Complex methods in real dynamics

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We met at the Groningen-Delft-Diepenbeek-Aachen *Dynamical Systems* ‘traveling seminar’ around 1990.

I worked in Dynamical Systems, initially in bifurcation theory and then in real one-dimensional dynamics.

I was aware of holomorphic dynamics, but as my colleague said: real man use $\mathbb{R}$ mathematics.

Walter gave a beautiful talk, and over the years I increasingly became aware that complex tools are extremely helpful even in real dynamics.
General questions in the field of dynamical systems

The theory of dynamical system is about
- a $C^r$ manifold $M$, $r = 1, 2, \ldots, \infty, \omega$;
- the space $C^r(M)$ of smooth maps $f : M \to M$;
- orbits $\{f^n(x)\}_{n \geq 0}$ of points $x \in M$.

To classify one says $f \sim g$ if $f, g$ is topologically conjugate (i.e. $\exists$ orientation preserving homeo $h$ so that $h \circ f = g \circ h$).

Key questions:
- describe the topological conjugacy class $T(f) = \{g; g \sim f\}$. Are they connected? Are they submanifolds of $C^r(M)$?
- Is the set of ‘hyperbolic’ maps dense (for hyperbolic maps $T(f)$ is open).
Density of hyperbolicity in one-dimensional dynamics

- When \( \dim(M) \geq 2 \) then answer: unknown resp. ‘no’.
- When \( M = [0, 1] \) or \( M = S^1 \) much more is known.

**Theorem**

*Hyperbolic maps are dense* \( C^r(M) \) for \( r = 1, 2, \ldots, \infty, \omega \).

**Equivalent Def. Hyperbolicity:**
- Non-wandering set is hyperbolic
- Each critical point is attracted to hyperbolic periodic attractor

**Simplification:** Of course, it is enough to show that each real polynomial can be approximated by a *hyperbolic* real polynomial, but possibly with *non-real critical points*.

**History:**
- Lyubich, Graczyk & Swiatek for real quadratic maps.
- Kozlovski, Shen & SvS general case.
Conjugacy classes are connected manifolds

Theorem (Trevor Clark & SvS)

Within the space of real analytic interval maps with only repelling periodic points, $T(f)$ is a connected manifold.

- Trevor and I think we are close to proving this also within corresponding space of $C^3$ maps.
- This is part of programme of questions posed by Sullivan.
- Important difficulty when solving this:
  - ¬ Measurable Riemann Mapping Theorem on the real line.
Quasisymmetric rigidity

The proofs of all these theorems rely on quasisymmetric rigidity:

- This means that one needs to show:
  \( f, g \) topologically conjugate on \( \mathbb{R} \) \( \implies \) \( f, g \) are qs-conjugate.

- Def: \( \exists \) conjugacy \( h: \mathbb{R} \to \mathbb{R} \) so that for all \( x, h, \)
  \[
  \frac{h(x + h) - h(x)}{h(x) - h(x - h)} \leq K.
  \]

- Sullivan proposed a research programme around this notion.
Completion of Sullivan’s programme: qs-rigidity

Theorem (Trevor Clark & SvS)

Assume $f, g: M \to M$ are $C^3$, ..., have at least one critical point and topologically conjugate. Assume conjug. is bijection of

- the set of critical points and the order of corresponding critical points is the same;
- the set of parabolic periodic points.

Then $f$ and $g$ are quasi-symmetrically conjugate.

New even in case of real polynomials with non-real critical points.
The previous result builds on previous results by Lyubich, Graczyk & Shen and particular Kozlovski & Shen & SvS.

- If $f, g$ are real analytic then no additional assumptions needed.
- If $f, g$ are $C^3$ then extra assumption: all critical points and parabolic points are non-flat.
- If $f, g$ are $C^3$ then the qs-conjugacy does not extend to a qc-conjugacy near $\mathbb{R}$.
- If $f, g$ are $C^2$ then probably the theorem is not true.

It is important to note that

- Thm says $\exists$ qs-conjugacy on $M$ not just on postcritical set $P$.
- To have qs on $M$ is important in applications
- Postcritical set $P(c)$ can be an interval, and then previous techniques entirely break down, except in very special cases.
How does complex analysis / complex dynamics enter?

If \( f \) is real analytic: Then \( f \) has a complex analytic extension \( f : U \rightarrow \mathbb{C} \).

- Unfortunately the domain of \( f^n \) shrinks as \( n \rightarrow \infty \);
- Need to use complex box mapping (also for non-recurrent critical points):

**Theorem (Trevor Clark, Sofia Trejo & SvS)**

\[ \exists \text{ box mapping as in figure (but domain can have } \infty \text{ many components).} \]
To construct qc homeomorphisms: when \( P(c) \)
- `strongly recurrent`: \( \exists \) use specific sequence of pullbacks of domains of complex box mappings: *enhanced nest.*
- otherwise, choose compatible good puzzle pieces (note: \( \nexists \) natural partition by rays, exponentials).

