Construction of transcendental entire functions of arbitrarily slow growth with prescribed polynomial dynamics

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Resonances in Complex Dynamics

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§1 Introduction

Definition 1:

\( f \) : a transcendental entire function, \( f^n \) : the \( n \)-th iterate of \( f \)

- \( F(f) := \{ z \in \mathbb{C} \mid \exists U : \text{nbd. of } z, \{ f^n|_U \}_{n=1}^{\infty} \text{ is normal} \} \) : Fatou set

\((\{ f^n|_U \}_{n=1}^{\infty} \text{ is normal} \)

\iff \forall \text{ subseq. of } \{ f^n|_U \}_{n=1}^{\infty} \text{ contains a local unif. convergent subseq.)}

- \( J(f) := \mathbb{C} \setminus F(f) \) : Julia set = \{repelling periodic points\}

- \( \text{sing}(f^{-1}) := \{ \text{all crit. & asympt. values and their accumulation pts} \} \)

- \( P(f) := \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1})) \) : post-singular set

- \( S := \{ f \mid f : \text{transcendental entire}, \#\text{sing}(f^{-1}) < \infty \} \)

- \( B := \{ f \mid f : \text{transcendental entire, sing}(f^{-1}) : \text{bounded} \} \)

(trivially \( S \subset B \))
There are two directions for research on dynamics of transcendental entire functions:

(I) Research on $f$ with similar behavior as polynomials,

(II) Research on phenomena which never occur for polynomials.

- For (I):

**Example 2:**

$e^z$, $ze^z$, $\int \limits^z P(z)e^{Q(z)}dz$ ($P, Q : \text{polyn.}$), $\sin z$, $\cos z \in S$, \( \frac{\sin z}{z} \in B \setminus S \)

**Theorem 3 (Goldberg-Keen, Eremenko-Lyubich):**

$f \in S \implies f$ has no wandering domains and Baker domains.

**Theorem 4 (Eremenko-Lyubich, 1992):**

$f \in B \implies \forall z \in F(f), f^n(z) \not\to \infty (n \to \infty)$. In particular, $f$ has no Baker domains.
For (II):

Theorem 5 (Baker, 1963+1976):

\[ \exists f(z) = cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right), \quad c > 1, \ 1 < r_1 < r_2 < \cdots \ (r_n \text{ satisfies some recursive formula}) \text{ has a multiply connected wandering domain.} \]

Theorem 6 (Herman, 1984):

\[ f(z) := z + 1 - e^z + 2\pi i \text{ has a simply connected wandering domain.} \]

Definition 7:

There exists an \( f \) of arbitrarily slow growth with property \( P \)

\[ \iff \text{For any monotone increasing function } \varphi(r) > 0 \ (r > 0) \text{ with } \lim_{r \to \infty} \varphi(r) = +\infty, \text{ there exists } f \text{ with the property } P \text{ and satisfies } \]

\[ \log M(r, f) < \varphi(r) \log r, \quad \forall r > r_0 \quad (M(r, f) := \max_{|z|=r} |f(z)|) \]

(Note that if \( \varphi(r) \equiv \text{const} \), then \( f \) is a polynomial.)
$f$ with sufficiently slow growth has similar properties with polynomials.

**Theorem 8 (Hayman, 1960):**

$$\log M(r, f) < A(\log r)^2 \implies |f(z)| > \exists K \text{ outside small neighborhoods of zeros of } f$$

On the other hand,

**Theorem 9 (Baker, 1984):**

There exists a transcendental entire function with arbitrarily slow growth which has a multiply-connected wandering domain.

**Theorem 10 (Bergweiler-Eremenko, 2000):**

There exists a transcendental entire function with arbitrarily slow growth which satisfies $J(f) = \mathbb{C}$. 
§2 Main Result

Theorem A:

For a given polynomial $P$ with $\deg P \geq 2$, there exists a transcendental entire function with arbitrarily slow growth which satisfies the following:

1. There exists a topological disk $U$ such that $(f|U, U, f(U))$ is polynomial-like and conjugate to $P$.

2. Periodic Fatou components of $(f|U, U, f(U))$, (which come from $P$) are the only periodic Fatou components of $f$ and any Fatou component of $f$ is eventually mapped to one of these components. In particular,
   
   (i) $f$ has no wandering domains.
   
