Smale's Mean Value Conjecture

and Related Problems

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C.C. Yang: My hearty congratulations to Walter on the occasion of his 60th birthday celebration !

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1. Smale's Mean Value Conjecture

In 1981, Stephen Smale proved the following

Theorem 1. *Let P be a non-linear polynomial and $a \in \mathbb{C}$ such that $P'(a) \neq 0$. Then there exists a critical point b of P such that*

$$\left|\frac{P(a) - P(b)}{a - b}\right| \leq 4|P'(a)| \tag{1.1}$$

Or equivalently, we have

$$\min_{b, P'(b)=0} \left|\frac{P(a) - P(b)}{a - b}\right| \leq 4|P'(a)| \tag{1.2}$$

Smale then asked whether one can replace the factor 4 in the upper bound in (1.1) by 1, or even possibly by $\frac{d-1}{d}$, where $d = \deg P$.

He also pointed out that the number $\frac{d-1}{d}$ would, if true, be the best possible bound here as it is attained (for any nonzero λ) when $P(z) = z^d - \lambda z$ and $a = 0$ in (1.1).

Note that if b_i are the critical points of $P(z) = z^d - \lambda z$ and $a = 0$, then

$$\left|\frac{P(b_1)}{b_1 P'(0)}\right| = \cdots = \left|\frac{P(b_{d-1})}{b_{d-1} P'(0)}\right| = \frac{d-1}{d}.$$

Q: Is it also true for **all** extremal polynomials ?

$$\min_{b, P'(b)=0} \left| \frac{P(a) - P(b)}{a - b} \right| \le 4|P'(a)| \tag{1.2}$$

Let M be the least possible values of the factor in the upper bound in (1.2) for all non-linear polynomials and M_d be the corresponding value for the polynomial of degree d.

Then Smale's theorem and example show that

$\frac{d-1}{d} \le M_d \le 4$ and $1 \le M \le 4$.

Smale's Mean Value conjecture:

$M = 1$ or even $M_d = \frac{d-1}{d}$, where $d = \deg P$.

• S. Smale, The fundamental theorem of algebra and complexity theory, *Bull. Amer. Math. Soc.* 4 (1981), 1-36.

• S. Smale, Mathematical Problems for the Next Century, *Mathematics: frontiers and perspectives*, eds. Arnold, V., Atiyah, M., Lax, P. and Mazur, B., Amer. Math. Soc., 2000.

Smale's mean value conjecture is equivalent to the following

Normalised conjecture : Let P be a monic polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) = 1$. Let $b_1, \ldots, b_{d-1}$ be its critical points. Then

$$\min_i \left| \frac{P(b_i)}{b_i P'(0)} \right| \leq \frac{d-1}{d} \qquad (*)$$

Estimates of M_d

Tischler (1989), Crane (2006), Sendov and Marinov (2006)

For $2 \le d \le 5$,

$$M_d = \frac{d-1}{d}.$$

Beardon, Minda and N. (2002)

$$M_d \le 4^{1-\frac{1}{d-1}} = 4 - \frac{4\log 4}{d} + O\left(\frac{1}{d^2}\right).$$

Conte, Fujikawa and Lakic (2007)

$$M_d \le 4\frac{d-1}{d+1} = 4 - \frac{8}{d} + O\left(\frac{1}{d^2}\right) .$$

Fujikawa and Sugawa (2006)

$$M_d \le 4\left(\frac{1+(d-2)4^{-1/(d-1)}}{d+1}\right) = 4-\frac{8+4\log 4}{d}+O\left(\frac{1}{d^2}\right).$$

Crane (2007)

For $d \geq 8$,

$$M_d \;\leq\; 4 - \frac{2}{\sqrt{d}}\,.$$

• Q.I. Rahman and G. Schmeisser, *Analytic theory of Polynomials*, Oxford University Press, Oxford, 2002.

• T. Sheil-Small, *Complex polynomials*. Cambridge Studies in Advanced Mathematics, **75**. Cambridge University Press, Cambridge, 2002.

Motivation.

Smale discovered the Mean Value theorem as a by product of his investigations of the efficiency of zero finding algorithms.

Newton's map of P: $N_P(z) = z - \frac{P(z)}{P'(z)}$.