If there are **no parabolic periodic points** then as in KSS
- Control shape of puzzle pieces. New: enhanced puzzle pieces are quasidiscs.
- Complicating issue: \( \exists \) critical points with different behaviour with intertwined behaviour. New: how to glue where \( \infty \)-renormalizable, \( P(c) \) is Cantor or \( P(c) \) interval.
- QC-criterion.
- Spreading principle.

If there are **parabolic periodic points** then
- introduce touching box mappings;
- construct box mappings with ranges compatible with touching box mapping.
Additional problems in the $C^3$ case.

If $f$ is $C^3$: then $\exists$ an extension $f : U \to \mathbb{C}$ which is asymptotically holomorphic of order 2:

$$\frac{\overline{\partial} f(z)}{|Im(z)|^2} \to 0 \text{ as } z \to \mathbb{R}.$$ 

- So extension is quasi-regular map,
- $f^n$ is NOT uniformly quasi-regular.
- In general, $Df^q(a)$ has two different eigenvalues at $q$-periodic points $a$.

- Certain puzzle pieces have extremely bad geometry but there we no longer use dynamics.
- Use QC-criterion, combined with by-hand constructions.
Theorem (Trevor Clark, Edson de Faria and SvS)

Let \( f \in C^{3+\alpha} (\alpha > 0) \) be a unimodal, \( \infty \)-renormalizable interval map of bounded type with critical point \( c \) of order \( d \in 2\mathbb{N} \). Let \( F \) be a \( C^{3+\alpha} \) extension of \( f \). Then \( \exists \) domains \( U_n \subset V_n \subset \mathbb{C} \) containing \( c \) and iterates \( q_n \) with:

1. \( G := F^{q_n} : U_n \rightarrow V_n \) is a degree \( d \), quasi-regular polynomial-like map.

2. For large \( n \), each periodic point in \( K_G := \{ z \in U_n; G^i(z) \in U_n \ \forall i \geq 0 \} \) is repelling.

3. \( \partial K_G = K_G \).

4. \( G \) is topologically conjugate to a polynomial mapping near \( K_G \). In particular, \( G \) has no wandering domains.

5. The Julia set \( J_G \) is locally connected.
Milnor’s conjecture: monotonicity of entropy

**Theorem (Henk Bruin & SvS)**

Consider the space $P^d$ of real polynomials with precisely $d$ critical points, all real and non-degenerate. Then the space $P_h = \{ f; h_{top}(f) = h \}$ is connected.

**Theorem (Henk Bruin & SvS)**

When $d \geq 5$, there exist (many) $h$ so that the space $P^d_h = \{ f; h_{top}(f) = h \}$ is not locally connected.

**Questions (Thurston):**

- Is there a dense set $H$ so that $h \in H \implies P^d_h$ is locally connected dense?
- Is there a dense set $H$ so that $h \in H \implies P^d_h$ contains a dense set of hyperbolic maps?
Theorem (Lasse Rempe & SvS)

Density of hyperbolicity within certain space of real transcendental entire functions, bounded on the real line, whose singular set is finite and real.

Example: Density of hyperbolicity within the family of Arnol’d maps:

\[ S^1 \ni x \mapsto x + a \sin(2\pi x) + b. \]

Question: Does density of hyperbolicity also hold when the singular set is non-real?
Consider the family $z \mapsto z^2 + c$ with $z \in \mathbb{R}$ and $c \in \mathbb{R}$.

For $c = -2$ it has periodic points of all periods and for $c \in \mathbb{R}$ large, all points go to infinity:

Theorem (Sullivan, Thurston, Douady, Tsujii, .... (1980’s))

As $c$ increases, periodic points disappear and no periodic points are created.

All these proofs use complex methods.
Sullivan’s approach: via quasi-symmetric rigidity

- Suppose not monotone. Then $\exists c_1 < c_2$ so that the maps $f_{c_1} = x^2 + c_1$ and $f_{c_2} = x^2 + c_2$ are topologically conjugate.
- Can assume that these maps are critically finite (i.e. the orbit of 0 is finite) and that there exists no $c \notin [c_1, c_2]$ for which $x^2 + c$ is conjugate to these maps.
- $\exists K$-qc map $h_0$ mapping postcritical set of $f_{c_1}$ to postcritical set of $f_{c_2}$
- $\exists K$-qc pullbacks $h_0, h_1, \ldots$ defined by $f_{c_1} \circ h_{n+1} = h_n \circ f_{c_2}$.
- $\implies f_{c_1}$ and $f_{c_2}$ are quasiconformally conjugate.
- Measurable Riemann Mapping Theorem $\implies \exists$ open interval $U \supset [c, c']$ so that for all $c \in U$, $f_c$ is conjugate to $f_{c_1}$.
- This gives contradiction with maximality of interval $[c_1, c_2]$. 
Assume that $f_{c^*}$ has 0 as a periodic point of (minimal) period $q$.

