Absence of absolutely continuous diffraction spectrum in inflation tilings

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Diffraction Spectrum

We want to study the diffraction of a mass density μ on a tiling,

$$\mu = \sum_i w_i \sum_{x \in \Lambda_i} \delta_x$$

where $\Lambda_i = \{\text{positions of tiles of type } i\}$. Using correlation coefficients

$$\nu_{ij}(z) = \frac{\mathrm{dens}\big(\Lambda_i \cap (\Lambda_j - z)\big)}{\mathrm{dens}(\Lambda)}$$

we get the autocorrelation measure γ of μ,

$$\gamma = \mathrm{dens}(\Lambda) \sum_{i,j} \overline{w}_i w_j \sum_{z \in \Lambda_i - \Lambda_j} \nu_{ij}(z) \delta_z$$

whose Fourier transform $\hat{\gamma}$ is the diffraction measure of μ.

Inflation Tilings

Under an inflation ρ, a tiling is expanded by a factor λ, and each expanded tile is replaced by a union of tiles. Their relative positions are given in

$$T_{ij} := \{\text{relative positions of tiles of type } i \text{ in a supertile of type } j\}$$

The inflation matrix M_ρ has leading eigenvalue λ^d and entries $\mathrm{card}(T_{ij})$. The same information is also encoded in the Fourier matrix

$$B_{ij}(k) = \sum_{t \in T_{ij}} \mathrm{e}^{2\pi \mathrm{i}\, t \cdot k}$$

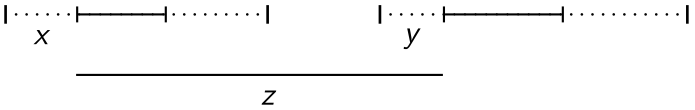
Note that ρ^n has Fourier matrix $B^{(n)}(k) = B(k)B(\lambda k) \dots B(\lambda^{n-1}k)$.

What is Known About the Spectral Type

- non-trivial pp part if and only if inflation factor λ is Pisot
- if λ is Pisot, support of pp part is known (up to extinctions)
- if λ is Pisot, one can algorithmically determine whether spectrum is pp (overlap algorithm by Solomyak and Akiyama–Lee, extending earlier work by Dekking for the constant-length case)
- for integer inflation factor (equivalent to constant-length case), all spectral components can be determined algorithmically (Bartlett, extending work of Queffélec)
- multiple-to-1 extensions of pp systems cannot be pp
- irreducible Pisot inflations are conjectured to be pp (Pisot substitution conjecture)

Recursion Relations for the Pair Correlation

The pair correlation satisfies exact recursion relations:



$$\nu_{mn}(z) \;=\; \frac{1}{\lambda^d} \sum_{i,j} \sum_{x \in T_{mi}} \sum_{y \in T_{nj}} \nu_{ij}\Big(\frac{z+x-y}{\lambda}\Big)$$

$$\Upsilon_{mn} \;=\; \sum_{z \in \Lambda_m - \Lambda_n} \nu_{mn}(z)\,\delta_z \;=\; \frac{1}{\lambda^d} \sum_{i,j} \sum_{x \in T_{mi}} \sum_{y \in T_{nj}} \delta_{y-x} * \big(f.\Upsilon_{ij}\big).$$

where $f(x) := \lambda x$ and $\big(f.\mu\big)(\mathcal{E}) = \mu\big(f^{-1}(\mathcal{E})\big)$ for any Borel set $\mathcal{E}$.

Fourier Transform of Pair Correlation

As $\widehat{f.\mu} = \frac{1}{\lambda^d}(f^{-1}.\widehat{\mu})$, we get

$$\widehat{\Upsilon_{mn}} = \frac{1}{\lambda^{2d}} \sum_{i,j} \sum_{x \in T_{mi}} \sum_{y \in T_{nj}} \mathrm{e}^{-2\pi \mathrm{i}(y-x)(.)} \big(f^{-1}.\widehat{\Upsilon_{ij}}\big).$$

Writing the $\widehat{\Upsilon_{mn}}$ in one big vector, we get $\widehat{\Upsilon} = \frac{1}{\lambda^{2d}} \boldsymbol{A}(.)\big(f^{-1}.\widehat{\Upsilon}\big)$, with $\boldsymbol{A}(k) = B(k) \otimes \overline{B(k)}$.

This relation is satisfied separately by each component of the decompositon

$$\widehat{\Upsilon} = \big(\widehat{\Upsilon}\big)_{\mathsf{pp}} + \big(\widehat{\Upsilon}\big)_{\mathsf{sc}} + \big(\widehat{\Upsilon}\big)_{\mathsf{ac}}.$$

AC Spectrum

Suppose now there is an ac part in the spectrum, described by a vector $\boldsymbol{h}(k)$ of density functions. This vector satisfies

$$\boldsymbol{h}(k) = \frac{1}{\lambda^d} \boldsymbol{A}(k)\, \boldsymbol{h}(\lambda k), \qquad \text{for a.e. } k$$

Note: one factor λ^{-d} has been eaten up by a change of variables.