Choose an initial point z_0 suitably and let

$$z_{n+1} = N_P(z_n) = z_n - \frac{P(z_n)}{P'(z_n)},$$

then the sequence $\{z_n\}$ will converge to a zero of P.

If we consider the Taylor's series of P at z_n, then we have

$$P(z_{n+1}) = P(z_n) + \sum_{i=1}^{d} (-1)^i \frac{P^{(i)}(z_n)}{i!} \left(\frac{P(z_n)}{P'(z_n)} \right)^i .$$

It follows that $\frac{P(z_{n+1})}{P(z_n)} = 1 + \sum_{i=1}^{d} (-1)^i \frac{P^{(i)}(z_n) P(z_n)^{i-1}}{i! P'(z_n)^i}$ and hence the efficiency of Newton's method mainly depends on the growth of

$$\frac{P^{(i)}(z_n) P(z_n)^{i-1}}{i! P'(z_n)^i},$$

$i = 2, 3, ..., d; n = 0, 1, ...$

By using Löwner's theorem, Smale proved the following result.

Theorem 2. (Smale, 1981) *Let a be any non-critical point of P. Then there exists a critical point b of P such that for each $k \geq 2$,*

$$\left|\frac{P^{(k)}(a)}{k!P'(a)}\right|^{\frac{1}{k-1}} |P(a) - P(b)| \leq 4|P'(a)| \qquad (**)$$

Let K be the least possible values of the factor in the upper bound in $(**)$ and $K_{d,i}$ be the corresponding value for the polynomial of degree d and $k = i$.

Smale suggested six open problems (Problem 1A-1F) related to the inequality (**).

Most of these problems are about the precise values of K and $K_{d,i}$.

Smale also gave an example to show that $1 \leq K \leq 4$ and conjectured that $K = 1$.

Problem 1A: Reduce K from 4.

Problem 1B, 1C and 1D are about $K_{d,2}$.

Porblem 1E is the mean value conjecture.

The constant K is quite important for estimating the efficiency of Newton's Method.

Theorem 3. (Smale,1981) *Let* $R_0 = \min_{b, P'(b)=0}\{|P(b)|\} > 0$. *If* $|P(w)| < \frac{R_0}{3K+1}$, *then the iterations of Newton's method starting at* w *will converge to some zero of* P. *In addition, if* $P(w) \neq 0$, *one has*

$$\frac{|P(w')|}{|P(w)|} < \frac{1}{2}$$

where $w' = w - \frac{P(w)}{P'(w)}$.

When $i=2$, Smale showed that for some critical point b,

$$\left|\frac{P^{(2)}(a)}{2P'(a)}\right| |P(a)-P(b)| = \left|\frac{1}{2}\sum_{j=1}^{d-1}\frac{1}{a-b_j}\right| |P(a)-P(b)| \leq 2|P'(a)|$$

Problem 1B asked whether 2 can further be reduced to $\frac{d-1}{2d}$, i.e.

$$\left|\sum_{j=1}^{d-1}\frac{1}{a-b_j}\right| |P(a)-P(b)| \leq \frac{d-1}{d}|P'(a)|.$$

- $K_{d,2}=\frac{d-1}{2d}$ when $d=2$, i.e. $K_{2,2}=\frac{1}{4}$.

For Problem 1B, Y.Y. Choi, P.L. Cheung and N. showed that

$$K_{3,2} = \frac{4}{6\sqrt{3}} = 0.3845... > \frac{2}{6}$$

$$K_{4,2} \geq 0.473... > \frac{3}{8}, \quad K_{d,2} = ?$$

For Problem 1A, we also showed that

$$K \leq 4^{\frac{d-2}{d-1}}.$$

$$\left|\frac{P^{(i)}(a)}{i!P'(a)}\right|^{\frac{1}{i-1}} |P(a)-P(b)| \leq K_{d,i}|P'(a)|.$$

For $i = d$, V.N. Dubinin (2006), applies the method of dissymmetrization to prove the **sharp** inequality.

$$\left|\frac{P^{(d)}(a)}{d!P'(a)}\right|^{\frac{1}{d-1}} |P(a)-P(b)| \leq \frac{d-1}{d^{\frac{d}{d-1}}}|P'(a)|.$$

Hence, $K_{d,d} = \frac{d-1}{d^{\frac{d}{d-1}}}$.