   (ii) If $J(P) = K(P) := \{z | P^n(z) \not\to \infty \ (n \to \infty)\}$, then $J(f) = \mathbb{C}$

3. $f$ has no asymptotic values and all the critical points of $f$ escape to $\infty$ under the iterate of $f$, except for the ones which correspond to the non-escaping critical points of $P$. 

7
(Outline of Proof):

Proposition B:

Suppose a given polynomial $P$ with $d = \deg P \geq 2$ and $z_1, z_2, \cdots, z_{k-1} \in \mathbb{C}$ satisfy the following:

(a) $P(0) = 0$, $P(1) = 1$,

(b) $z_1, z_2, \cdots, z_{k-1} \in K(P)$

(c) Let $c_1, c_2, \cdots, c_l$ be the distinct critical points of $P$, then $c_1, \cdots, c_m \in K(P)$, $c_{m+1}, \cdots, c_l \in \mathbb{C} \setminus K(P)$.

Then for any given $z_k \in \mathbb{C} \setminus K(P)$, $\varepsilon > 0$ and $R > 0$, there exist a polynomial $Q$ and $z'_1, z'_2, \cdots, z'_k$ which satisfy the following:

(1) $\deg Q = d + 1$

(2) $Q(0) = 0$, $Q(1) = 1$

(3) There exists a quasiconformal map $\varphi$ and a topological disk $U$ such that $K(P) \subset \varphi(U)$ and $Q|U \sim \varphi P$. 
(4) \( z'_j := \varphi^{-1}(z_j) \ (1 \leq j \leq k) \) satisfy
\[
|z_j - z'_j| < \varepsilon \ (1 \leq j \leq k), \quad z'_1, z'_2, \ldots, z'_{k-1}, z'_k \in K(Q) \quad \text{(in fact \( \exists m, \ Q^m(z'_k) = 0 \))}
\]

(5) \(|P(z) - Q(z)| < \varepsilon \) for \(|z| < R\)

(6) \( c'_j := \varphi^{-1}(c_j) \ (1 \leq j \leq l) \) and \( \exists c'_{l+1} \) are the distinct critical points of \( Q \) and satisfy
\[
|c_j - c'_j| < \varepsilon, \quad c'_1 \sim c'_m \in K(Q), \quad c'_{m+1} \sim c'_l, \quad c'_{l+1} \in \mathbb{C} \setminus K(Q)
\]

(7) Let \( a_1, \ldots, a_d \) be the zeros of \( P \), then the zeros of \( Q \) are \( a'_1, \ldots, a'_d, a'_{d+1} \) and satisfy
\[
|a_j - a'_j| < \varepsilon \ (1 \leq j \leq d), \quad |a'_{d+1}| > R
\]

(Outline of Proof of Proposition B):

(Construction of \( Q(z) \) :
[1] Define the quasiregular map $Q_1(z)$ as follows:

$$Q_1(z) = \begin{cases} 
P(z) & |z| \leq r \\
\psi(z) & r < |z| < 2r \\
\tilde{P}(z) := zd(a - az/P^n(z_k)) & |z| \geq 2r
\end{cases}$$

$$P(z) = zd(a + h_1(z)), \quad \tilde{P}(z) = zd(a + h_2(z)) \quad (\text{i.e. } h_2(z) = -az/P^n(z_k))$$

$$\psi(z) := zd(a + h(z)), \quad h(z) := \left(2 - \frac{|z|}{r}\right) h_1(z) + \left(\frac{|z|}{r} - 1\right) h_2(z)$$

[2] Construction of $Q_1$-invariant ellipse field:

Since $\forall z \in \{r \leq |z| \leq 2r\}$ satisfies

$$|Q_1(z)| > 2r, \quad Q_1^n(z) \to \infty \ (n \to \infty),$$

the orbit of $\forall z \in \mathbb{C}$ passes the region $\{r \leq |z| \leq 2r\}$, where $Q_1$ is not holomorphic, at most once. Then define

$$X_{\mu_n} := (Q_1^n)^*(X_0), \quad X_0 : \text{circle field}$$
(i.e. $X_{\mu_n} = \text{pull back of } X_0 \text{ by } Q^n_1$).

\[
\begin{array}{c}
\text{z} \xrightarrow{Q_1} \text{.} \xrightarrow{Q_1} \text{.} \xrightarrow{Q_1} \ldots \xrightarrow{Q_1} Q^n_1(z)
\end{array}
\]

It follows that $X_{\mu_n} \to \exists X_\mu \ (n \to \infty)$ and $X_\mu$ is a $Q_1$-invariant ellipse field by the construction.