- **Prove “Positive” transversality:**

  \[
  \frac{d}{dc} f_c^q(0) \bigg|_{c=c^*} = \frac{q-1}{Df_{c^*}^{q-1}(f_{c^*}(0))} = \sum_{n=0}^{q-1} \frac{1}{Df_{c^*}^i(f_{c^*}(0))} > 0. \quad (1)
  \]

- Since $f$ has minimum at 0, if $x \mapsto f_c^q(x)$ has local max (min) at 0 then $Df_{c^*}^{q-1}(f_{c^*}(0)) < 0$ (resp. $> 0$).

  By pos. transv.

  \[
  \frac{d}{dc} f_c^q(0) \bigg|_{c=c^*} < 0 \quad \text{if } f_{c^*}^q \text{ has a local maximum at 0},
  \]

  \[
  \frac{d}{dc} f_c^q(0) \bigg|_{c=c^*} > 0 \quad \text{if } f_{c^*}^q \text{ has a local minimum at 0}.
  \]

- $\implies$ (using real arguments) periodic orbits cannot be reborn.
Tsujii’s vs Douady-Hubbard approach

Compare with Douady-Hubbard approach:

- **Douady-Hubbard**: \( c \mapsto \lambda(c) \) is univalent in each hyperbolic component of the family of quadratic maps.
- **Tsujii’s approach** \( c \mapsto \lambda(c) \) is increasing.

All known proofs of

- monotonicity require maps to be entire (and usually even that maps are quadratic).
- transversality proofs as based on the finite dimensionality of some Teichmüller space.
What about families which are not holomorphic?

Fix \( \ell \geq 1 \), \( b > 2(\ell e) \) and consider

\[
f_c(x) = \begin{cases} 
  be^{-1/|x|^\ell} + c & \text{for } x \in \mathbb{R} \setminus \{0\}, \\
  c & \text{for } x = 0.
\end{cases}
\]

Or

\( \ell > 1 \) and

\[
f_c(x) = |x|^\ell + c.
\]
To develop transversality results in one-dimensional dynamics for maps which are only locally holomorphic

Examples of objectives:

- Show that when \( \ell > 1 \) (not necessarily an integer) the topological entropy of the map \( \mathbb{R} \ni x \mapsto |x|^\ell + c \) depends monotonically on \( c \in \mathbb{R} \).
- Give a natural class of unimodal functions \( f : \mathbb{R} \to \mathbb{R} \) for which the topological entropy of \( f_\lambda(x) = \lambda f(x) \) is monotone in \( \lambda \)?

Approach: using holomorphic motions we obtain results on the spectrum of some transfer operator.

Set-up: consider a map \( F \) defined in a neighbourhood \( U \) of a finite set \( P \), so that \( F(P) \subset P \).
Currently: A map $g$ defined in a neighbourhood $U$ of a finite set $P$, so that $g(P) \subset P$.

Work in progress: Show that positive transversality holds at parameters with parabolic periodic point.

In that case,

- critical orbit is infinite and
- no growth condition on $Df^n(c)$ holds.
Theorem (Genadi Levin, Weixiao Shen and SvS)

Suppose that $g$ extends to a holomorphic map $g : U_g \to V$ where

- $U_g$ is a bounded open set in $\mathbb{C}$ such that $U_g \supset P \setminus \{0\}$ and $0 \in \overline{U_g}$;
- $V$ is a bounded open set in $\mathbb{C}$ such that $c_1 := g(0) \in V$;
- $g : U_g \setminus \{0\} \to V \setminus \{c_1\}$ is an unbranched covering.