2. Introduction to theory of amoeba

Let $f = f(z_1, ..., z_n)$ be a non-constant polynomial.

Let $Z_f = \{(z_1, ..., z_n) \in \mathbb{C}_*^n \,|\, f(z_1, .., z_n) = 0\}$ be the hypersurface defined by f.

The amoeba $\mathcal{A}_f$ is defined to be the image of Z_f under the map Log : $\mathbb{C}_*^n \to \mathbb{R}^n$ defined by

$$\mathrm{Log}(z_1, ..., z_n) = (\log|z_1|, ..., \log|z_n|).$$

• Introduced by Gelfand, Kapranov and Zelevinsky in 1994.

I.N. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Math. Theory Appl., Birkhauser, Boston, 1994.

M. Forsberg, M. Passare, and A. Tsikh, Laurent determinants and arrangements of hyperplane amoebas, *Adv. Math.* **151** (2000), 45–70.

Components of the complement

Theorem (GKZ, 1994). $\mathcal{A}_f$ is closed and any connected component of $\mathcal{A}_f^c = \mathbb{R}^n \backslash \mathcal{A}_f$ is convex.

Ronkin function for the hypersurface, $N_f : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$N_f(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\mathsf{Log}^{-1}(\mathbf{x})} \log |f(\mathbf{z})| \frac{\mathsf{d}z_1}{z_1} \wedge \ldots \wedge \frac{\mathsf{d}z_n}{z_n}.$$

Theorem (Ronkin,2001). N_f is convex. It is affine on each connected component of $\mathcal{A}_f^c$ and strictly convex on $\mathcal{A}_f$.

3. A problem on extremal polynomials.

Recall that we can always assume that the polynomials are **monic**. Note that any monic polynomial with zero constant term is determined uniquely by its critical points.

Let $B = (b_1, \ldots, b_{d-1}) \in \mathbb{C}^{d-1}$ and $P_B(z)$ be a degree d monic polynomial whose critical points are $b_1, \ldots, b_{d-1}$.

If $P_B(0) = 0$, then $P_B(z) = d \int_0^z (w - b_1) \cdots (w - b_{d-1}) dw$.

Assume that 0 is not be a critial point of $P_B(z)$.Then, $P_B'(0) \neq 0$ or $\prod b_i \neq 0$.

Let $\lambda \neq 0$. Consider

$$P_{\lambda B}(z) = d\int_0^z (w-\lambda b_1)\cdots(w-\lambda b_{d-1})dw.$$

Then,

$$\frac{P_{\lambda B}(\lambda b_i)}{\lambda b_i P'_{\lambda B}(0)} = \frac{P_B(b_i)}{b_i P'_B(0)}.$$

Therefore, we may further assume that B is in the set

$$E = \left\{(z_1, ..., z_{d-1}) \in \mathbb{C}^{d-1} | \prod z_i = \frac{(-1)^{d-1}}{d}\right\}$$

so that $P'_{\lambda B}(0) = 1$.

Define $S_i : E \to \mathbb{C}$ by

$$S_i(B) = S_i(b_1, \ldots, b_{d-1}) = \frac{P_B(b_i)}{b_i P_B'(0)} = \frac{P_B(b_i)}{b_i}.$$

To solve Smale's conjecture, we need to show that

$$\sup_{B \in E} \{ \min_{1 \le i \le d-1} |S_i(B)| \} = \frac{d-1}{d}$$

- Not clear if a maximum point exists.

Theorem 4. (Crane, 2006). *There exists some* B^* *such that*

$$\max_{B\in E}\{\min_{1\le i\le d-1}|S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

Theorem 5. (Crane, 2006) *If* $M_{d+1} > M_d$*, then for* **all** *degree* d *extremal polynomial* P_{B^*},

$$\max_{B\in E}\{\min_{1\le i\le d-1}|S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

Conjecture 1: For **all** degree d extremal polynomial P_{B^*}, we have

$$\max_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

- True when $2 \le d \le 5$.

Crane noticed that if Conjecture 1 is true, then for each $d \ge 2$, the set of all degree d extremal polynomials P_{B^*} is **compact**.

