The quasi-Fatou set in quasiregular dynamics

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Talk overview

- Quick introduction to quasiregular maps on $\mathbb{R}^d$. These generalize analytic functions on $\mathbb{C}$.
- Survey some results from “quasiregular dynamics”. We seek an iterative theory parallel to complex dynamics.
- Discuss some specific results on the quasi-Fatou set.

Quasiregular mappings

Definition

A continuous $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is *quasiregular* (qr) if $f \in W^1_{d,\mathrm{loc}}(\mathbb{R}^d)$ and there exists $K_O \geq 1$ such that

$$\|Df(x)\|^d \leq K_O J_f(x) \qquad \text{a.e.}$$

where $\|Df(x)\|$ is the norm of the derivative and $J_f(x)$ is the Jacobian.

- Informally, a qr map sends infinitesimal spheres to infinitesimal ellipsoids of bounded eccentricity.
- A mapping is called K-qr if the local distortion is $\leq K$.
- Holomorphic functions on $\mathbb{C}$ are 1-qr.
- The iterates of a qr map are qr, but in general if f is K-qr then f^n may be K^n-qr.

Quasiregular mappings

Quasiregular functions on $\mathbb{R}^d$ generalize analytic functions on $\mathbb{C}$.

Theorem (Reshetnyak, 1967-68)

Non-constant quasiregular maps are open, discrete and almost everywhere differentiable.

Definition

A non-constant qr map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is called *polynomial type* if

$$\lim_{x\to\infty} |f(x)| = \infty.$$

Otherwise, this limit does not exist and f is *transcendental type*.

Can also consider quasiregular self-maps of $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ that are analogous to rational functions on $\overline{\mathbb{C}}$.

Some easily-stated results

Let f be quasiregular on $\mathbb{R}^d$ of transcendental type.

Theorem (Siebert, 2004)

f has infinitely many periodic points of every period $p \geq 2$.

Theorem (Bergweiler, Fletcher, Langley, Meyer, 2009)

The escaping set is non-empty; that is,

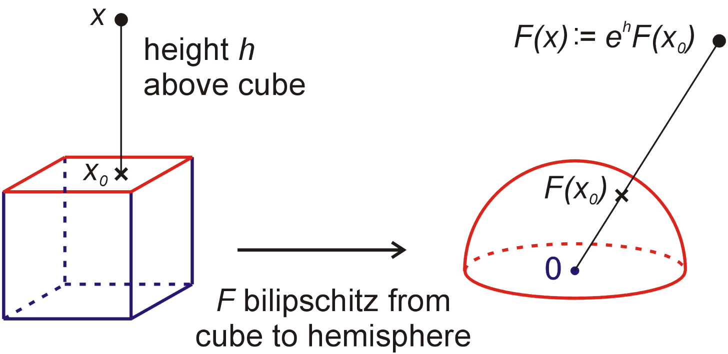
$$I(f) := \{x : f^n(x) \to \infty \text{ as } n \to \infty\} \neq \emptyset.$$

Theorem (Bergweiler, Fletcher, Drasin, 2014)

The fast escaping set $A(f) \neq \emptyset$. In fact, all components are unbounded.

An example of a quasiregular map

Bergweiler and Eremenko defined a qr “trig function analogue” on $\mathbb{R}^d$ as follows:



Extend to a map $\mathbb{R}^d \to \mathbb{R}^d$ by reflecting in hyperplanes.

For large enough λ, the map $S := \lambda F$ is locally uniformly expanding.

An application of quasiregular dynamics(!)

By iterating S, they obtained a strong Karpińska paradox in $\mathbb{R}^d$:

Theorem (Bergweiler, Eremenko, 2011)

Let $d \geq 2$. $\mathbb{R}^d$ can be expressed as an uncountable union of hairs such that

- *any two hairs intersect only at a common endpoint (if at all); and*
- *the union of hairs without their endpoints has Hausdorff dim 1. (It follows that the set of endpoints has Hausdorff dim d.)*

Remark: the hairs minus endpoints lie in the escaping set $I(S)$.

Vogel, 2015: $I(S)$ has positive Lebesgue measure.

Analogues of Picard's and Montel's theorem

Theorem (Rickman, 1980)

For $d \geq 2$ and $K \geq 1$ there exists a constant $q = q(d, K)$ with the following property:
every K-qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ that omits q values must be constant.

Miniowitz used Rickman's theorem to obtain an analogue of Montel's theorem:

Theorem (Miniowitz, 1982)

Let $\mathcal{F}$ be a family of K-qr maps on a domain $D \subset \mathbb{R}^d$. If there exist distinct points $a_1, \ldots, a_q$ that are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

Uniformly quasiregular maps

- If every iterate f^n is K-quasiregular with the same K, then f is called *uniformly quasiregular* (uqr).
- For uqr maps, the usual definition of Fatou and Julia sets via normality works well.

