Resonances of complex dynamics : polynomials to rational dynamics

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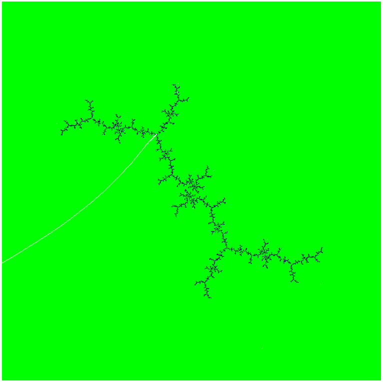
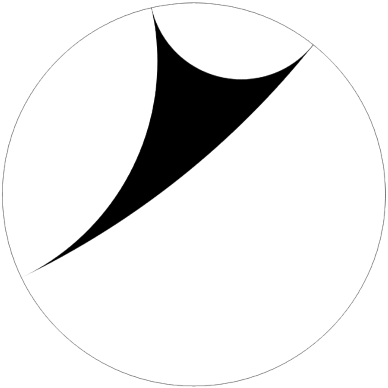
In the dynamics of polynomials, external rays play a fundamental role.

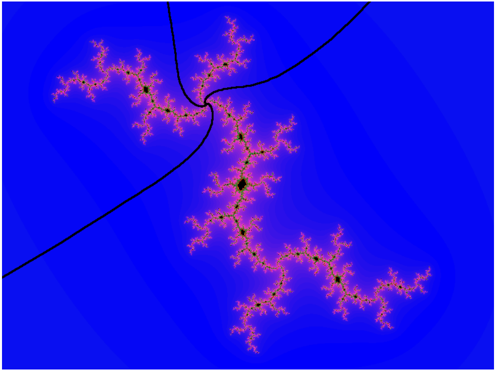


The Böttcher map conjugate the dynamics on $B(\infty)$ to z^d on $\mathbf{C} \setminus \overline{\mathbf{D}}$.

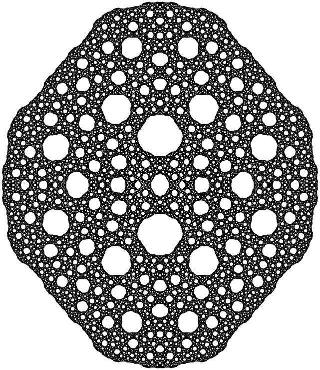
If the Julia set — $J = \partial B(\infty)$ — is locally connected this conjugacy extends to the boundary.

The dynamics on J is then semi-conjugated to the multiplication by 2 on $(\mathbb{R}/\mathbb{Z})/\sim$





Julia set of a rational maps



more complicate

One can understand the dynamics using a partition of the dynamical plane

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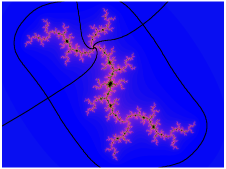
with equipotentials for Cantor sets

Tool : Branner-Hubbard-Yoccoz jigsaw puzzles for polynomials.



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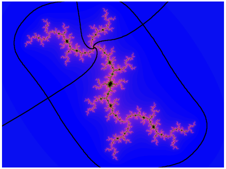

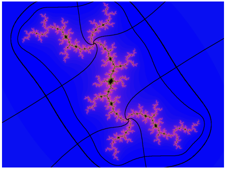

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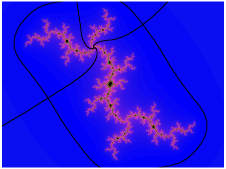 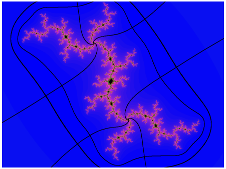 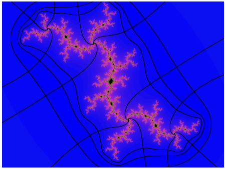

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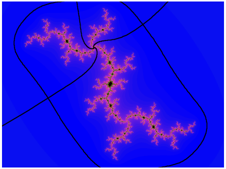

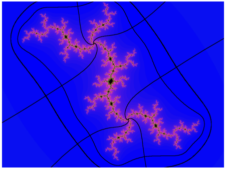

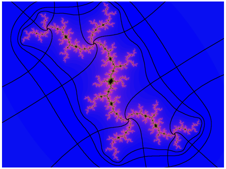

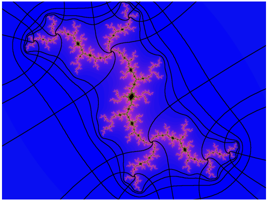

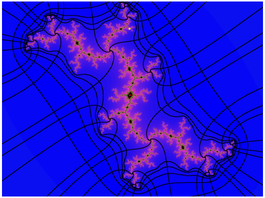

Yoccoz Theorem : The map is renormalizable or the impression of puzzle pieces is one point

How to generalize to rational maps a partition in order to understand the dynamics ?

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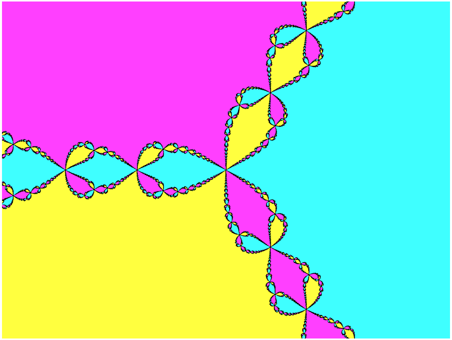
No general result

First example : Newton map

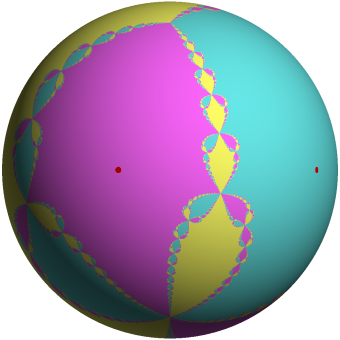
.

The *Newton's method* N_P of a polynomial P is defined by

$$N_P(z) = z - \frac{P(z)}{P'(z)}.$$

The roots of P are super-attracting fixed points of N_P.

The Newton maps can be viewed as a dynamical system



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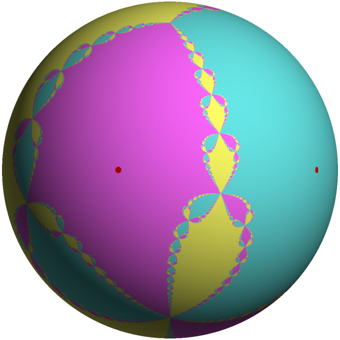

as well as a root-finding algorithm.

The Julia set is defined as the unique minimal compact subset of the Riemann sphere $\widehat{C}$ totally invariant (by N and N^{-1}) containing at least 3 points.

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- With Magnus Aspenberg we generalize this to non-postcritcally finite cases with some assumptions using puzzles.
- Lodge, Mikulich and Schleicher gave a combinatorial classification of post-critically finite Newton maps.