An amoeba approach

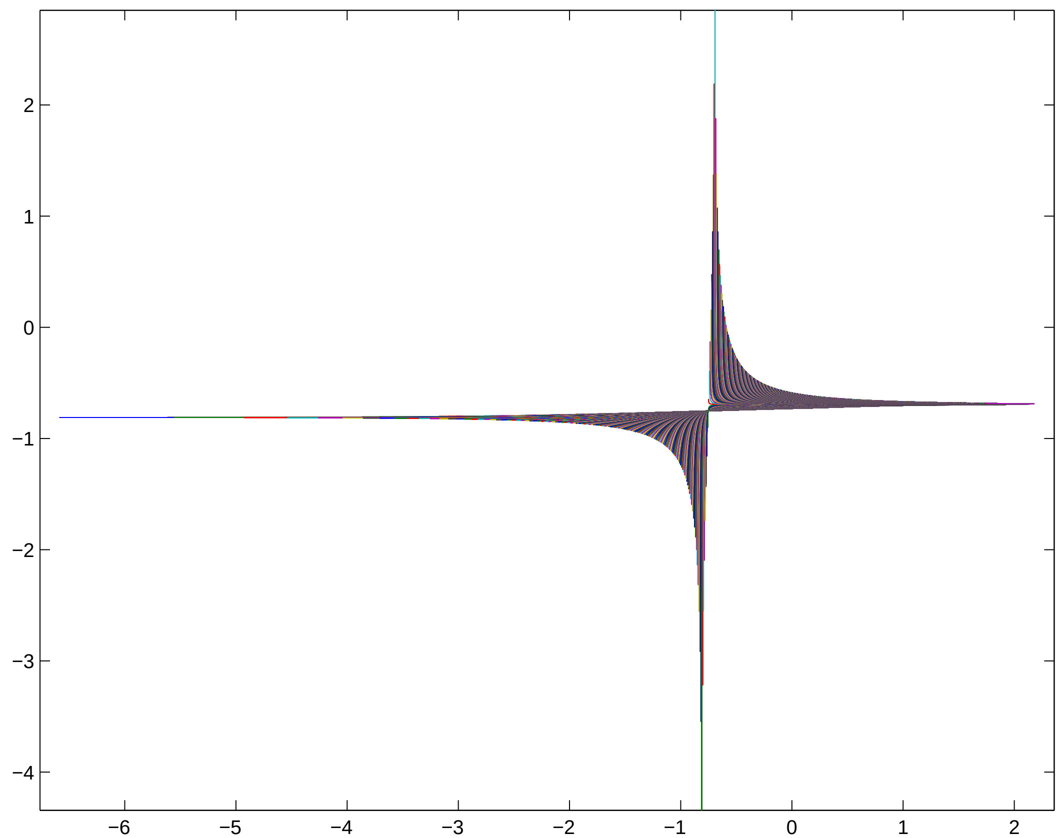
Note that we have $d-1$ homogeneous $S_i(b_1, ..., b_{d-1})$ polynomial in $d-1$ variables, so there is a unique non-constant irreducible symmetric complex polynomial $f = f_d$ such that

$$f(S_1(B), \ldots, S_{d-1}(B)) = 0$$

whenever $B \in \mathbb{C}_*^{d-1}$.

Let $\mathcal{A}_f$ be the amoeba of f. It follows from Smale's theorem that for all $t > 4$, $\mathbf{t} = (\log t, ..., \log t)$ lies in $\mathcal{A}_f^c$.

For $d = 3$, $f(z_1, z_2) = 18z_1z_2 - 9z_1 - 9z_2 + 4$



Theorem 6. *(N.) f has a leading term of the form $z_1^k \cdots z_{d-1}^k$ for some $k \in \mathbb{N}$. Let U be the unbounded component of $\mathcal{A}_f^c$ containing $(\log 4, ..., \log 4) + \mathbb{R}_+^{d-1}$ and $\mathbf{d} = (\log \frac{d-1}{d}, ..., \log \frac{d-1}{d})$. Then the following are equivalent:*

1) Smale's mean value conjecture is true for degree d;

2) U contains $\mathbf{d} + \mathbb{R}_+^{d-1}$;

3) U contains the ray $\{t(1, ..., 1) : t > \log \frac{d-1}{d}\}$;

4) $\mathbf{d}$ is a boundary point of U;

5) $N_f(\mathbf{d}) = k(d-1) \log \frac{d-1}{d}$.