Writing $\boldsymbol{h}(k)$ again as entries of a (Hermitian, positive semi-definite) matrix $\mathcal{H}(k)$, we get

$$\mathcal{H}(k) = \lambda^{-d} B(k) \mathcal{H}(\lambda k) B^{\dagger}(k).$$

$\mathcal{H}(k)$ can be decomposed into a sum of terms $\mathcal{H}_i(k) = v^{(i)}(k) \left(v^{(i)}\right)^{\dagger}(k)$ of rank 1, and we can study the growth of each $v^{(i)}(k)$ separately.

Lyapunov Exponents

Assuming that $B(k)$ is invertible, we must study the growth of the iteration

$$v(\lambda k) = \lambda^{d/2} B^{-1}(k) v(k),$$

The maximal and minimal growth is governed by the Lyapunov exponents

$$\chi_{\max}(k) = \log \lambda^{d/2} + \limsup_{n\to\infty} \frac{1}{n} \log \left\| B^{-1}(\lambda^{n-1}k) \cdots B^{-1}(k) \right\|_F$$
$$\chi_{\min}(k) = \log \lambda^{d/2} + \liminf_{n\to\infty} \frac{1}{n} \log \left\| B(k) B(\lambda k) \cdots B(\lambda^{n-1}k) \right\|_F^{-1}.$$

As $v(k)$ must be translation bounded, it can be non-trivial only if it is possible to have no growth, i.e., if $\chi_{\min}(k) \leqslant 0$.

Estimates on Lyapunov Exponents

Setting

$$\chi^B(k) := \limsup_{n\to\infty} \frac{1}{n} \log \left\| B^{(n)}(k) \right\|_F$$

we get $\chi_{\min}(k) = \log \lambda^{d/2} - \chi^B(k)$.

In order to show absence of ac spectrum, we need $\chi^B(k) < c \cdot \log \lambda^{d/2}$ for almost all k and some $c < 1$.

Fortunately, for a sub-multiplicative norm like the Frobenius norm, we get

$$\chi^B(k) = \limsup_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(k)\|_F \leqslant \frac{1}{N}\mathbb{M}(\log \|B^{(N)}(k)\|_F),$$

for any fixed N, where $\mathbb{M}(f)$ is the mean of the quasiperiodic function f.

Estimates on Lyapunov Exponents II

To compute the mean of the quasiperiodic function $\log \|B^{(n)}(k)\|_F$, or rather its square, $\log \|B^{(n)}(k)\|_F^2$, we observe that it can be lifted to a section through a periodic function, and the mean can be computed as an integral over its unit cell:

$$\frac{1}{N}\mathbb{M}\big(\log \|B^{(N)}(.)\|_{\mathrm{F}}^2\big) \;=\; \frac{1}{N}\int_{\mathbb{T}^D} \log\Big(\sum_{i,j}\big|P_{ij}^{(N)}(\tilde{k})\big|^2\Big)\,\mathrm{d}\tilde{k}.$$

For each N, this yields an upper bound for $\chi^B(k)$, which is readily computable for many examples, and can serve as a criterion for the absence of ac spectrum: $2\chi^B(k) < c \cdot 2\log\lambda^{d/2}$, $c < 1$.

Three Examples with Bar Swap Symmetry

TSM:	$a \to ab\bar{a}, \quad b \to \bar{a}$	Eigenvalues:	$1 \pm \sqrt{2}, \pm i$
F2:	$a \to ab\bar{a}, \quad b \to ab$	Eigenvalues:	$\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(1 \pm \sqrt{5})$
F3:	$a \to a\bar{b}a\bar{a}\bar{b}, \quad b \to aba$	Eigenvalues:	$2 \pm \sqrt{5}, 1 \pm 2i$

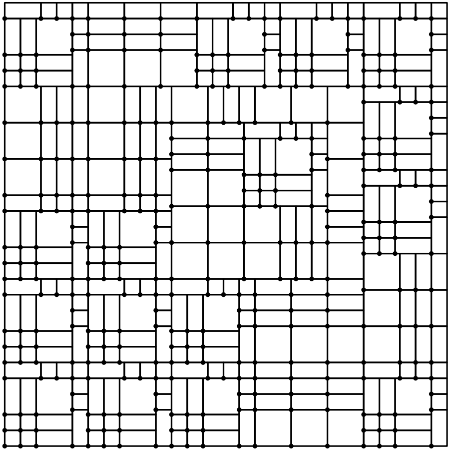
All three are almost 2-to-1 extensions of a system with pp spectrum.