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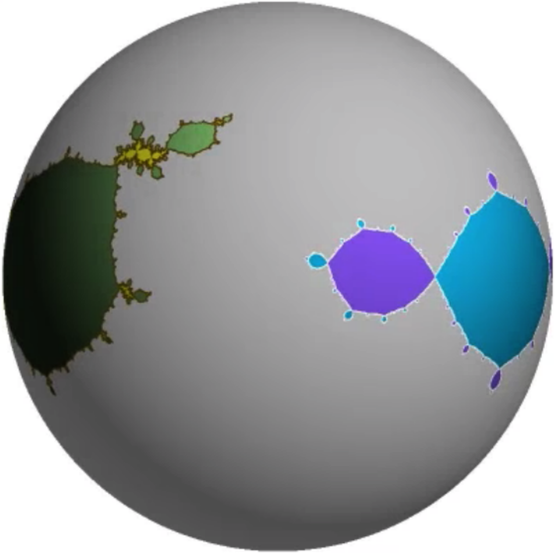
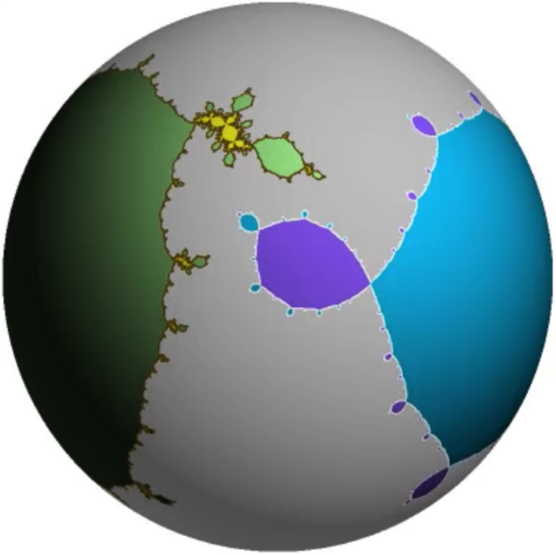
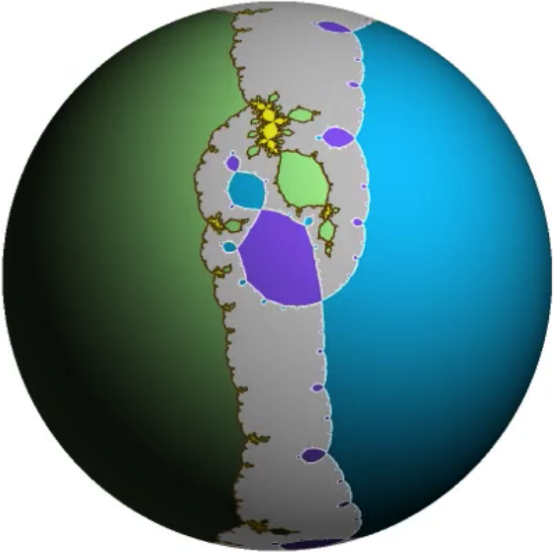
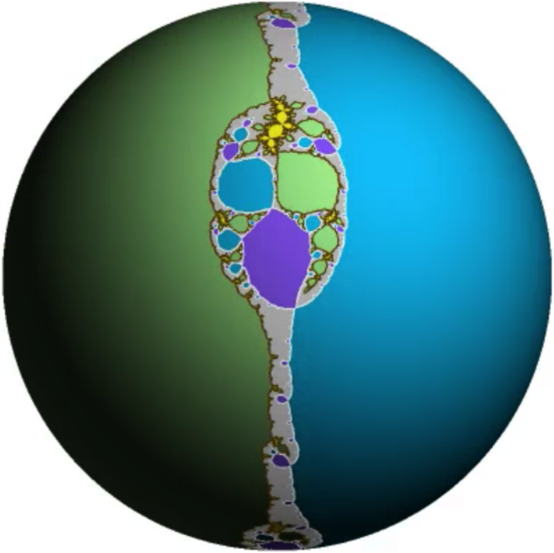
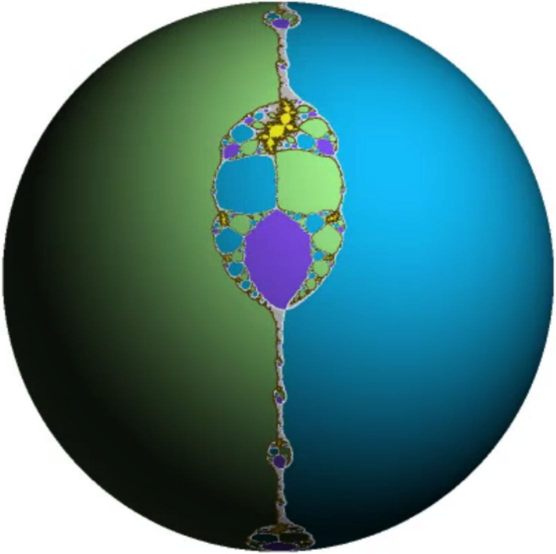
- For the higher dimensional cases, Hubbard and Papadopol , Roeder studied the Newton's methods for two complex variables.





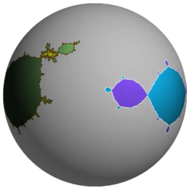






Sketch of the mating (images courtesy of A. Chéritat)

Understand rational map via the two polynomials



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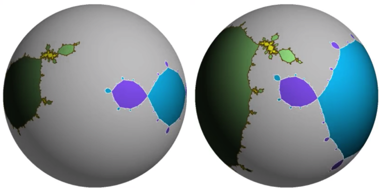

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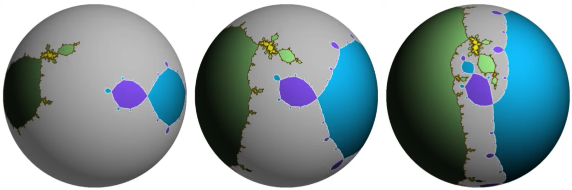

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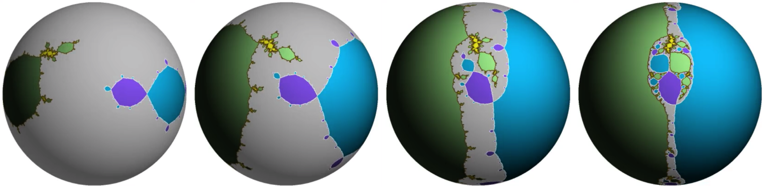

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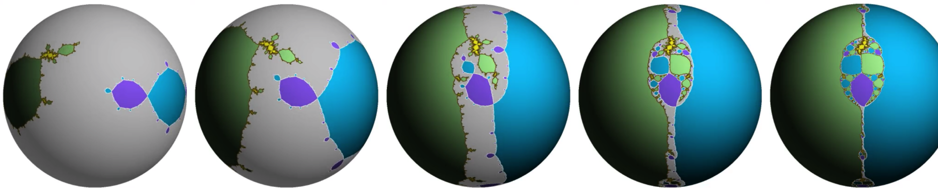
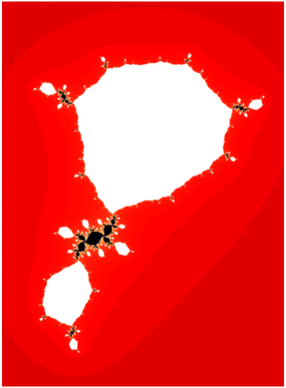