In principle, one can use $N_f(\mathbf{d}) = k(d-1)\log\frac{d-1}{d}$ to verify Smale's mean value conjecture for small degree d. For example, when $d=3$, $k=1$ and when $d=4$, $k=3$.

$$N_f(\mathbf{x}) = \frac{1}{(2\pi i)^n}\int_{\mathsf{Log}^{-1}(\mathbf{x})} \log|f(\mathbf{z})|\frac{\mathsf{d}z_1}{z_1}\wedge\ldots\wedge\frac{\mathsf{d}z_n}{z_n}.$$

Theorem 5 (Crane, 2006). There exists some B^* such that

$$\max_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

Max-Min and Min-Max problem on hypersurfaces in $\mathbb{C}^n$

For a non-constant polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ and the hypersurface $Z_f \subset \mathbb{C}_*^n$ defined by f, let

$$C(f) = \sup_{\mathbf{z} \in Z_f} \left(\min_{1 \le i \le n} |z_i| \right) .$$

Problem: Characterize those polynomials f for which $C(f)$ is finite, and for such a polynomial to determine whether the bound is attained by some point $\mathbf{x} \in \mathbb{C}^k$.

A monomial term of the polynomial f is the **dominant monomial** of f if it is of maximal degree in each variable separately.

Theorem 7. (Crane, 2006) *$f \in \mathbb{C}[z_1, \ldots, z_n]$ has a dominant monomial if and only if $C(f) < \infty$.*

Theorem 8. (Crane, 2006) *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant. If $C(f) < \infty$, then there exists some $(z_1, \ldots, z_n) \in Z_f$ such that*

$$|z_1| = \cdots = |z_n| = C(f).$$

Related to the above max-min problem, we consider the dual min-max problem.

For a non-constant polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ and the hypersurface $Z_f \subset \mathbb{C}_*^n$ defined by f, let

$$D(f) = \inf_{\mathbf{z} \in Z_f} \left(\max_{1 \le i \le n} |z_i| \right) .$$

$$D(f) = \inf_{\mathbf{z} \in Z_f} \left(\max_{1 \le i \le n} |z_i| \right)$$

Theorem 9. *(N.)* $f \in \mathbb{C}[z_1, \ldots, z_n]$ *has a non-zero constant term if and only if* $D(f) > 0$*.*

Theorem 10. *(N.) If* $D(f) > 0$*, then there exists at least one* $(z_1, \ldots, z_n) \in Z_f$ *such that*

$$|z_1| = \cdots = |z_n| = D(f).$$

$$f_d(S_1(B),\ldots,S_{d-1}(B))=0.$$

Note that f_d has a non-zero constant term. Apply the previous result to $f=f_d$.

