Fatou components of transcendental maps and singularities

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Resonances of Complex Dynamics
July 9-13, 2018

History

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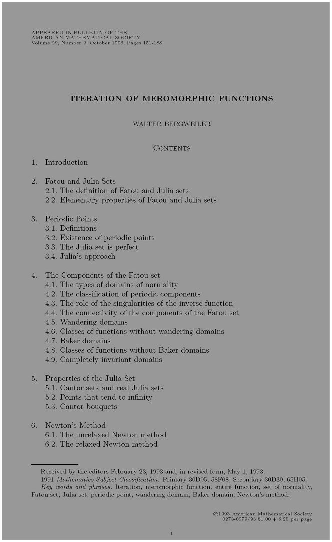

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ITERATION OF MEROMORPHIC FUNCTIONS

WALTER BERGWEILER

CONTENTS

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1

THE SURVEY

- 277 citations (MathSciNet)
- 607 citations (Google Sch.)

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Question 4. Let f be a meromorphic function with a cycle of Baker domains that does not contain a point of $\operatorname{sing}(f^{-1})$. Is there some relation between $\operatorname{sing}(f^{-1})$ and the boundaries of the domains of this cycle?

Question 5. Is it possible that a meromorphic function f has Baker domains if $O^+(z)$ is bounded for all $z \in \operatorname{sing}(f^{-1})$?

Question 8. Let U be a wandering domain of the transcendental meromorphic function f. Does there exist a sequence (n_k) such that $f^{n_k}|_U \to \infty$ as $k \to \infty$?

Question 10. Can a meromorphic function f have wandering domains if all (or all but finitely many) points of $\operatorname{sing}(f^{-1})$ are contained in preperiodic domains?

Question 11. Let f be a meromorphic function with a wandering domain U such that $U_n \cap \operatorname{sing}(f^{-1}) = \emptyset$ for all $n \geq 0$. Is there some relation between ∂U_n and $\operatorname{sing}(f^{-1})$?

Question 17. Let g be a meromorphic function, and let f be defined by (11). Does the convergence of $f^n(z)$ for all $z \in \operatorname{sing}(f^{-1})$ imply the convergence of $f^n(z)$ (to zeros of g) for all $z \in F(f)$?

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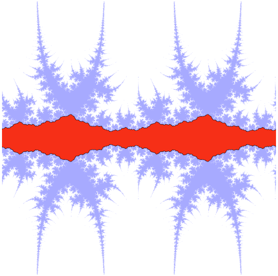

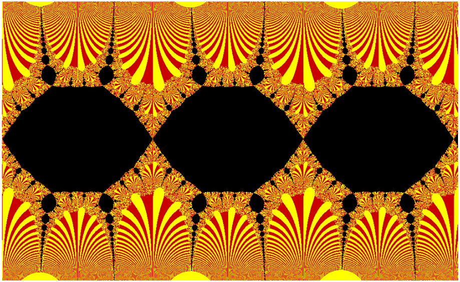

$z + a + b\sin(z)$ [Figures: Christian Henriksen] $z + 2\pi + \sin(z)$

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Holomorphic maps are local homeomorphisms everywhere except at the critical points

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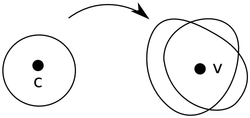
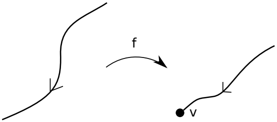
The set of singular values $S(f) = \mathrm{Sing}(f^{-1})$, consists of points for which some local branch of f^{-1} fails to be well defined.

These can be

- Critical values $CV = \{v = f(c) | c \in Crit(f)\}$;
- Asymptotic values $AV = \{a = \lim_{t\to\infty} f(\gamma(t)); \gamma(t) \to \infty\}$.



critical value



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- $f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$ is a covering map of infinite degree.