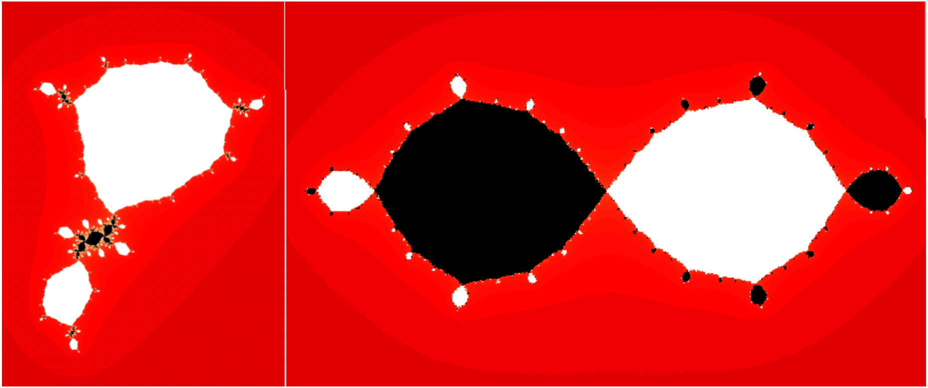
Definition

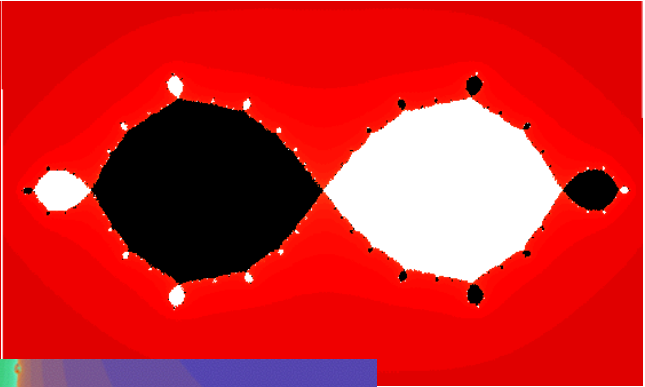
Two polynomials f_1 and f_2 are said mateable, if there exist a rational map R and two semi-conjugacies $\phi_j : K_j \to \hat{\mathbb{C}}$ conformal on the interior of K_j, such that $\phi_1(K_1) \cup \phi_2(K_2) = \hat{\mathbb{C}}$ and

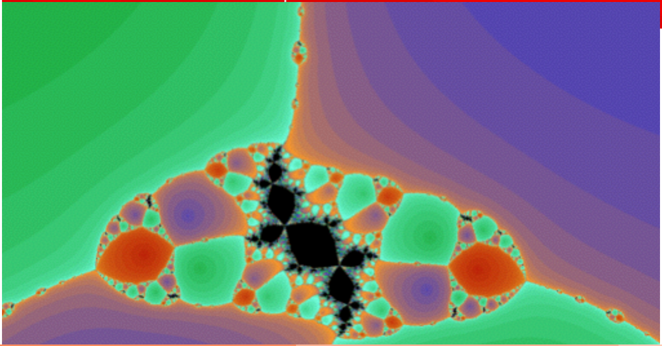
$$\forall (z, w) \in K_i \times K_j, \quad \phi_i(z) = \phi_j(w) \iff z \sim_r w.$$

The relation $\sim_r$ is generated by :
the landing point of $R_1(t)$ is equivalent to the landing point of $R_2(-t)$.





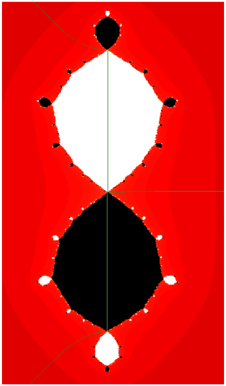


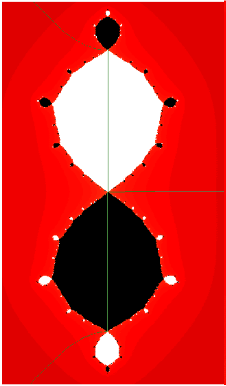


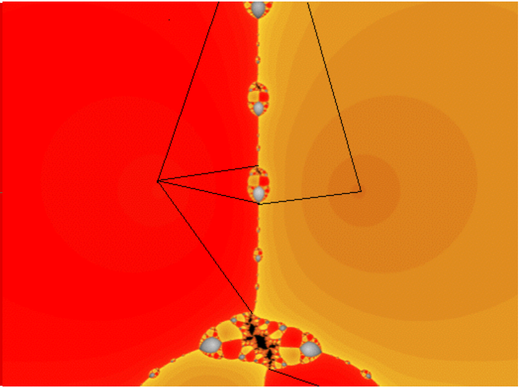
Theorem (Aspenberg, R)

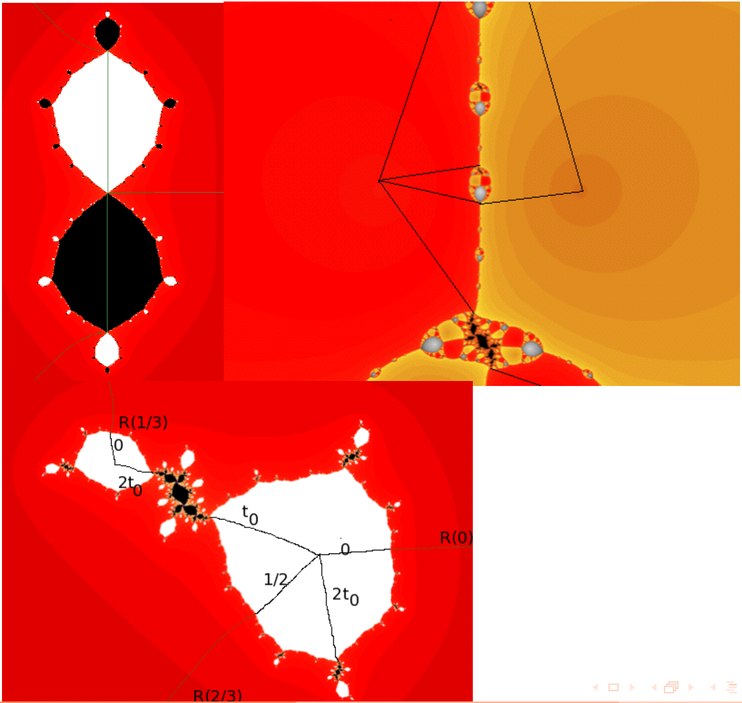
There exists a subset RC of renormalizable cubic polynomials, a subset RN of renormalizable cubic Newton maps and a map $M : RC \to RN$ which is onto and such that $M(f)$ is the mating of f with the polynomial $f_\infty(z) = z(z^2 + \frac{3}{2})$.

One can understand the dynamics of N through the dynamics of the polynomials. But there is no external rays any more.

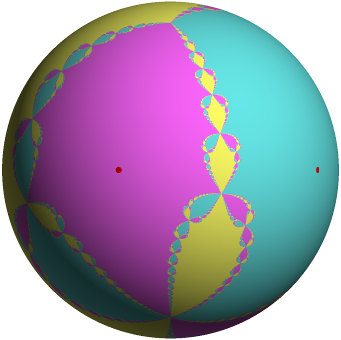







R(1/3)
0
2t0
t0
0
R(0)
1/2
2t0
R(2/3)

Idea of the proof : use puzzle. Cut the Julia set in small pieces. Need to construct the equivalent to external ray.