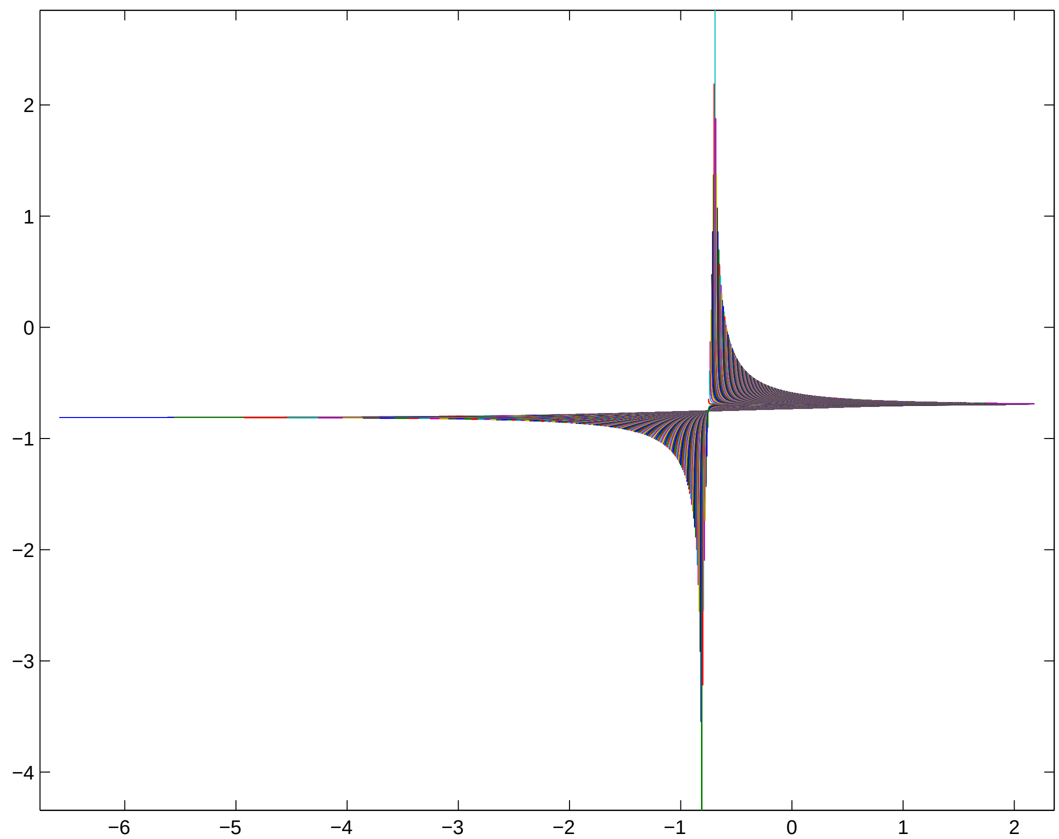
Theorem 11. *(N.) There exists some $N_d>0$ such that if P be a monic polynomial of degree $d\geq 2$ with $P(0)=0$ and $P'(0)=1$ and $b_1,\ldots,b_{d-1}$ are its critical points, then*

$$\max_i\left|\frac{P(b_i)}{b_iP'(0)}\right|\geq N_d\,.$$

Moreover, at least one of the extremal polynomials for N_d satisfies the condition

$$\left|\frac{P(b_1)}{b_1 P'(0)}\right| = \cdots = \left|\frac{P(b_{d-1})}{b_{d-1} P'(0)}\right|. \tag{$*$}$$

For $d = 3$, $f(z_1, z_2) = 18z_1z_2 - 9z_1 - 9z_2 + 4$



Dual Mean Value Conjecture:

Let P be a monic polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) = 1$. Let $b_1, \ldots, b_{d-1}$ be its critical points. Then

$$\max_i \left| \frac{P(b_i)}{b_i P'(0)} \right| \geq \frac{1}{d}\,.$$

It is conjectured that the extremal polynomial should be $P(z) = (z-a)^d - (-a)^d$, where a is some non-zero complex number.

Note that Dubinin and Sugawa (2009) have also discovered this dual mean value conjecture around the same time independently and they are able to show that $N_d \geq 1/(d4^d)$.

Theorem 13 (N. and Y.Q. Zhang (2017))

$$N_d > \frac{1}{4^d}.$$

To prove this, we need to consider similar conjecture for finite Blaschke products.

Finite Blaschke products.

A *finite Blaschke product* of degree n is a rational function of the form

$$B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}$$

where α is a real number and $z_1, \ldots, z_n$ are complex numbers on the standard unit disk $\mathbb{D} = \{z : |z| < 1\}$.

First noted by Walsh (1952) that finite Blaschke products can be viewed as $non\text{-}euclidean\ polynomials$ in $\mathbb{D}$ and he proved a version of Gauss-Lucas Theorem for finite Blaschke products.

This point of view was also propagated by Beardon and Minda (2004), as well as Singer (2006).

Recently, a dictionary between polynomials and finite Blaschke products has be established by N. and Tsang (2013) based on the joint works with my former students M.X. Wang. C.Y. Tsang and P.L. Cheung.

Smale's mean value conjecture for finite Blaschke products (Sheil-Small, 2002):

Let $B(z) = z\prod_{i=1}^{d-1}\left(\frac{z-a_i}{1-\overline{a_i}z}\right)$ and has critical points $w_1,\ldots,w_{d-1}$, $d\geq 2$. Then

$$\min_i \left|\frac{B(w_i)}{w_i B'(0)}\right| \leq 1$$

- There are examples showing that the constant 1 on the right hand side cannot be reduced.