[3] By measurable Riemann mapping theorem, $\exists \phi$ with $\mu_\phi(z) = \mu(z)$.

[4] $Q = \phi \circ Q_1 \circ \phi^{-1}$ is holomorphic on $\mathbb{C}$ and since it is finite to one, $Q$ is a polynomial. □

Take a dense subset $\{z_j\} \subset \mathbb{C}$ with $z_1 = 0$. Also take a sequence $\{R_n\}$ with $R_n \nearrow \infty$, $R_1 >> 1$ and $\sum \frac{1}{R_n} < \infty$. Take $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\sum \varepsilon_n < \exists \varepsilon_\infty$ (small enough). By starting with $P_0(z) := P(z)$, $z_1$, $\varepsilon_1 > 0$, $R_1 > 0$ and applying Proposition B over and over again, we get $\{P_n(z)\}_{n=0}^\infty$ with $\deg P_n = d + n$ which converges to an $f(z)$. 11
Since $P_n(0) = 0$ and $P_n(1) = 1$, we have

$$P_n(z+1) = \prod_{k=1}^{n} \left( 1 - \frac{z}{c_{n,k}} \right), \quad c_{n,k} := a_{n,k} - 1$$

Then $\exists c_k := \lim_{n \to \infty} c_{n,k}$ and $|c_k| > R_k - 2$ and we have

$$f(z+1) = \prod_{k=1}^{\infty} \left( 1 - \frac{z}{c_k} \right).$$

This shows that $f$ has genus 0 and if we choose $\{R_k\}$ so that it increases rapidly, then $f$ has arbitrarily slow growth (by a standard theory of entire functions). Also it is shown by the construction that preimages of $\text{int}K(f|_U)$ (if any) are dense in $\mathbb{C}$. So there are no wandering domains. \(\square\)
§3 Applications

Corollary 11 = Theorem 10 (Bergweiler-Eremenko, 2000) :

There exists a transcendental entire function with arbitrarily slow growth which satisfies \( J(f) = \mathbb{C} \).

( \because P(z) := 4z^2 - 3z \) (or in general, \( P \) with \( K(P) = J(P) \))

Corollary 12 (Baker (2001), Boyd (2002)):

There exists a transcendental entire function with arbitrarily slow growth which satisfies the following:

1. 0 is an attracting fixed point.
2. \( F(f) = \) the attractive basin of 0.

( \because P(z) := z^2 \) (or in general, \( P \) with only one attractive fixed point) )
Corollary C (K, 2013):

There exists a transcendental entire function with arbitrarily slow growth such that $J(f)$ is a Sierpiński carpet.

(\(\therefore P(z) = \text{polynomial with } m \text{ attractive fixed points and their immediate basins have no common boundary points.}\) )

(pictures : by S. Morosawa)

Remark 13:

Since $P_n \to f$ locally uniformly on $\mathbb{C}$ and $F(f)$ consists only of attractive
basins, it follows that $J(P_n) \to J(f)$ wrt Hausdorff metric (K, 1996). Note that $J(P_n)$ is disconnected and therefore it is not locally connected at any point, while $J(f)$ is locally connected.

**Corollary D:**
There exists a transcendental entire function with arbitrarily slow growth which has prescribed finite number of attracting, parabolic, Siegel and Cremer cycles.

(∵ $P(z) = \text{polynomial with prescribed finite number of attracting, parabolic, Siegel and Cremer cycles.}$)

**Corollary E:**
There exists a transcendental entire function with arbitrarily slow growth such that $f$ has a Cremer point but $J(f)$ is locally connected.

(∵ $P(z) = \text{polynomial with a Cremer point and an attractive fixed point which satisfies some condition.}$)
Thank you for your attention and
Herzlichen Glückwunsch zum 60 geburtstag, Walter!!