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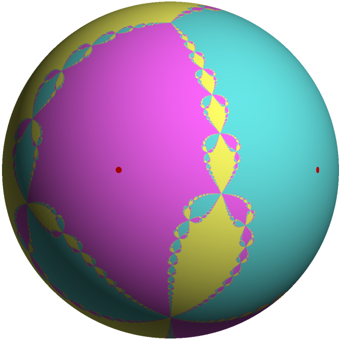
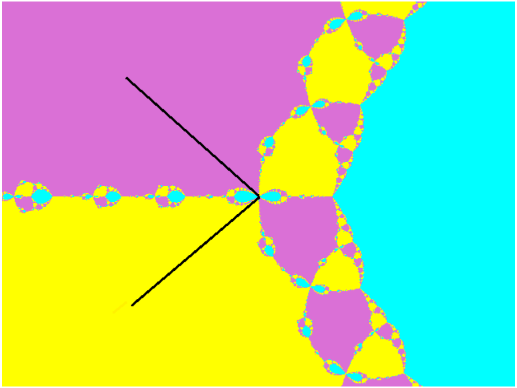


There are 3 basins corresponding to the 3 roots of P, ∞ is a common point, landing of fixed internal rays in the basins.

Except in the symmetric case, only two basins intersect and there is a last angle of intersection



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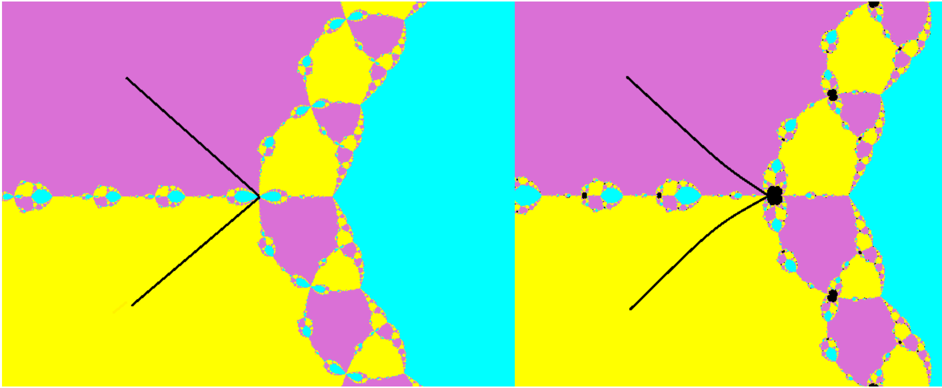
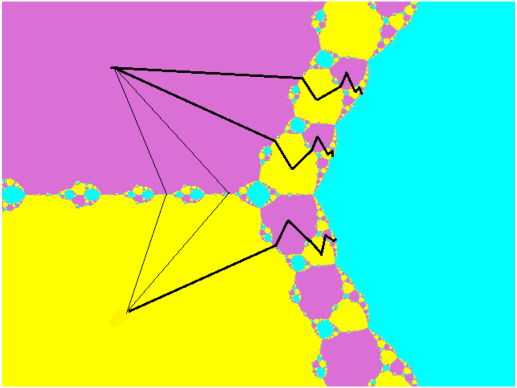
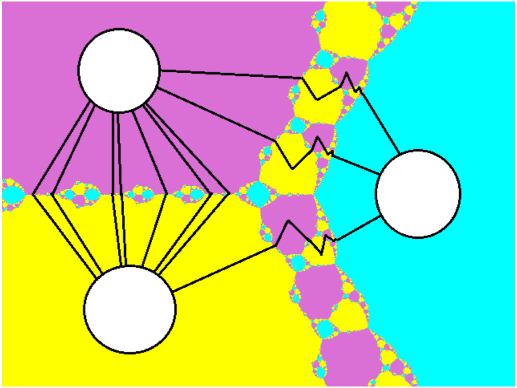


Construction of articulated rays by iterated pull back



It is a curve γ such that $f^k(\gamma) = \gamma \cup R_1(t) \cup \overline{R_2(-t)}$. It consists in infinitely many internal rays alternating from basin 1 et 2.

Using the following two graphs,



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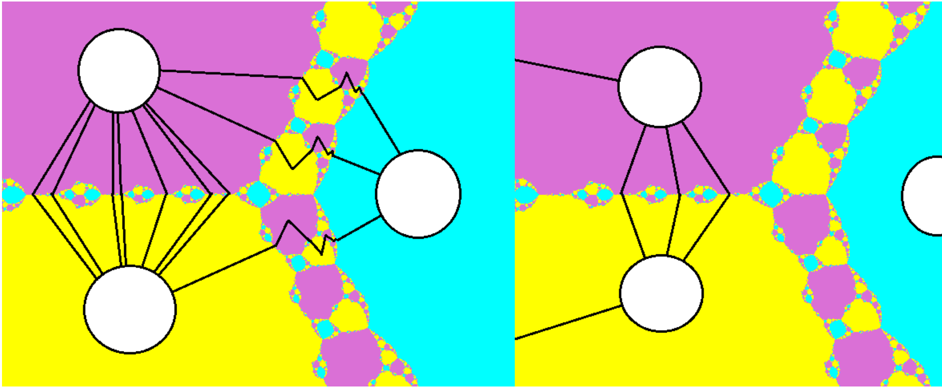
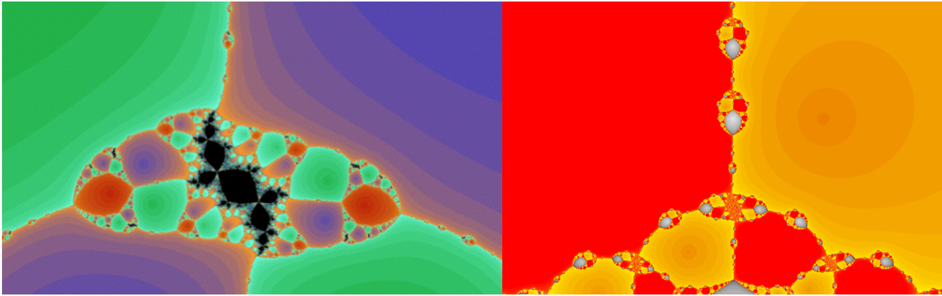


Theorem (R)

The intersection of the puzzle piece is either a point or the homeomorphic image of the filled Julia set of a quadratic polynomial.

Using similar puzzles for the cubic polynomials Julia sets, we can construct the semi-conjugacies to the Julia set of the Newton map.

Difference



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Except some definite particular cases the Julia set is locally connected.