- For degree two finite Blaschke products, no extremal finite Blaschke products exist.

- Sheil-Small (2002) showed that the conjecture for finite Blaschke products implies that of polynomials.

- The two Smale's mean value conjectures are actually connected through suitable **rescalings**.

Connection through rescalings

Given $P(z) = z \prod_{i=1}^{d-1} (z - a_i)$ with critical points c_i.

Define $B_n(z) = z \prod_{i=1}^{d-1} \left(\dfrac{z - \frac{a_i}{n}}{1 - \frac{\overline{a_i}}{n} z} \right)$ so that when n is sufficiently large, B_n is a finite Blaschke product.

Let $f_n(z) = n^d B_n \left(\dfrac{1}{n} z \right) = z \prod_{i=1}^{d-1} \left(\dfrac{z - a_i}{1 - \frac{\overline{a_i}}{n} z} \right)$.

Then $f_n(z) \to P(z)$ locally uniformally on $\mathbb{C}$ as $n \to \infty$.

If $c_{n,i}$ are the critical points of B_n, then $d_{n,i} = nc_{n,i}$ are the critical points of f_n and

$$\frac{f_n(d_{n,i})}{d_{n,i}f_n'(0)} = \frac{n^d B_n(\frac{1}{n}d_{n,i})}{d_{n,i}n^{d-1}B_n'(0)} = \frac{B_n(c_{n,i})}{c_{n,i}B_n'(0)}.$$

So if for each n, there exists some $1 \leq j \leq d-1$ such that

$$\left|\frac{B_n(c_{n,j})}{c_{n,j}B_n'(0)}\right| \leq K.$$

Then as $n \to \infty$, we have there exists some $1 \le k \le d-1$ such that

$$\left|\frac{P(c_k)}{c_k P'(0)}\right| \le K.$$

Theorem 14 [N. & Y.Q. Zhang (2017)] Let $B(z) = z\prod_{i=1}^{d-1}\left(\frac{z-a_i}{1-\overline{a_i}z}\right)$ and has critical points $w_1,\ldots,w_{d-1}$, $d \geq 2$.

Then

$$\min_i \left|\frac{B(w_i)}{w_i B'(0)}\right| \leq 2^{\frac{2d-1+(2d-3)4^{1/(1-d)}}{2d-1}}.$$

and

$$\max_i \left|\frac{B(w_i)}{w_i B'(0)}\right| > \frac{1}{4^d}$$

- Hence, we can have $N_d > \frac{1}{4^d}$ instead of $N_d \geq \frac{1}{d4^d}$ for polynomials.

- As $2^{\frac{2d-1+(2d-3)4^{1/(1-d)}}{2d-1}}$ is of the form $4 - O(\frac{1}{d})$, we do not get a better upper bound M_d for polynomials.

Applications of amoeba theory to $K_{d,i}$

Note that the existence of the extremal polynomials for any $K_{d,i}$ has never been proven and it is not clear if they exist at all because the parameter space for the normalized polynomials is not compact.

Using the amoeba theory, one can prove that for each $K_{d,i}$, at least one extremal polynomial exists .

When $i = 2$, Smale showed that for some critical point b_i,

$$T_i := \left| \frac{P^{(2)}(0)}{2P'(0)} \right| \frac{|P(b_i)|}{|P'(0)|} = \left| \frac{1}{2} \sum_{j=1}^{d-1} \frac{1}{b_j} \right| \frac{|P(b_i)|}{|P'(0)|} \leq 2.$$

One may ask if there is a dual inequality, i.e. there exists some positive $L_{d,2}$ such that one can always find some b_k so that

