The quasi-Fatou set in quasiregular dynamics

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Talk overview

• Quick introduction to quasiregular maps on $\mathbb{R}^d$. These generalize analytic functions on $\mathbb{C}$.

• Survey some results from “quasiregular dynamics”. We seek an iterative theory parallel to complex dynamics.

• Discuss some specific results on the quasi-Fatou set.
Quasiregular mappings

Definition

A continuous $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is quasiregular (qr) if $f \in W^1_{d,\text{loc}}(\mathbb{R}^d)$ and there exists $K_O \geq 1$ such that

$$\|Df(x)\|^d \leq K_O J_f(x) \quad \text{a.e.}$$

where $\|Df(x)\|$ is the norm of the derivative and $J_f(x)$ is the Jacobian.

- Informally, a qr map sends infinitesimal spheres to infinitesimal ellipsoids of bounded eccentricity.
- A mapping is called $K$-qr if the local distortion is $\leq K$.
- Holomorphic functions on $\mathbb{C}$ are 1-qr.
- The iterates of a qr map are qr, but in general if $f$ is $K$-qr then $f^n$ may be $K^n$-qr.
Quasiregular mappings

Quasiregular functions on $\mathbb{R}^d$ generalize analytic functions on $\mathbb{C}$.

**Theorem (Reshetnyak, 1967-68)**

Non-constant quasiregular maps are open, discrete and almost everywhere differentiable.

**Definition**

A non-constant qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *polynomial type* if

$$\lim_{x \to \infty} |f(x)| = \infty.$$  

Otherwise, this limit does not exist and $f$ is *transcendental type*.

Can also consider quasiregular self-maps of $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ that are analogous to rational functions on $\overline{\mathbb{C}}$. 
Some easily-stated results

Let \( f \) be quasiregular on \( \mathbb{R}^d \) of transcendental type.

**Theorem (Siebert, 2004)**

\( f \) has infinitely many periodic points of every period \( p \geq 2 \).

**Theorem (Bergweiler, Fletcher, Langley, Meyer, 2009)**

The escaping set is non-empty; that is,

\[
I(f) := \{ x : f^n(x) \to \infty \text{ as } n \to \infty \} \neq \emptyset.
\]

**Theorem (Bergweiler, Fletcher, Drasin, 2014)**

The fast escaping set \( A(f) \neq \emptyset \). In fact, all components are unbounded.
An example of a quasiregular map

Bergweiler and Eremenko defined a qr “trig function analogue” on $\mathbb{R}^d$ as follows:

Extend to a map $\mathbb{R}^d \rightarrow \mathbb{R}^d$ by reflecting in hyperplanes.

For large enough $\lambda$, the map $S := \lambda F$ is locally uniformly expanding.
An application of quasiregular dynamics(!)

By iterating $S$, they obtained a strong Karpińska paradox in $\mathbb{R}^d$:

**Theorem (Bergweiler, Eremenko, 2011)**

Let $d \geq 2$. $\mathbb{R}^d$ can be expressed as an uncountable union of hairs such that

- any two hairs intersect only at a common endpoint (if at all); and
- the union of hairs without their endpoints has Hausdorff dim 1. (It follows that the set of endpoints has Hausdorff dim $d$.)

Remark: the hairs minus endpoints lie in the escaping set $I(S)$.

Vogel, 2015: $I(S)$ has positive Lebesgue measure.
Analogues of Picard’s and Montel’s theorem

Theorem (Rickman, 1980)

For \( d \geq 2 \) and \( K \geq 1 \) there exists a constant \( q = q(d, K) \) with the following property:

every \( K \)-qr map \( f : \mathbb{R}^d \to \mathbb{R}^d \) that omits \( q \) values must be constant.

Miniowitz used Rickman’s theorem to obtain an analogue of Montel’s theorem:

Theorem (Miniowitz, 1982)

Let \( \mathcal{F} \) be a family of \( K \)-qr maps on a domain \( D \subset \mathbb{R}^d \). If there exist distinct points \( a_1, \ldots, a_q \) that are omitted by every \( f \in \mathcal{F} \), then \( \mathcal{F} \) is a normal family.
Uniformly quasiregular maps

- If every iterate $f^n$ is $K$-quasiregular with the same $K$, then $f$ is called uniformly quasiregular (uqr).
- For uqr maps, the usual definition of Fatou and Julia sets via normality works well.