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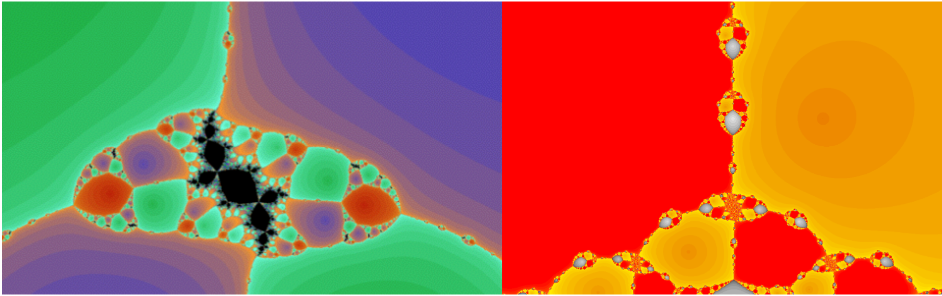


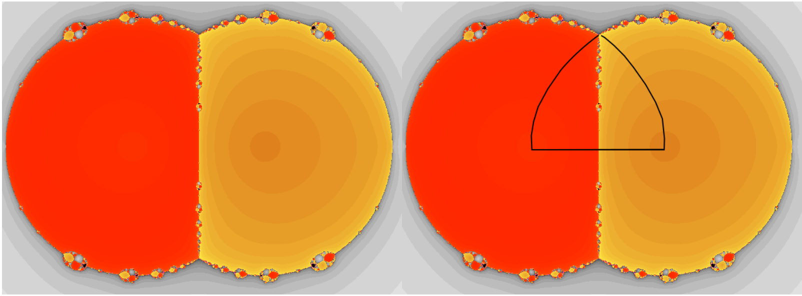
Theorem (R)

Except some definite particular cases the Julia set is locally connected.

Theorem (R)

In particular $J(N) \supset h(J(P))$ where $J(P)$ is a non locally connected Julia set of quadratic polynomials P and $J(N)$ is locally connected.

To find the cubic Newton map, one has to investigate the space of cubic Newton map.



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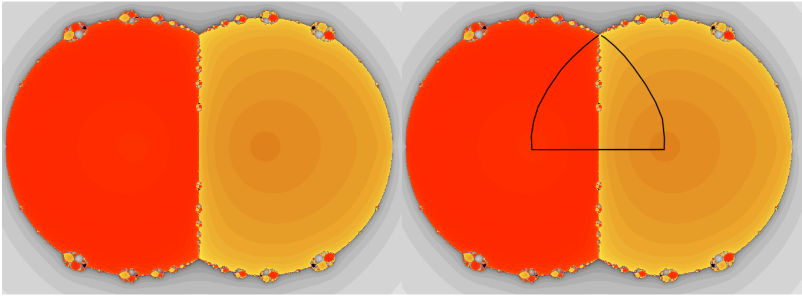
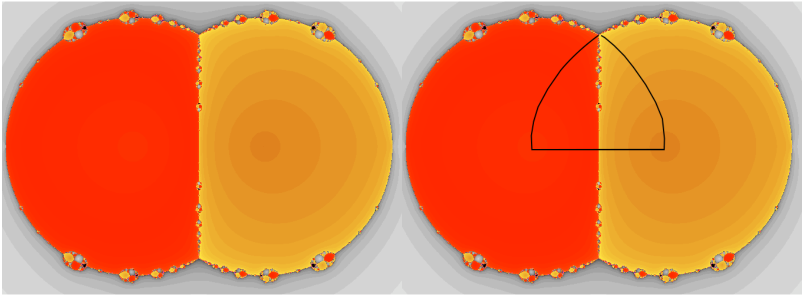


It is a one parameter slice with symmetries.
More precisely any Newton map is conjugate to one of the form

$$N_\lambda(z) = \frac{2z^3 - (\lambda^2 - \frac{1}{4})}{3z^2 - (\lambda^2 + \frac{3}{4})} \text{ with } \lambda \in \mathbb{C} \setminus \{\pm\frac{3}{2}, 0\}$$

To find the cubic Newton map, one has to investigate the space of cubic Newton map.



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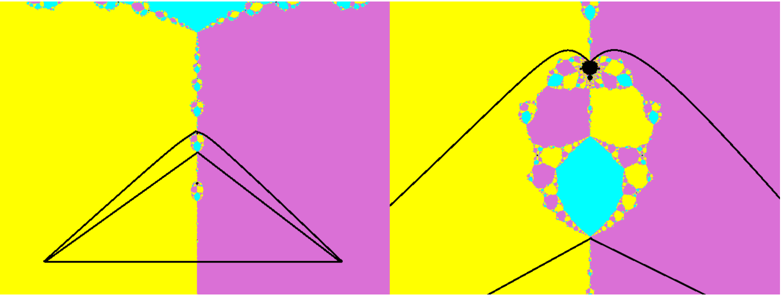
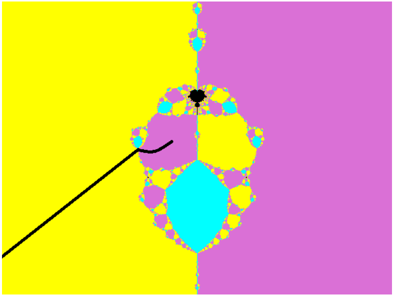
The graphs exist and define puzzles in some precise regions of the parameter plane called para-puzzle pieces.



To define them one has to transfer to the parameter plane the articulated



rays and all the pre-images.

Para-puzzles are technical. New technics : rigidity to investigate the parameter plane of cubic Newton method.

Para-puzzles are technical. New technics : rigidity to investigate the parameter plane of cubic Newton method.

It has the advantage that it can be generalized to higher degree Newton maps.

Theorem (Wang, R, Yin)

Any ray in any hyperbolic component lands. The boundary of any hyperbolic component is a Jordan curve.

It generalizes the proof done with para-puzzle pieces of the following

Theorem (R)

The boundary of the principal hyperbolic components are Jordan curves.

Sketch of the proof in the case of the principal hyperbolic component:

- Assume λ_1 and λ_2 are two accumulation points of an irrational ray so that $R_{\lambda_i}(t)$ lands at the free critical point of N_{λ_i}.
- Then the Newton maps N_{λ_1} and N_{λ_2} share the combinatorial dynamics with respect to the puzzles constructed with the same angles.
- There is a topological conjugacy ψ between N_{λ_1} and N_{λ_2}, which is holomorphic in the Fatou set of N_{λ_1}.
- The conjugacy is a quasi-conformal map.
- The Lebesgue measure of $J(N_{\lambda_i})$ is zero (Lyubich, Shishikura arguments on rational like maps with an admissible puzzle)
- The distortion on puzzle pieces based on $J(N_{\lambda_1})$ is bounded.
- The conjugacy is a Möbius transformation

More recent progress in the dynamical plane

Theorem (Wang, Yin, Zeng)

Let f_p be the Newton map for any non-trivial polynomial p. Then the boundary of any immediate root basin B is locally connected. Moreover, ∂B is a Jordan curve if and only if $deg(f^p_{|B}) = 2$.

This is proved by generalizing the work for cubic Newton maps. Namely the puzzles.
As a corollary this puzzle allows to get the rigidity for higher degree Newton maps.

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This is proved by generalizing the work for cubic Newton maps. Namely the puzzles.
As a corollary this puzzle allows to get the rigidity for higher degree Newton maps.

Theorem (Drach, Lodge, Schleicher, Sowinski)

There exists an invariant graph for higher degree Newton maps that gives a Fatou-Shihikura inequality.

McMullen maps

We consider the maps

$$f_\lambda : z \mapsto z^n + \frac{\lambda}{z^n}.$$

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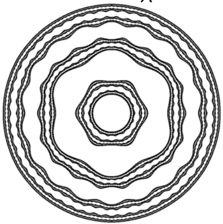
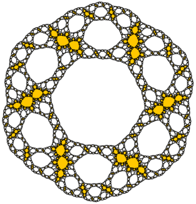
McMullen showed that the Julia set of f_λ is a Cantor set of simple closed curves provided
$n \neq 1, 2$ and λ is small.



