0-extendibility result can be viewed as a stability result, which can be understood completely independently of the singular nature of the boundary.
- The singularity in the interior of the black hole is highly related to the **decay rate** of the scalar field in the exterior region. As a consequence of our proof, we also establish a **lower bound** of such decay rate in the generic case.
- The lower bound of the decay rate is in turn a consequence of the geometry of the asymptotically flat end.
- See work of Van de Moortel (2017,2018) on generalizations to a more realistic spherically symmetric problem.

Stability of Kerr exterior conjecture

- We now study the problem with no symmetry assumptions.
- One may first try to study the strong cosmic censorship conjecture in a neighborhood of the Kerr solution.
- The exterior region of Kerr is widely believed to be stable — but even this remains an open problem (see the next talk)!

Conjecture (Stability of Kerr exterior)

Given an initial data set to the Einstein vacuum equation which is globally close to Kerr initial data with $0 \leq |a| < M$. The maximal future Cauchy development has an exterior region which converges to a nearby Kerr spacetime.

Stability of Kerr exterior conjecture

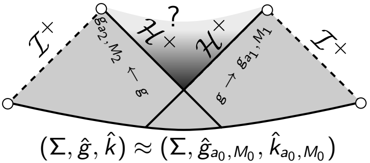


Figure: The conjectured stability of the Kerr exterior

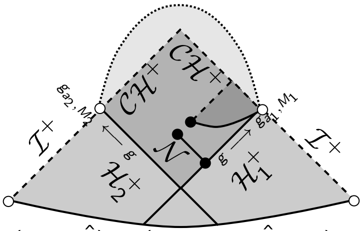
Stability of the Kerr Cauchy horizon

We turn to the **nonlinear** problem **with no symmetry assumptions**. For this purpose, we **assume** the validity of the stability of Kerr exterior conjecture.

Theorem (Dafermos–L., 2017)

If the stability of Kerr exterior conjecture is true (with quantitative decay rates), then the maximal future Cauchy development to any small perturbation of Kerr data (with $0 < |a| < M$) has a bifurcate Cauchy horizon. Moreover, the metric is continuously extendible to the Cauchy horizon and (in appropriate coordinate systems) is C^0-close to the Kerr metric.

Stability of the Kerr Cauchy horizon



$(\Sigma, \hat{g}, \hat{k}) \approx (\Sigma, \hat{g}_{a_0,M_0}, \hat{k}_{a_0,M_0})$

Figure: The global stability of the Kerr Penrose diagram

C^0 formulation of strong cosmic censorship is false!

As a consequence, we also have the following conditional result:

Corollary

If the stability of Kerr exterior conjecture is true (with quantitative decay rates), then the C^0 formulation of the strong cosmic censorship conjecture is **false***.*

Instability conjectures

While we showed that the Cauchy horizon is C^0 stable, our theorem is consistent with higher derivatives of the metric blowing up.

Conjecture (Christodoulou)

For generic perturbations as in the theorem before, the metric cannot be extended in $W^{1,2}_{loc}$ and hence spacetime cannot be extended as weak solutions to Einstein equations.

Linear instability on Kerr spacetime

As a first step, one can study instability for the **linear** wave equation $\Box_g \phi = 0$ on a **fixed** Kerr spacetime:

Theorem (Dafermos–Shlapentokh-Rothman (2016), L.–Sbierski (2016), L.–Oh–Shlapentokh-Rothman (to appear))

Linear instability for the wave equation holds on any fixed subextremal, strictly rotating Kerr spacetime.

Idea of proof I: Model problem for null singularities

- One main difficulty is that the linear theory already suggests a blowup at the Cauchy horizon.
- In fact linear theory suggests that the metric $g \in (C^0 \cap W^{1,1}) \setminus \cup_{\alpha>0,\, p>1}(C^\alpha \cup W^{1,p})$.
- This is way below the threshold for solving the Einstein equations in general.

Idea of proof I: Model problem for null singularities

We need to take advantage of specific properties of the singularities (as suggested by linear theory):

- Away from the Cauchy horizon, the spacetime is smooth.
- Only the derivatives of the metric transversal to the Cauchy horizon are singular.
- The derivative of the metric in the "worst direction" is still integrable, and is moreover "bounded above" by an explicit integrable function.

Let us first consider a model problem with a spacetime singularity capturing these features.