Theorem (Hinkkanen, Martin, Mayer, 2004)

For a non-injective uniformly quasiregular map $f\colon \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$

- $J(f^p) = J(f)$;
- $J(f)$ *is perfect;*
- $J(f)$ *is the smallest closed completely invariant set with* $> q$ *points;*
- *classification of periodic points and periodic Fatou components;*
- $J(f) =$ *boundary of any attracting basins.*

Uniformly quasiregular maps

Question: For uqr maps, is the Julia set the closure of the repelling periodic points?

Theorem (Siebert, 2004)

Let $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ be uqr.

$$J(f) \not\subset \overline{\{\textit{post-branch set}\}} \implies J(f) = \overline{\{\textit{repelling periodic points}\}}.$$

Note:

- Uqr maps in dimension 2 are quasiconformally conjugate to rational/analytic maps.
- No examples are known of transcendental type uqr maps in dimension ≥ 3.

Non-uniformly quasiregular dynamics

Now let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ or $\overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ be K-qr, but not assumed uqr.

Extending an idea of Sun and Yang (c.1999) we use a 'blowing-up' property to *define* the Julia set:

$$J(f) := \left\{ x : \text{for every nhd } U \text{ of } x,\ \mathbb{R}^d \setminus O^+(U) \text{ is small} \right\}.$$

Here "small" means conformal capacity zero.
It follows immediately that $J(f)$ is closed and completely invariant.

Theorem (Bergweiler 2013, Bergweiler, N. 2014)

The definition of $J(f)$ above agrees with the usual one if f is uqr.
If $\deg(f) > K_I$, then $J(f) \neq \emptyset$ and, in fact, $J(f)$ is infinite.

Example: For the qr sine analogue, $J(S) = \mathbb{R}^d$ (Fletcher, N. 2013).

A conjecture

Assume $f\colon \mathbb{R}^d \to \mathbb{R}^d$ or $\overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is K-qr, with $\deg(f) > K_I$.

$$J(f) := \left\{ x : \text{for every nhd } U \text{ of } x,\ \mathbb{R}^d \backslash O^+(U) \text{ is small} \right\}.$$

Conjecture

- Equivalent to replace "small" by "finite" in $J(f)$ definition.
- $J(f)$ is perfect and $J(f^p) = J(f)$.

The conjecture is open in general, but holds under a variety of extra hypotheses. In particular, it holds in two dimensions or if f is Lipschitz.

Warren defines $J(f)$ for quasimeromorphic maps (with poles) of trans type. $J(f) \neq \emptyset$ and the analogous conjecture holds for such maps.

The quasi-Fatou set

The rest of this talk considers the *quasi-Fatou set* of a qr map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ of trans type,

$$\mathcal{QF}(f) := \mathbb{R}^d \setminus J(f).$$

Note: no normality assumption!

Notation: the maximum modulus $M(r,f) = \max\{|f(x)| : |x| = r\}$.

Theorem (Bergweiler, Fletcher, N. 2014)

Let f be qr such that

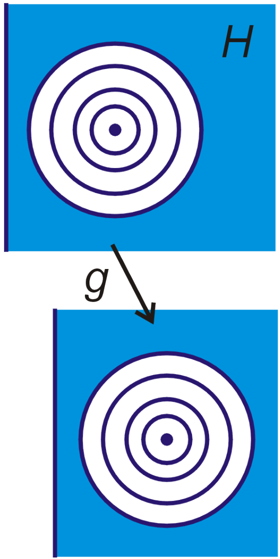
$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} = \infty. \tag{1}$$

Then $J(f) = \partial A(f)$, where $A(f)$ is the fast escaping set.

Corollary: If (1) holds and a quasi-Fatou component meets $A(f)$, then it is contained in $A(f)$. (A 'normality' property!)

This is not true for the escaping set $I(f)$ ...

Example: Modifying Fatou's function



Fatou's function $g(z) = z + 1 + e^{-z}$ has a right half-plane $H \subset I(g) \cap F(g)$ on which $g(z) \approx z + 1$.

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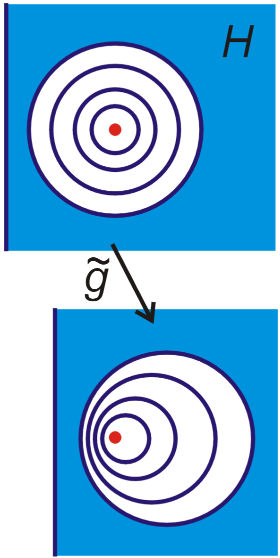

Fatou's function $g(z) = z + 1 + e^{-z}$ has a right half-plane $H \subset I(g) \cap F(g)$ on which $g(z) \approx z + 1$.

Modifying g in a disc, as shown, gives a qr map $\tilde{g}$ with a fixed point in H.

We still have $H \subset \mathcal{QF}(\tilde{g})$ because

$$\tilde{g}(H) \subset H \Rightarrow \text{no blowing-up in } H.$$

But H contains escaping and non-escaping points of $\tilde{g}$.