We restrict to $n \geq 3$.

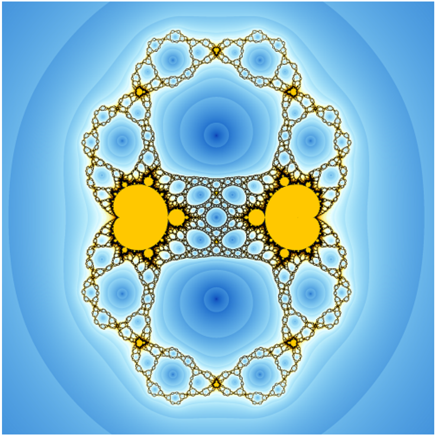
There exist also maps which are renormalizable and contain copies of



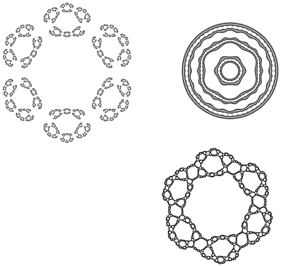
polynomial Julia sets.

In the parameter plane appear :

- the unbounded component which is the Cantor set region
- the neighborhood of 0 where $J(f_\lambda)$ is a Cantor set of circles
- the other "holes" where the Julia set is a Sierpinsky carpet.



$n = 3$



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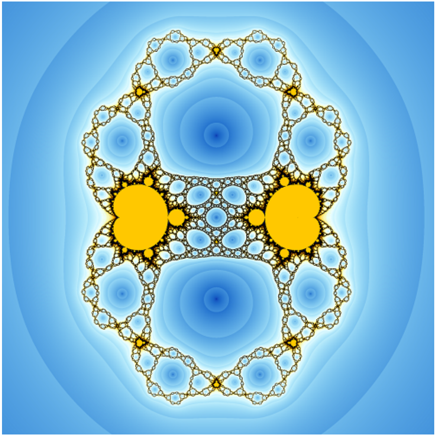

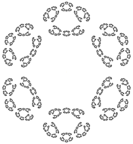

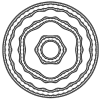

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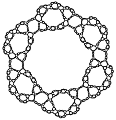
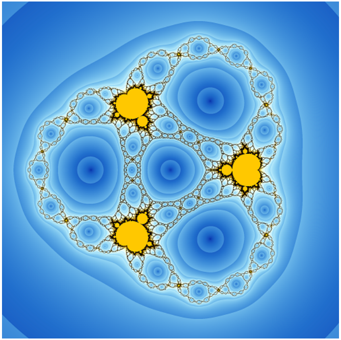
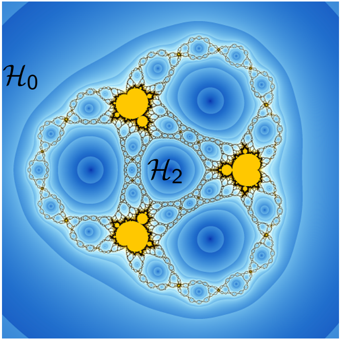


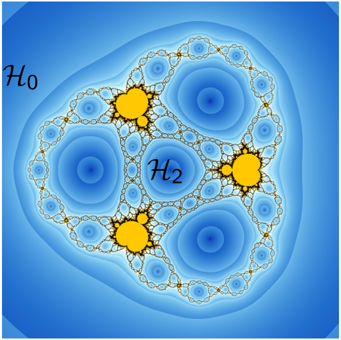
$\mathcal{H}_\infty$: the set of λ so that the critical points converge to ∞.





$\mathcal{H}_0$ is the unbounded component

$\mathcal{H}_2$ is the component containg 0



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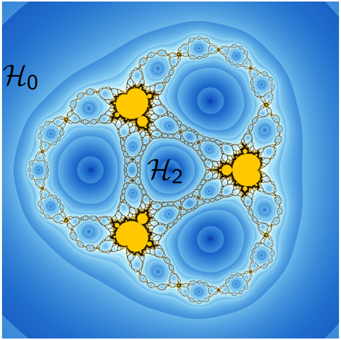

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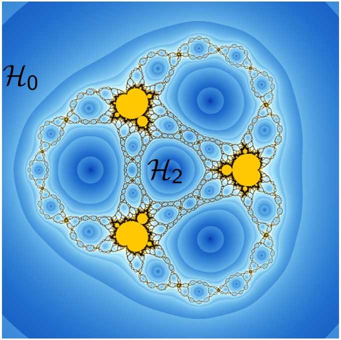
Precisely,

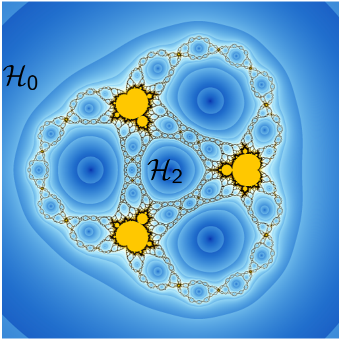
Theorem (Devaney-Look-Uminsky; Devaney-Russell)

- *If $\lambda \in \mathcal{H}_0$, then $J(f_\lambda)$ is a Cantor set ;*
- *If $\lambda \in \mathcal{H}_2 \setminus \{0\}$, then $J(f_\lambda)$ is homeomorphic to the product of a Cantor set and a circle ;*
- *If $\lambda \in \mathcal{H}_\infty \setminus (\mathcal{H}_0 \cup \mathcal{H}_2)$, then $J(f_\lambda)$ is a Sierpinsky carpet ;*
- *If $\lambda \notin \mathcal{H}_\infty$ then $J(f_\lambda)$ is connected.*



Theorem (Devaney)

The boundary of $\mathcal{H}_2$ is a Jordan curve.



Theorem (Devaney)

The boundary of $\mathcal{H}_2$ is a Jordan curve.

Conjecture (Devaney)

The boundary of any connected component of $\mathcal{H}_\infty$ is a Jordan curve.

Theorem (Qiu, Rœsch, Wang, Yin)

Let $\mathcal{H}$ be any connected component of $\mathcal{H}_\infty$. Then $\mathcal{H}$ is a Jordan domain.

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Proposition (Qiu, Rœsch, Wang, Yin)

The parametrization extends to the boundary as a function $\nu(\theta)$.