- Define the postsingular set of f as

$$P(f) = \overline{\cup_{s\in S} \cup_{n\geq 0} f^n(s)}.$$

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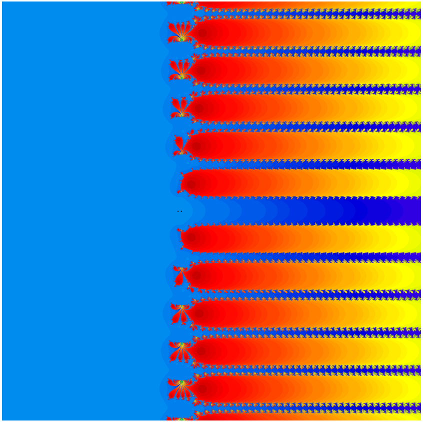
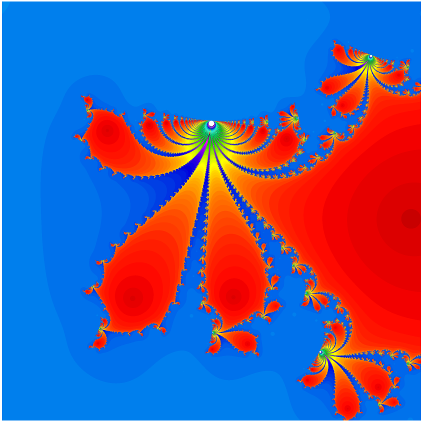
What about Baker domains and wandering domains?

Newton's method of entire transcendental maps

Can these map have a wandering domain?

Newton's method for $F(z) = z + e^z$.





Newton's method of entire transcendental maps

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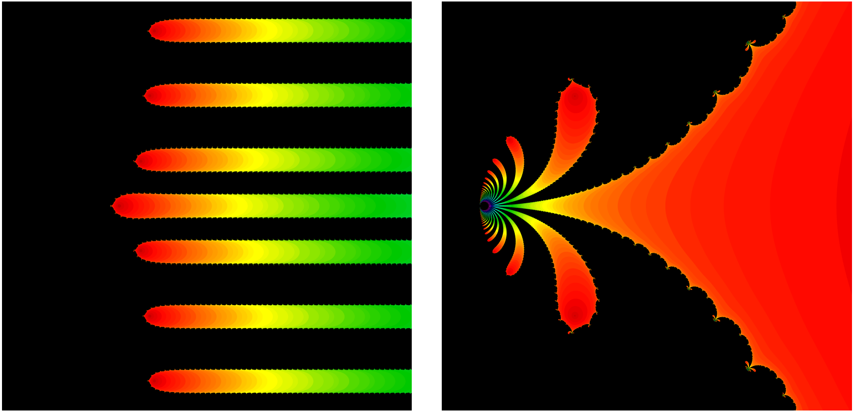

Walter's questions

We can find these type of questions in Walter's survey.

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Baker domains

The best result for Baker domains is the following, and answers Q4 - Q5.

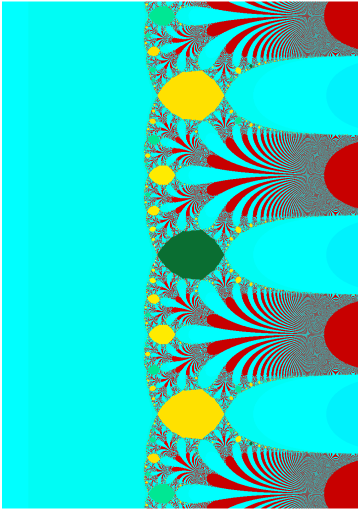
Theorem (Bergweiler'95, Mihaljevic-rempe'13, Baranski-F-Jarque-Karpinska'17)

f transcendental meromorphic, U invariant Baker domain, $U \cap S(f) = \emptyset$. Then $\exists p_n \in P(f)$ st

1. $|p_n| \to \infty$
2. $\left|\frac{p_{n+1}}{p_n}\right| \to 1$
3. $\frac{\mathrm{dist}(p_n, U)}{|p_n|} \to 0$

The theorem is sharp: there exists an (ETF) example for which $\mathrm{dist}(p_n, U) > c > 0$.

Walter's example



$$f(z) = 2 - \log 2 + 2z - e^z$$

Lift of a function of "finite type" with two superattracting fixed points (and no other singular values).

Special classes

Some classes of maps are singled out depending on their singular values.

- The Speisser class or finite type maps:

$$\mathcal{S} = \{f \text{ ETF (or MTF) such that } S(f) \text{ is finite}\}$$

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- The Eremenko-Lyubich class

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Maps in class $\mathcal{B}$ have no Baker domains and NO ESCAPING WANDERING DOMAINS (Escaping set $\subset J(f)$).