Theorem (Hinkkanen, Martin, Mayer, 2004)

For a non-injective uniformly quasiregular map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$

- $J(f^p) = J(f)$;
- $J(f)$ is perfect;
- $J(f)$ is the smallest closed completely invariant set with $> q$ points;
- classification of periodic points and periodic Fatou components;
- $J(f) =$ boundary of any attracting basins.
Uniformly quasiregular maps

Question: For uqr maps, is the Julia set the closure of the repelling periodic points?

Theorem (Siebert, 2004)

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be uqr.

\[ J(f) \not\subset \{ \text{post-branch set} \} \implies J(f) = \{ \text{repelling periodic points} \}. \]

Note:

- Uqr maps in dimension 2 are quasiconformally conjugate to rational/analytic maps.
- No examples are known of transcendental type uqr maps in dimension $\geq 3$. 
Non-uniformly quasiregular dynamics

Now let $f: \mathbb{R}^d \to \mathbb{R}^d$ or $\overline{\mathbb{R}}^d \to \overline{\mathbb{R}}^d$ be $K$-qr, but not assumed uqr.

Extending an idea of Sun and Yang (c.1999) we use a ‘blowing-up’ property to define the Julia set:

$$J(f) := \{ x : \text{for every nhd } U \text{ of } x, \mathbb{R}^d \setminus O^+(U) \text{ is small} \}.$$

Here “small” means conformal capacity zero.
It follows immediately that $J(f)$ is closed and completely invariant.

**Theorem (Bergweiler 2013, Bergweiler, N. 2014)**

*The definition of $J(f)$ above agrees with the usual one if $f$ is uqr. If $\deg(f) > K_I$, then $J(f) \neq \emptyset$ and, in fact, $J(f)$ is infinite.*

Example: For the qr sine analogue, $J(S) = \mathbb{R}^d$ (Fletcher, N. 2013).
A conjecture

Assume $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ or $\overline{\mathbb{R}}^d \rightarrow \overline{\mathbb{R}}^d$ is $K$-qr, with $\deg(f) > K_I$.

$J(f) := \left\{ x : \text{for every nhd } U \text{ of } x, \mathbb{R}^d \setminus O^+(U) \text{ is small} \right\}.$

Conjecture

• Equivalent to replace “small” by “finite” in $J(f)$ definition.
• $J(f)$ is perfect and $J(f^p) = J(f)$.

The conjecture is open in general, but holds under a variety of extra hypotheses. In particular, it holds in two dimensions or if $f$ is Lipschitz.

Warren defines $J(f)$ for quasimeromorphomeric maps (with poles) of trans type. $J(f) \neq \emptyset$ and the analogous conjecture holds for such maps.
The quasi-Fatou set

The rest of this talk considers the quasi-Fatou set of a qr map $f: \mathbb{R}^d \to \mathbb{R}^d$ of trans type,

$$QF(f) := \mathbb{R}^d \setminus J(f).$$

Note: no normality assumption!

Notation: the maximum modulus $M(r, f) = \max\{|f(x)| : |x| = r\}$.

Theorem (Bergweiler, Fletcher, N. 2014)

Let $f$ be qr such that

$$\liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log \log r} = \infty. \quad (1)$$

Then $J(f) = \partial A(f)$, where $A(f)$ is the fast escaping set.

Corollary: If (1) holds and a quasi-Fatou component meets $A(f)$, then it is contained in $A(f)$. (A ‘normality’ property!)

This is not true for the escaping set $I(f) \ldots$
Example: Modifying Fatou’s function

Fatou’s function \( g(z) = z + 1 + e^{-z} \) has a right half-plane \( H \subset I(g) \cap F(g) \) on which \( g(z) \approx z + 1 \).
Example: Modifying Fatou’s function

Fatou’s function \( g(z) = z + 1 + e^{-z} \) has a right half-plane \( H \subset \text{I}(g) \cap \text{F}(g) \) on which \( g(z) \approx z + 1 \).

Modifying \( g \) in a disc, as shown, gives a qr map \( \tilde{g} \) with a fixed point in \( H \).

We still have \( H \subset \text{QF}(\tilde{g}) \) because

\[
\tilde{g}(H) \subset H \Rightarrow \text{no blowing-up in } H.
\]

But \( H \) contains escaping and non-escaping points of \( \tilde{g} \).
Full periodic domains vs. Baker domains

- We say that a domain in $\mathbb{R}^d$ is full if its complement has no bounded components; otherwise, it is hollow.
- A component $U$ of $QF(f)$ is called $p$-periodic if $f^p(U) \subset U$.