Idea of proof I: Model problem for null singularities

As a model problem, let us consider on $\mathbb{R}^{3+1}$

$$\Box\phi = Q_0(\partial\phi, \partial\phi) = (\partial_t\phi)^2 - \sum_{i=1}^{3}(\partial_{x^i}\phi)^2.$$

Let $v = t + x^1$ and $u = t - x^1$. Consider a characteristic initial problem where the data

$$|\partial_v\phi|(u=0) \lesssim (v_* - v)^{-1}\log^{-2}(1/(v_* - v)),$$

$$|\partial_u\phi|(v=0) \lesssim (u_* - u)^{-1}\log^{-2}(1/(u_* - u))$$

and one has a similar profile after taking higher derivatives with respect to x^2, x^3.

Idea of proof I: Model problem for null singularities

- Note the already non-trivial fact that this can be solved locally if ϕ is independent of x^2 and x^3.
- The equation takes the form

$$\partial_u \partial_v \phi = -(\partial_u \phi)(\partial_v \phi).$$

- A null condition present in the nonlinearity. It takes the form $\partial_u \phi \partial_v \phi$, not $(\partial_u \phi)^2$ or $(\partial_v \phi)^2$.
- We want to show that the profile "persists", i.e. for all u, $|\partial_v \phi|$ is bounded by the same profile.
- Note that

$$|\partial_u \partial_v \phi| \lesssim (v_* - v)^{-1} \log^{-2}(1/(v_* - v))(u_* - u)^{-1} \log^{-2}(1/(u_* - u)).$$

Idea of proof I: Model problem for null singularities

- Without symmetry one necessarily uses L^2-based estimates, but one can build in the profile into the norms, e.g.

$$\int_0^{v_*}\int_{\mathbb{R}^2} f(v)(\partial_v\phi)^2 + h(u)\sum_{i=2}^{3}(\partial_{x^i}\phi)^2\,dx^2\,dx^3\,dv,$$

$$\int_0^{u_*}\int_{\mathbb{R}^2} h(u)(\partial_u\phi)^2 + f(v)\sum_{i=2}^{3}(\partial_{x^i}\phi)^2\,dx^2\,dx^3\,du,$$

  for some f, $h \to 0$ towards the singularity.
  - These norms also generate "good terms" when the weights are differentiated.
  - To close the problem, we need higher derivatives of ϕ: It suffices to take ∂_{x^2} and ∂_{x^3} derivatives for which no additional singular terms are generated.
  - Sufficient to handle nonlinear terms due to null condition.

Idea of proof II: Null singularities for Einstein equations

- In view of the significance of null coordinates even in the model problem, we introduce the double null coordinates:

$$-4\Omega^2 du\, dv + \gamma_{AB}(d\theta^A - b^A dv)(d\theta^B - b^B dv).$$

- Correspondingly, define a double null frame

$$e_3 = \partial_u, \quad e_4 = \frac{1}{\Omega^2}(\partial_v + b^A \partial_\theta^A), \quad e_A = \partial_{\theta^A}.$$

- Define frame coefficients and curvature components with respect to $\{e_\mu\}$:

$$\Gamma = g(D_{e_\mu} e_\nu, e_\beta), \quad \Psi = R(e_\mu, e_\nu, e_\beta, e_\alpha).$$

- Need to control these quantities for a metric singular in u, v similar to the model problem.

Idea of proof II: Null singularities for Einstein equations

- The Einstein equations can be recast as a system for Γ and Ψ.
  - Γ satisfies nonlinear transport equations and elliptic equations with Ψ as a source.
  - Ψ satisfies a system of hyperbolic equations with Γ as coefficients.
- Ψ are second derivatives of metric — considerably more singular!
- Consider only a subset of (more regular) components and introduce a **renormalization** $\check{\Psi} = \Psi + \Gamma \cdot \Gamma$. $\check{\Psi}$ still satisfies a hyperbolic system.

Idea of proof II: Null singularities for Einstein equations

We then use the insights from the model problem:

- The singular profiles are incorporated into the energy norms.
  - Uncontrollable terms that are generated have good signs!
- Take higher derivatives only in the θ^A directions.
  - No additional singular terms are generated by differentiation!

Remarkably, due to the null structure of the Einstein equations, these weak norms are already sufficient to control the nonlinearities.

Thank you!