The spectrum is either pp + sc or pp + ac.

Estimates on Lyapunov exponents χ^B show it is pp+sc in all three cases.

Other large eigenvalues are unrelated to λ (which is Pisot), and do not prevent the existence of a pp part.

Frank–Robinson Tiling



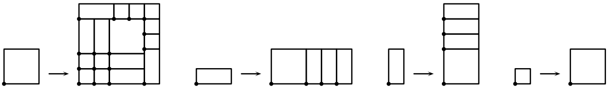
- Non-Pisot:

  $\lambda = (1 + \sqrt{13})/2$

- Non-FLC, but finitely many tiles up to translation
- Expectation:

  singular continuous spectrum

Frank–Robinson Inflation



Inflation factor $\lambda = (1+\sqrt{13})/2$ (non-Pisot)

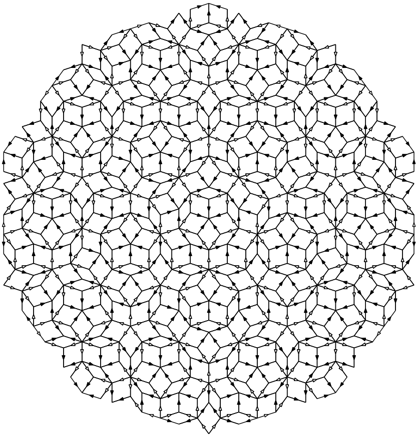
$\longrightarrow$ Bragg peak at 0 + continuous spectrum

Estimates for Lyapunov exponent $2\chi^B$ drop below $2\log(\lambda) \approx 1.668$:

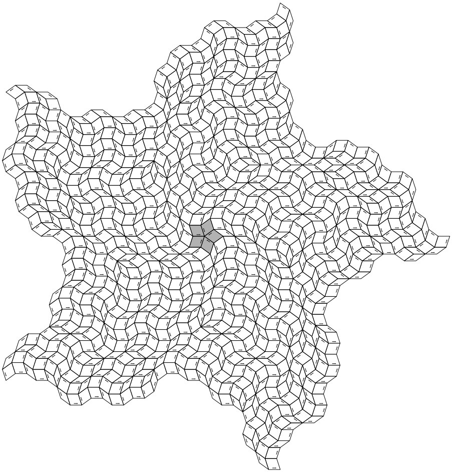
N	5	6	7	8	9	10
$\frac{1}{N}\mathbb{M}\big(\log \lVert B^{(N)}(.)\rVert_{\mathrm{F}}^2\big)$	1.752	1.695	1.653	1.621	1.596	1.576

Continuous spectrum is singular continuous.

Penrose and Lançon–Billard Tilings

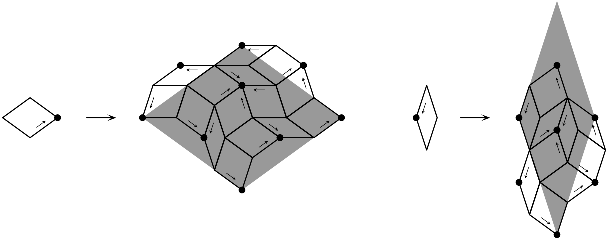


Penrose



Lançon–Billard

Lançon-Billard Inflation



Inflation factor $2+\tau$ (non-Pisot) $\longrightarrow$ Bragg peak at 0 + cont. spectrum

Estimates for Lyapunov exponent $2\chi^B$ drop below $2\log(2+\tau) \approx 2.57186$:

N	9	10	11	12	13	14
$\frac{1}{N}\mathbb{M}\big(\log \lVert B^{(N)}(.)\rVert_{\mathrm{F}}^2\big)$	2.594	2.563	2.537	2.516	2.498	2.482

Continuous spectrum is singular continuous.

Example with ac Spectrum: Rudin–Shapiro

$$\text{RS:}\quad a \to ab, \quad b \to a\bar{b}, \quad \bar{a} \to \bar{a}\bar{b}, \quad \bar{b} \to \bar{a}b$$

A decoration odd under the bar swap yields a structure with ac spectrum.

Due to the bar swap symmetry, the Fourier matrix can be brought to block diagonal form:

$$B(k) = \left(\begin{array}{cc|cc} 1+e^{2\pi ik} & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{O}$}} \\ -e^{2\pi ik} & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\boldsymbol{O}$}} & 1 & 1 \\ & & e^{2\pi ik} & -e^{2\pi ik} \end{array}\right)$$

Second block is $\sqrt{2} \times$ unitary, so $\chi^B = \log\sqrt{2}$, and ac spectrum is possible.