**Theorem (Baker, 1988)**

*If $f$ is trans entire on $\mathbb{C}$ and $U$ is a $p$-periodic Fatou component, then, for $z \in U$,*

$$\log |f^{np}(z)| = O(n) \quad \text{as } n \to \infty.$$  

Periodic Fatou components are always full (simply-conn) and are called *Baker domains* if $U \cap l(f) \neq \emptyset$. Baker domains cannot meet $A(f)$.

**Theorem (N., Sixsmith, 2017)**

*If $f$ is trans type qr on $\mathbb{R}^d$ and $U$ is a full $p$-periodic quasi-Fatou component, then, for $x \in U$,*

$$\log \log |f^{np}(x)| = O(n) \quad \text{as } n \to \infty.$$  

Moreover, $U \cap A(f) = \emptyset$. 
Examples

Our first example shows we cannot improve the qr result to be as good as the entire one.

1. On $\mathbb{C}$, let $g(z) = z + 1 + e^{-z}$ and $\phi(z) = |z| z$. Then $f = g \circ \phi$ is qr and has a full 1-periodic $QF$ component $U$ containing a right half-plane $H$ such that

$$\log \log |f^n(x)| \sim n \log 2 \quad \text{for } x \in H.$$

Theorem (N., Sixsmith, 2017)

2. There exists a trans type qr map $G: \mathbb{R}^3 \to \mathbb{R}^3$ equal to the identity in a half-space.

3. There exists a trans type qr map $f: \mathbb{R}^3 \to \mathbb{R}^3$ for which $QF(f)$ is a full domain in which $f^n \to \infty$ locally uniformly.

Remark: $f = G + \text{constant}$. 
For each $d \geq 2$, there exists a quasiregular $f$ on $\mathbb{R}^d$ of trans type such that $\mathcal{QF}(f)$ has a hollow component.

For any such $f$, either

- $\mathcal{QF}(f)$ has a sequence of wandering, bounded hollow components; or
- only one component of $\mathcal{QF}(f)$ is hollow and this is unbounded.

But, for the $f$ we construct, we don’t know which!
Hollow quasi-Fatou components — How many holes?

- Next, aim to generalize Kisaka and Shishikura’s result on the connectivity of Fatou components.
- For a domain $U \subset \mathbb{R}^d$, denote by $cc(U)$ the number of components of $\mathbb{R}^d \setminus U$.
- In the plane, $cc(U)$ is the connectivity of $U$.
- In general, $cc(U) = 1 \iff U$ is full.

**Theorem (N., Sixsmith, 2017)**

Let $U_0$ be a quasi-Fatou component of a trans type qr map $f$. Denote by $U_n$ the component of $Q\mathcal{F}(f)$ that contains $f^n(U_0)$. Then

- $cc(U_{n+1}) \leq cc(U_n)$ for all $n \geq 0$;
- If $cc(U_0) = 1$ or $\infty$, then $cc(U_n) = cc(U_0)$ for all $n$;
- If $cc(U_0) \neq 1$ or $\infty$, then $cc(U_n) = 2$ for all large $n$. 
Suppose that \( f \) is trans type qr on \( \mathbb{R}^d \) and that \( U_0 \) is a bounded hollow quasi-Fatou component. Again, let \( U_n \) denote the component of \( QF(f) \) containing \( f^n(U_0) \).

**Theorem (N., Sixsmith, 2017)**

- \( U_n = f^n(U_0) \) and is bounded and hollow for all \( n \);
- \( U_{n+1} \) surrounds \( U_n \) for all large \( n \);
- \( \text{dist}(0, U_n) \to \infty \) as \( n \to \infty \);
- \( \overline{U}_n \subset A(f) \);
- the ‘inner’ and ‘outer’ boundaries of \( U_n \) are far apart for large \( n \);
- \( \lim_{n \to \infty} \frac{1}{n} \log \log(\text{meas}(U_n)) = \infty. \)
Unbounded hollow quasi-Fatou components?

Let $f$ be trans type qr on $\mathbb{R}^d$.

**Theorem (N., Sixsmith, 2017)**

If $f$ has an unbounded hollow quasi-Fatou component $U$, then

- $U$ is completely invariant;
- all components of $\mathbb{R}^d \setminus U$ are bounded;
- any other quasi-Fatou components are full.

Question: Can a trans type qr map ever have an unbounded hollow quasi-Fatou component?

If the answer is “no”, the next result becomes very interesting!

**Theorem (N., Sixsmith, 2017)**

If $f$ does not have an unbounded hollow quasi-Fatou component, then $J(f)$ is perfect and contains continua, $J(f) = \partial A(f)$ and $J(f^p) = J(f)$.