- If θ is periodic then the dynamical ray lands at a parabolic point.
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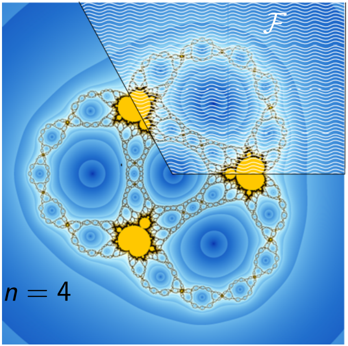
Corollary

The cusps are dense in the boundary of $\mathcal{H}_0$.

$$f_\lambda(\bar{z}) = \overline{f_{\bar{\lambda}}(z)} \quad \text{and} \quad f_\lambda(\omega z) = \omega f_{\lambda\omega^{-2}}(z) \quad \text{where} \quad \omega = e^{\frac{2i\pi}{n-1}}.$$

We will always restrict to the fundamental domain :



$$\mathcal{F} = \{\lambda \in \mathbf{C}^* \mid 0 \le arg\lambda < \frac{2\pi}{n-1}\}$$

Some dynamics

The maps $f_\lambda(z) = z^n + \lambda/z^n$ are the composition of two simple maps

$$z \mapsto z + \frac{\lambda}{z} \quad \text{and} \quad z^n.$$

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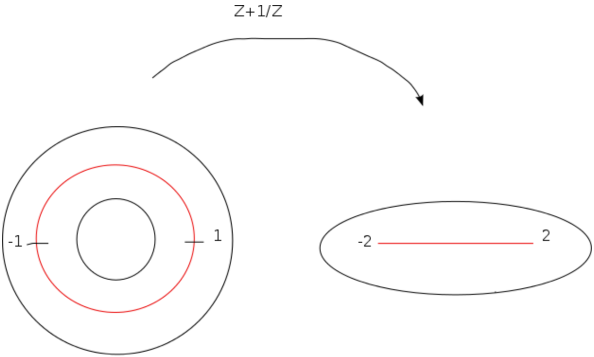
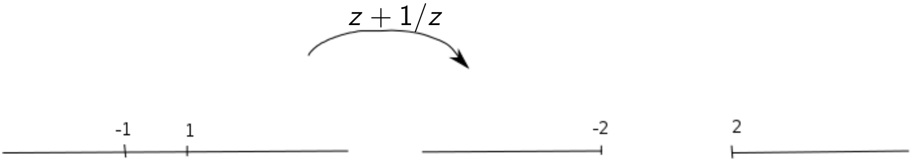
The map

$$z \mapsto z + \frac{\lambda}{z}$$

is just conjugated to

$$z \mapsto z + \frac{1}{z}.$$


Z+1/Z
-1
1
-2
2

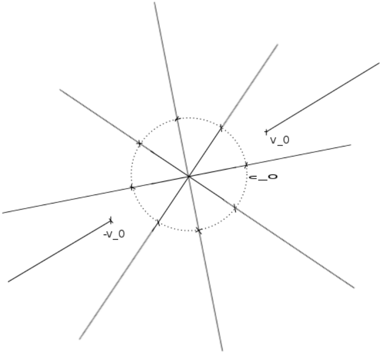

$z + 1/z$
-1
1
-2
2

The critical set of the map $f_\lambda(z) = z^n + \lambda/z^n$ is

$$Crit = \{0, \infty\} \cup C_\lambda$$

where

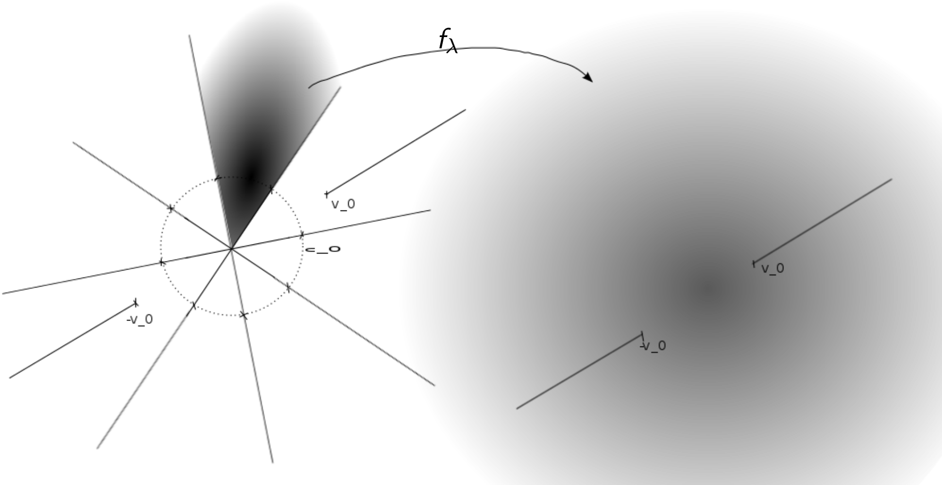
$$C_\lambda = \{c \mid c^{2n} = \lambda\} = \{c_0 e^{\frac{ik\pi}{n}} \mid k \in [0, .., 2n-1]\}$$



$n = 4$

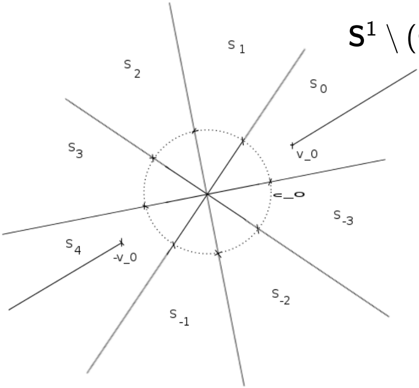
.

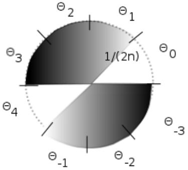
In each sector the map is one to one onto $\mathbf{C} \setminus \pm v_0[1, +\infty]$.



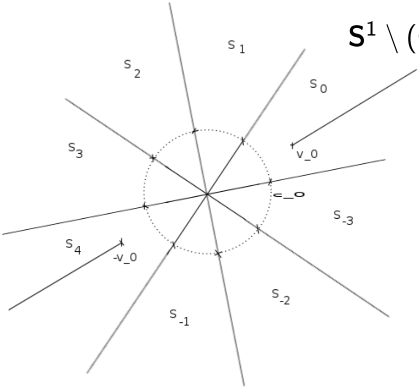
On can pull back any sector except the ones containing $\pm v_0$.

$$\mathbf{S}^1 \setminus (\Theta_0 \cup \Theta_n) = \left(\frac{1}{2n}, \frac{1}{2}\right] \cup \left(\frac{1}{2} + \frac{1}{2n}, 1\right]$$





$$\tau(\theta) = n\theta \mod 1.$$



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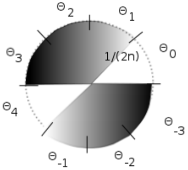

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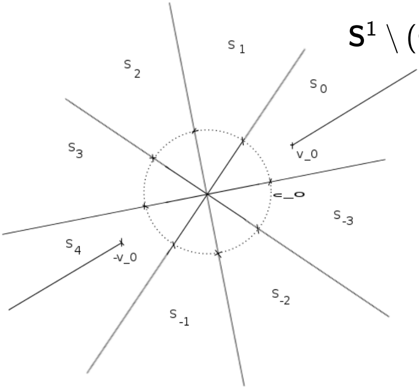
θ has itinerary $(s_0, \cdots, s_k, \cdots)$

if $\tau^k(\theta) \in \Theta_{s_k}$



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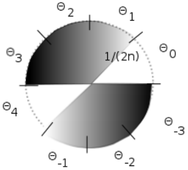