If U is a wandering domain, and $L(U)$ is the set of limit functions of f^n on U, then, all limit functions are constant and

$$U \text{ is} \begin{cases} \text{escaping} & \text{if } L(U) = \{\infty\} \\ \text{oscillating} & \text{if } \{\infty, a\} \subset L(U) \text{ for some } a \in \mathbb{C}. \\ \text{“bounded”} & \text{if } \infty \notin L(U). \end{cases}$$

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Open question Q17

Does there exist a map with a "bounded" wandering domain?

Examples of wandering domains

Examples of wandering domains are not abundant. Usual methods are:

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- Quasiconformal surgery [Kisaka-Shishikura'05, Bishop'15, Martí-Pete+Shishikura'18].

Wandering domains and singularities: Motivating examples

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Wandering domains and singularities: Motivating examples

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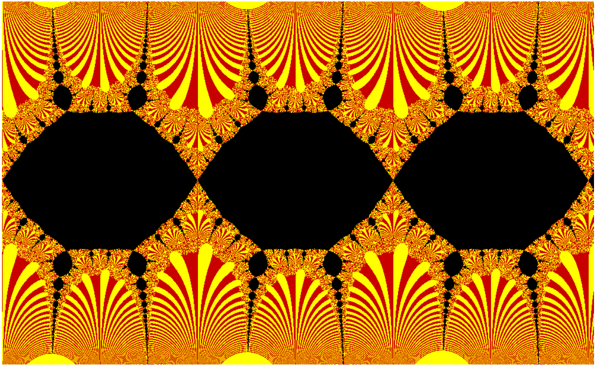
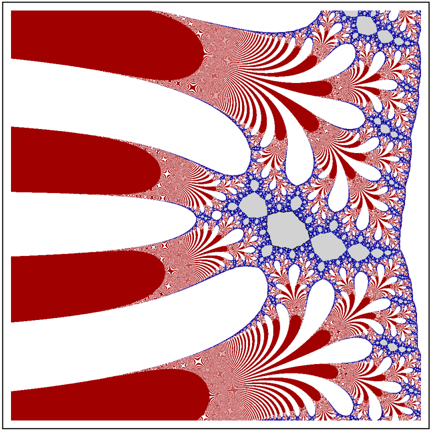
Example 1 (escaping):

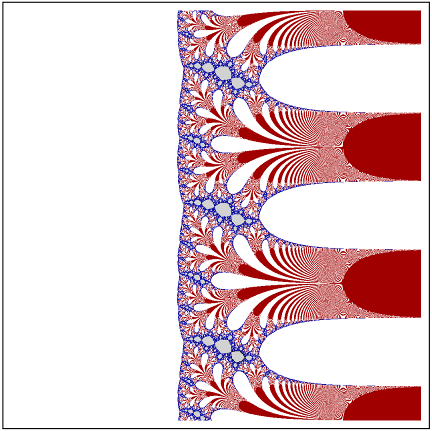


$z \mapsto z + 2\pi + \sin(z)$ One critical point in each WD.

Wandering domains and singularities: Examples

Example 2 (escaping and Univalent, $\partial U \subset P(f)$):





Left: Siegel disk of $g(w) = \frac{e^{2-\lambda}}{2-\lambda} w^2 e^{-w}$ with $\lambda = e^{2\pi i(1-\sqrt{5})/2)}$, around $w = 2 - \lambda$. Right: Lift to a wandering domain U.

Wandering domains and singularities: Examples

Example 3 [Kisaka-Shishilkura'05, Bergweiler-Rippon-Stallard'13] [Q10]. Wandering orbit of annuli such that

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Answer: yes. [F. - Jarque - Lazebnik'18 (to appear)]

Known results

Recall, for U a wandering domain, the set of limit functions

$$L(U) = \{a \in \widehat{\mathbb{C}} \mid f^{n_k}|_U \rightrightarrows a \text{ for some } n_k \to \infty\}.$$

Theorem (Bergweiler *et al*'93, Baker'02, Zheng'03)

Let f be a MTF with a wandering domain U. If $a \in L(U)$ then $a \in P(f)' \cap J(f)$.

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Theorem (Mihaljevic-Rempe'13)

If $f \in \mathcal{B}$ and $f^n(S(f)) \rightrightarrows \infty$ uniformly (+ extra geometric assumption), then f has no wandering domains.