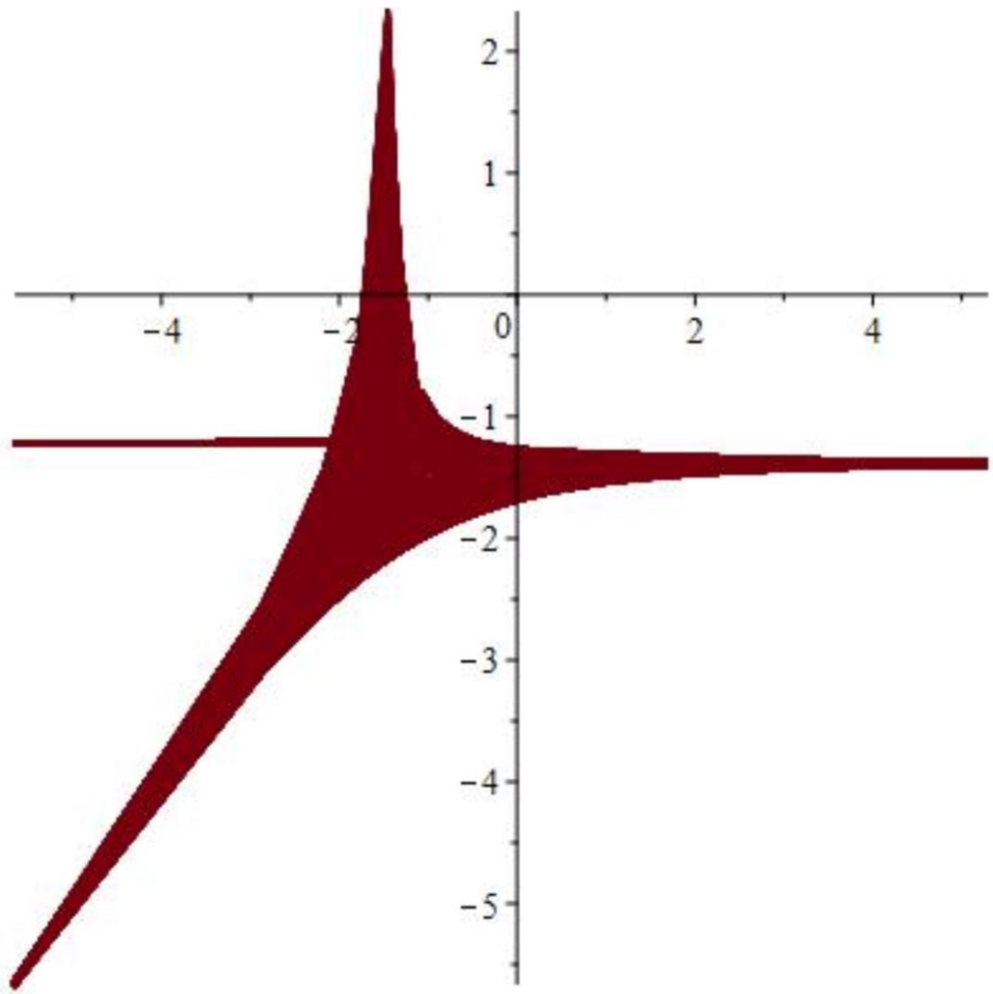
$$T_k = \left| \frac{1}{2} \sum_{j=1}^{d-1} \frac{1}{b_j} \right| \frac{|P(b_k)|}{|P'(0)|} \geq L_{d,2} > 0$$

For $g(z_1, z_2) = 432z_1^2z_2^2 - 216z_1^2z_2 - 216z_1z_2^2 + 27z_1^2 + 27z_2^2 + 90z_1z_2 - 8z_1 - 8z_2$, we have

$$g(T_1, T_2) = 0.$$

Since g has no constant term, no dual inequality for $d = 3$ (also true for general d).

The amoeba of g is given in the following figure.



2
1
−4
−2
0
2
4
−1
−2
−3
−4
−5

Applications to Pareto optimal points

Recall that

$$S_i(B) = S_i(b_1, \ldots, b_{d-1}) = \frac{P_B(b_i)}{b_i P'_B(0)}.$$

In Problem 1D, Smale suggested to look for the *Pareto optimal points* of those attain the following optimization problem:

$$\max_{B \in E} \{ \min_{1 \leq i \leq d-1} |S_i(B)| \}$$

Definition: $B^* = (b_1^*, \ldots, b_{d-1}^*) \in \mathbb{C}_*^{d-1}$ is a *Pareto optimal point* if there is no $B \in \mathbb{C}_*^{d-1}$ such that $S_j(B) \geq S_j(B^*)$ for all $1 \leq j \leq d-1$ with strict inequality for some j.

For the past thirty years, no one knows if a Pareto optimal point exists.

Using the amoeba theory, one can show that such a Pareto optimal point **does exist** if the set of extremal polynomials is **compact**.