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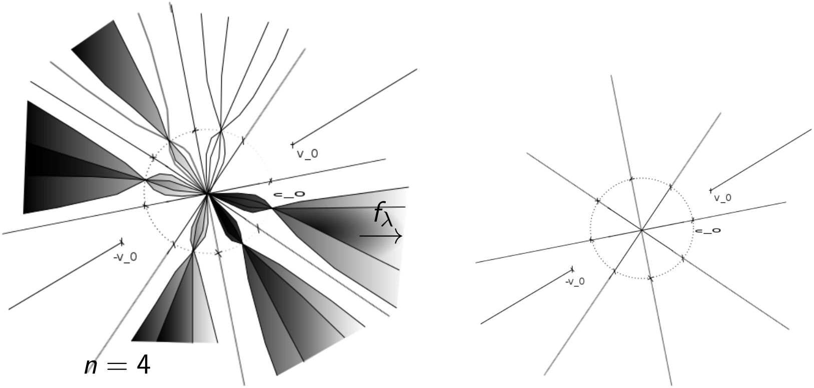
$$\Theta = \left\{\theta \mid \tau^k(\theta) \in \mathbf{S}^1 \setminus (\Theta_0 \cup \Theta_n) \quad \forall k \geq 0\right\}$$

Pulling back to the sectors without critical values



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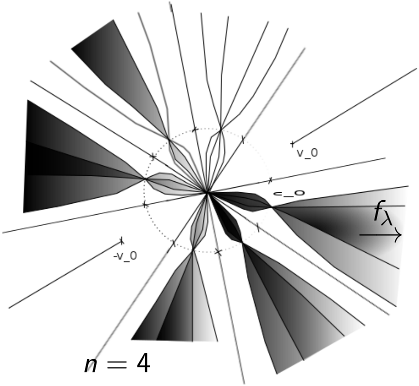
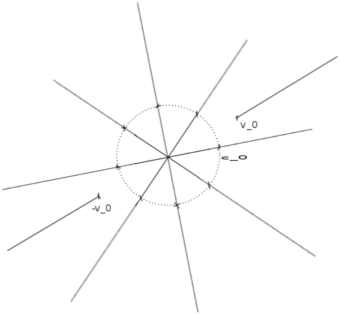




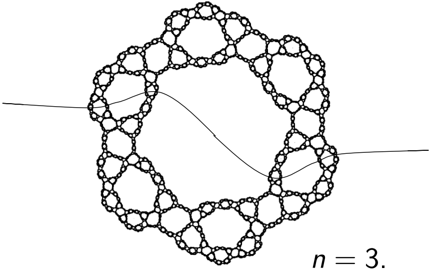
The intersection of a decreasing sequence of sectors shrinks to a curve in some cases.

Theorem (Devaney, Qiu-Wang-Yin)

For any λ in the interior of the fundamental domain $\mathcal{F}$ and for any $\theta \in \Theta$ with itinerary $(s_0, s_1, \cdots,)$ the set

$$\Omega_\lambda^\theta := \bigcap_{k \geq 0} f_\lambda^{-k}(S_{s_k}^\lambda \cup S_{-s_k}^\lambda)$$

is a Jordan curve intersecting the Julia set under a Cantor set.



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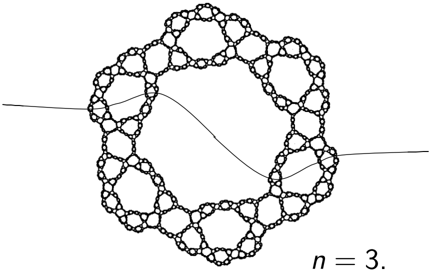
"cut rays"



$\Omega_\lambda^1 = \Omega_\lambda^{1/2}$

$n = 3.$

There is a similar construction for $\lambda \in \mathbf{R}$.

The "cut rays" are used in order to construct a puzzle.

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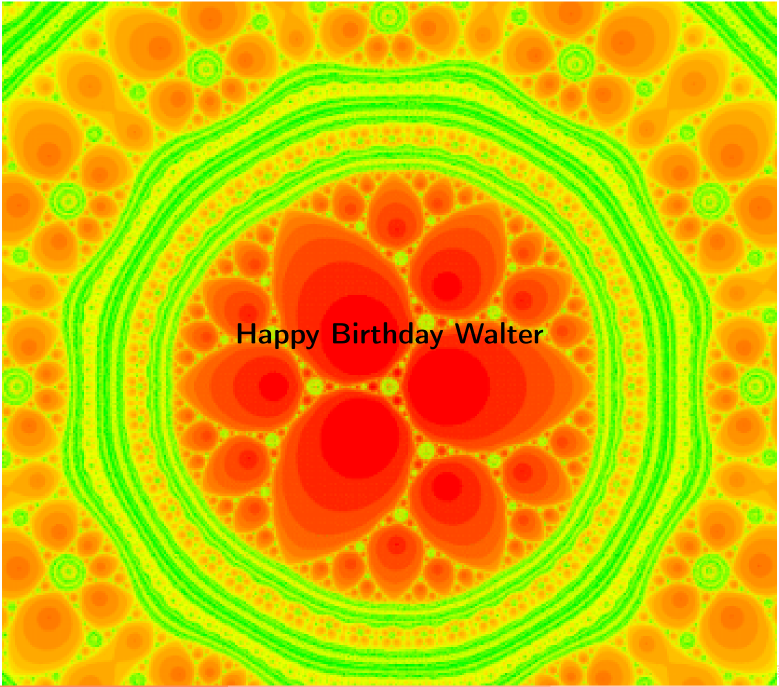
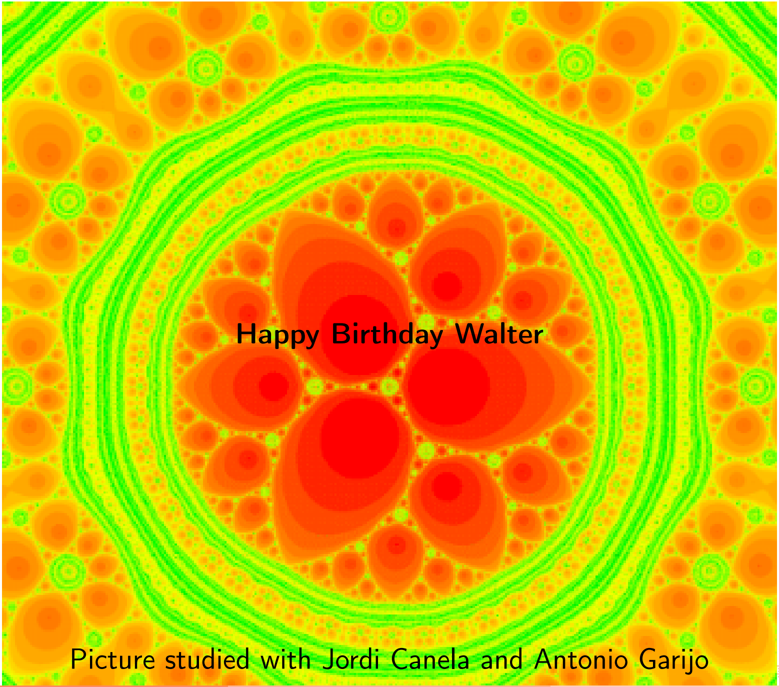
Theorem (Qiu-Wang-Yin)

If $J(f_\lambda)$ is not a Cantor set, then the boundary of B_λ is a Jordan curve.

The result in parameter plane uses rigidity arguments.


Happy Birthday Walter



Picture studied with Jordi Canela and Antonio Garijo