Wandering domains and singular orbits

Theorem B (Baranski-F-Jarque-Karpinska'17 [Q11])

Let f be a MTF and U be a wandering domain of f. Let U_n be the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence $p_n \in P(f)$ such that

$$\frac{\operatorname{dist}(p_n, U_n)}{\operatorname{dist}(f^n(z), \partial U_n)} \to 0 \quad \text{as } n \to \infty.$$

In particular, if for some $d > 0$ we have $\operatorname{dist}(f^n(z), \partial U_n) < d$ for all n (for instance if the diameter of U_n is uniformly bounded), then $\operatorname{dist}(p_n, U_n) \to 0$ as n tends to ∞.

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Proof: normal families argument, hyperbolic geometry.... Based on the improvement of a technical lemma from Bergweiler on Baker domains. Compare also [Mihaljevic-Rempe'13]. More details.

Application: Topologically hyperbolic functions

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- Topologically hyperbolic maps do not possess parabolic cycles, rotation domains or wandering domains which do not tend to infinity
- Examples include many Newton's methods of entire functions.

Application: Topologically hyperbolic functions

Corollary C

Let f be a MTF topologically hyperbolic. Let U be a wandering domain s.t. $U_n \cap P(f) = \emptyset$ for $n > 0$. Then for every compact set $K \subset U$ and every $r > 0$ there exists n_0 such that for every $z \in K$ and every $n \geq n_0$,

$$\mathbb{D}(f^n(z), r) \subset U_n.$$

In particular,

$$\operatorname{diam} U_n \to \infty \qquad \textit{and} \qquad \operatorname{dist}(f^n(z), \partial U_n) \to \infty$$

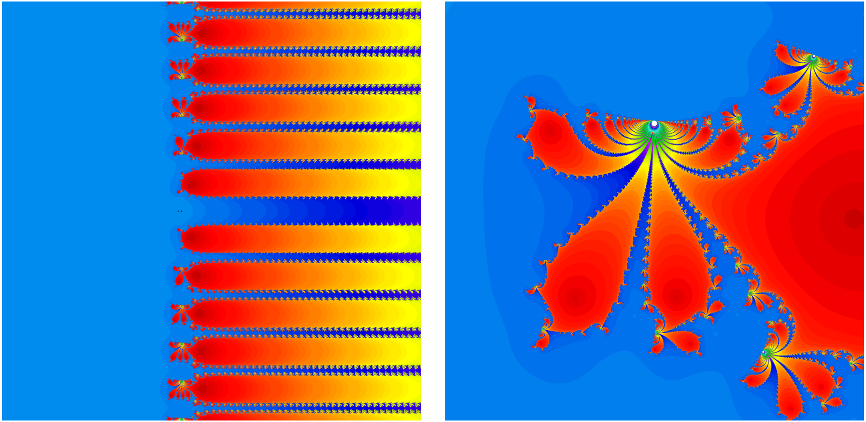
for every $z \in U$, as $n \to \infty$.

This can be applied to show that many functions, including Newton's method of $h(z) = ae^z + bz + c$ with $a, b, c \in \mathbb{R}$, have no wandering domains

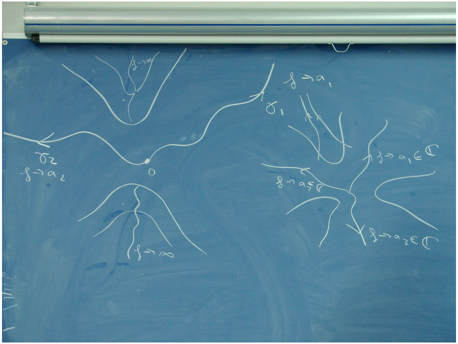
[c.f. Bergweiler-Terglane,Kriete].

No wandering domains

Newton's method for $F(z) = z + e^z$.



Happy birthday Walter!



Topics in Complex Dynamics 2007

Technical lemma

The technical lemma on the proof is the following.

Lemma

f TMF, U wandering domain, $U_n = f^n(U)$. Then,
$\forall K$ compact, $\varepsilon > 0$, $M \geq 1$,
there exists n_0 such that
for all $n > n_0$, $z \in K$, γ curve connecting $f^n(z)$ to $w \in \partial U$ with

$$\mathrm{length}(\gamma) \leq M \, \mathrm{dist}(f^n(z), \partial U_n)$$

there exists

$$p \in \mathbb{D}(\gamma, \varepsilon \, \mathrm{length}(\gamma)) \cap P(f).$$

▸ Go back