If the robust separation property

$$V \supset B(c_1; \text{diam}(U_g)) \supset U_g$$

holds, and $g^q(0) = 0$ and $g^k(0) \neq 0$ for each $1 \leq k < q$, then

$$\det(I - \rho A) = \sum_{i=0}^{q-1} \frac{\rho^i}{Dg^i(c_1)} \neq 0$$

holds for all $|\rho| \leq 1$. $\Rightarrow$ positive transversality.
Prototype application to family of non-analytic maps

**Theorem**

*Fix real numbers* \( \ell \geq 1 \) *and* \( b > 2(\ell e)^{1/\ell} \) *and consider the family*

\[
f_c(x) = be^{-1/|x|^\ell} + c, \quad c \in \mathbb{R}
\]

*of unimodal maps. Let* \( \beta \in (0, \ell^{1/\ell}) \) *be the solution of the equation*

\[
f_\beta(\beta) = \beta \quad \text{i.e.} \quad f_\beta(0) = -\beta, \quad f_\beta^2(0) = f_\beta(\beta) = \beta.
\]

*Then the kneading sequence* \( K(f_c) \) *is monotone decreasing in* \( c \in [-\beta, \infty) \) *and the positive transversality condition (3) below.*

- **“Positive” transversality:** if \( f_{c_*} \) has 0 as a periodic point of period \( q \), then

\[
\left. \frac{d}{dc} f_{c_*}^q(0) \right|_{c=c_*} = \sum_{n=0}^{q-1} \frac{1}{Df_{c_*}^n(f_{c_*}(0))} > 0. \tag{3}
\]
\[ \ell = 1, \quad b = 2(e\ell)^{1/\ell} - 0.1 \]

\[ \ell = 1, \quad b = 2(e\ell)^{1/\ell} + 0.1 \]
Families of the form \(af(x)\)

For which \(f\), the entropy of \(x \mapsto af(x)\) is monotone in \(a\)?

This question is subtle: many conjectures stated in literate turned out to be wrong.

As usual we say that \(v \in \mathbb{C}\) is a **singular value** of a holomorphic map \(f : D \to \mathbb{C}\) if it is a critical value, or an asymptotic value \((\exists \gamma : [0, 1) \to D \text{ so that } \gamma(t) \to \partial D \text{ and } f(\gamma(t)) \to v \text{ as } t \uparrow 1)\).

Consider the **class** \(\mathcal{E}\) of holomorphic maps \(f : D \to \mathbb{C}\) such that:

(a) \(D\) is a domain which is symmetric w.r.t. \(\mathbb{R}\) and \(D \cap \mathbb{R} = I\) where \(I\) is an open interval (finite or infinite), \(0 \in I\);

(b) \(f(I) \subset \mathbb{R}, f(D) = \mathbb{C}\) and the only possible (finite) asymptotic value of \(f\) is 0;

(c) \(f(0) = 0\);

(d) the only critical values of \(f\) are 1 and, perhaps, 0 and \(\exists\) minimal \(c > 0\) such that \(f\) has a positive local maximum at \(c\).
Simple examples of entire functions $f$ of the class $\mathcal{E}$ are

- $f(z) = 4z(1 - z)$,
- $f(z) = 4 \exp(z)(1 - \exp(z))$,
- $f(z) = \sin(z)^2$,
- $f(z) = m^{-m}(ez)^m \exp(-z)$ when $m$ is a positive even integer.

We also have a similar class $\mathcal{E}_o$:

- $f(z) = \sin(z)$ and
- $f(z) = (m/2)^{-m/2}e^{m/2}z^m \exp(-z^2)$ when $m$ is a positive odd integer. (in fact, $z^m \exp(-z^2)$ and $z^m \exp(-z)$ are conjugated by $z \mapsto 2z^2$).
Families of the form $af(x)$

**Theorem**

Assume that $f \in \mathcal{E} \cup \mathcal{E}_o$ and that the critical point $c > 0$ is either periodic or eventually periodic for $f_a(x) = af(x)$ at $a = a_* > 0$. Then

- positively oriented transversality holds for $f_a(x)$ at $a = a_*$.  
- $\implies$ monotonicity within these families.
Theorem (The family $x \mapsto |x|^{\ell} + c$)

Let $\ell_-, \ell_+ \geq 1$ and consider the family of unimodal maps

$$f_c(x) = \begin{cases} |x|^{\ell_-} + c & \text{if } x \leq 0 \\ |x|^{\ell_+} + c & \text{if } x \geq 0. \end{cases}$$

For any integer $L \geq 1$ there exists $\ell_0 > 1$ so that for any $q \geq 1$ periodic kneading sequence $i = i_1i_2\cdots \in \{-1,0,1\}^{\mathbb{Z}^+}$ so that

$$\#\{1 \leq j < q; i_j = -1\} \leq L,$$

and any pair $\ell_-, \ell_+ \geq \ell_0$ there is at most one $c_* \in \mathbb{R}$ for which the kneading sequence of $f_c$ is equal to $i$. Moreover, one has positive transversality

$$\sum_{n=0}^{q-1} \frac{1}{Df^n_{c_*}(c_*)} > 0.$$
Walter, Happy Birthday!!

Organisers: many thanks for organising this nice conference!