Full periodic domains vs. Baker domains

- We say that a domain in $\mathbb{R}^d$ is *full* if its complement has no bounded components; otherwise, it is *hollow*.
- A component U of $\mathcal{QF}(f)$ is called p-periodic if $f^p(U) \subset U$.

Theorem (Baker, 1988)

If f is trans entire on $\mathbb{C}$ and U is a p-periodic Fatou component, then, for $z \in U$,

$$\log |f^{np}(z)| = O(n) \quad \text{as } n \to \infty.$$

Periodic Fatou components are always full (simply-conn) and are called *Baker domains* if $U \cap I(f) \neq \emptyset$. Baker domains cannot meet $A(f)$.

Theorem (N., Sixsmith, 2017)

If f is trans type qr on $\mathbb{R}^d$ and U is a full p-periodic quasi-Fatou component, then, for $x \in U$,

$$\log\log |f^{np}(x)| = O(n) \quad \text{as } n \to \infty.$$

Moreover, $U \cap A(f) = \emptyset$.

Examples

Our first example shows we cannot improve the qr result to be as good as the entire one.

1. On $\mathbb{C}$, let $g(z) = z + 1 + e^{-z}$ and $\phi(z) = |z|z$.
   Then $f = g \circ \phi$ is qr and has a full 1-periodic $\mathcal{QF}$ component U containing a right half-plane H such that

$$\log\log|f^n(x)| \sim n \log 2 \quad \text{for } x \in H.$$

Theorem (N., Sixsmith, 2017)

2. *There exists a trans type qr map $G\colon \mathbb{R}^3 \to \mathbb{R}^3$ equal to the identity in a half-space.*
3. *There exists a trans type qr map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ for which $\mathcal{QF}(f)$ is a full domain in which $f^n \to \infty$ locally uniformly.*

Remark: $f = G +$ constant.

Hollow quasi-Fatou components — Existence

Theorem (N., Sixsmith, 2017)

For each $d \geq 2$, there exists a quasiregular f on $\mathbb{R}^d$ of trans type such that $\mathcal{QF}(f)$ has a hollow component.

For any such f, either

- $\mathcal{QF}(f)$ has a sequence of wandering, bounded hollow components; or
- only one component of $\mathcal{QF}(f)$ is hollow and this is unbounded.

But, for the f we construct, we don't know which!

Hollow quasi-Fatou components — How many holes?

- Next, aim to generalize Kisaka and Shishikura's result on the connectivity of Fatou components.
- For a domain $U \subset \mathbb{R}^d$, denote by $\mathrm{cc}(U)$ the number of components of $\overline{\mathbb{R}^d} \backslash U$.
- In the plane, $\mathrm{cc}(U)$ is the connectivity of U.
- In general, $\mathrm{cc}(U) = 1 \iff U$ is full.

Theorem (N., Sixsmith, 2017)

Let U_0 be a quasi-Fatou component of a trans type qr map f. Denote by U_n the component of $\mathcal{QF}(f)$ that contains $f^n(U_0)$. Then

- $\mathrm{cc}(U_{n+1}) \leq \mathrm{cc}(U_n)$ *for all* $n \geq 0$;
- *If* $\mathrm{cc}(U_0) = 1$ *or* ∞, *then* $\mathrm{cc}(U_n) = \mathrm{cc}(U_0)$ *for all* n;
- *If* $\mathrm{cc}(U_0) \neq 1$ *or* ∞, *then* $\mathrm{cc}(U_n) = 2$ *for all large* n.

Bounded hollow quasi-Fatou components

Suppose that f is trans type qr on $\mathbb{R}^d$ and that U_0 is a bounded hollow quasi-Fatou component.

Again, let U_n denote the component of $\mathcal{QF}(f)$ containing $f^n(U_0)$.

Theorem (N., Sixsmith, 2017)

- $U_n = f^n(U_0)$ *and is bounded and hollow for all* n*;*
- U_{n+1} *surrounds* U_n *for all large* n*;*
- $\operatorname{dist}(0, U_n) \to \infty$ *as* $n \to \infty$*;*
- $\overline{U_n} \subset A(f)$*;*
- *the 'inner' and 'outer' boundaries of* U_n *are far apart for large* n*;*
- $\lim_{n\to\infty} \frac{1}{n} \log\log(\operatorname{meas}(U_n)) = \infty$.

Unbounded hollow quasi-Fatou components?

Let f be trans type qr on $\mathbb{R}^d$.

Theorem (N., Sixsmith, 2017)

If f has an unbounded hollow quasi-Fatou component U, then

- *U is completely invariant;*
- *all components of $\mathbb{R}^d \setminus U$ are bounded;*
- *any other quasi-Fatou components are full.*

Question: Can a trans type qr map ever have an unbounded hollow quasi-Fatou component?

If the answer is "no", the next result becomes very interesting!

Theorem (N., Sixsmith, 2017)

If f does not have an unbounded hollow quasi-Fatou component, then $J(f)$ is perfect and contains continua, $J(f) = \partial A(f)$ and $J(f^p) = J(f)$.