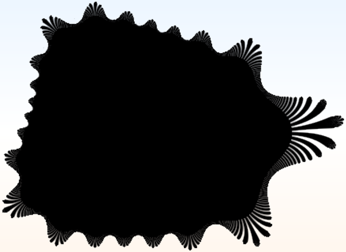
Similarities between hedgehogs of quadratic polynomials and iteration of exponentials

Arnaud Chéritat

CNRS & Institut de Mathématiques de Toulouse

Resonances of complex dynamics,
ICMS, Edinburgh, July 2018

Teaser



Disclaimer

This talk is mainly about heuristics and contains no result proved by the speaker.

Cantor bouquets in exponential dynamics

Consider $z \mapsto f(z) = e^{z-1.5}$. It has an attracting fixed point $a \approx 0.301$ and a singular value $z = 0$.

Cantor bouquets in exponential dynamics

Consider $z \mapsto f(z) = e^{z-1.5}$. It has an attracting fixed point $a \approx 0.301$ and a singular value $z = 0$.

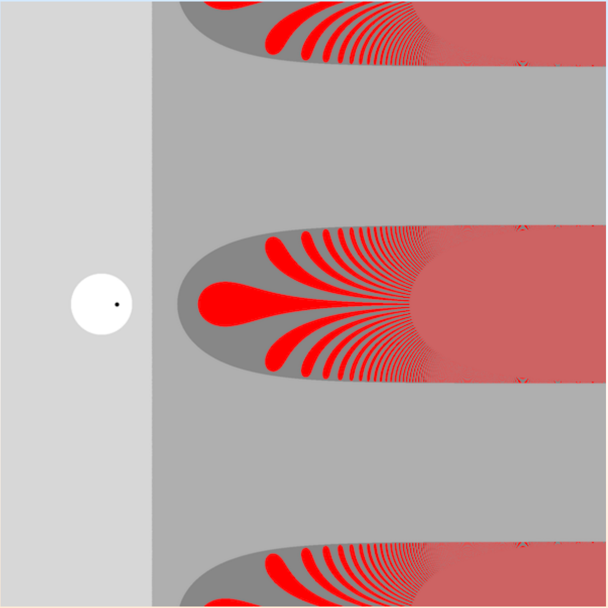
By a theorem of Fatou, 0 is in the immediate basin of a.

Cantor bouquets in exponential dynamics

Consider $z \mapsto f(z) = e^{z-1.5}$. It has an attracting fixed point $a \approx 0.301$ and a singular value $z = 0$.

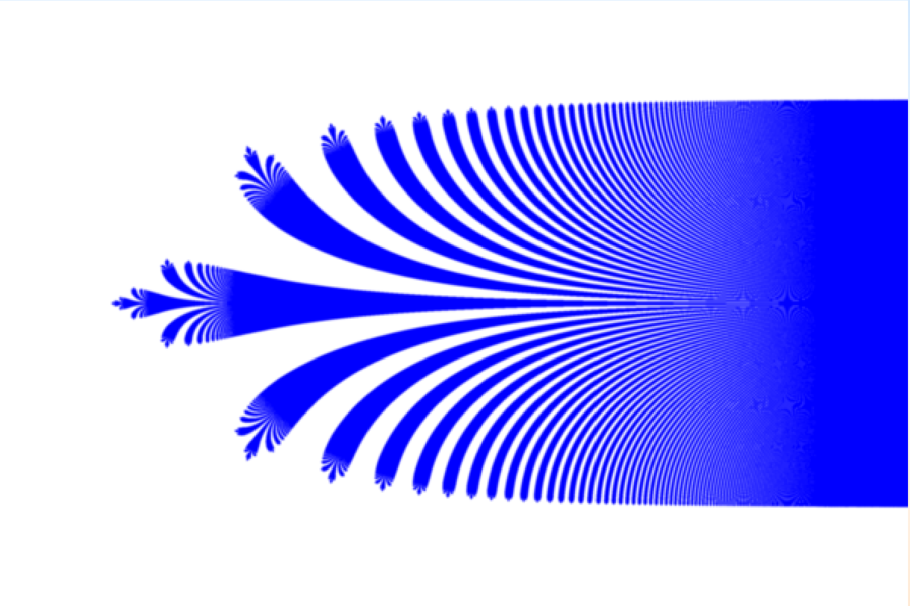
By a theorem of Fatou, 0 is in the immediate basin of a.

In fact the basin of a is connected and simply connected. Its complement is non-empty and equal to the Julia set of f.



Known facts

- (McMullen) The Lebesgue measure of the Julia set is 0.
- (Devaney) The Julia set is an uncountable union of disjoint hairs, each homeomorphic to $[0, +\infty[$, all its points escape to infinity except maybe its endpoint. A hair is non-empty iff its itinerary is less than the iterated exponential of some real number x.
- (Karpinska) The set of endpoints has Hausdorff dimension equal to 2 but the Julia set minus all endpoints has Hausdorff dimension equal to 1.
- (Viana) Hairs minus endpoints are smooth curves.



Indifferent fixed points

Yoccoz used *sector renormalization* to prove several important theorems concerning the following question:

> *Given a holomorphic map with an indifferent fixed point of given rotation number:* $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$*, how small can its Siegel disk be?*

Indifferent fixed points

Yoccoz used *sector renormalization* to prove several important theorems concerning the following question:

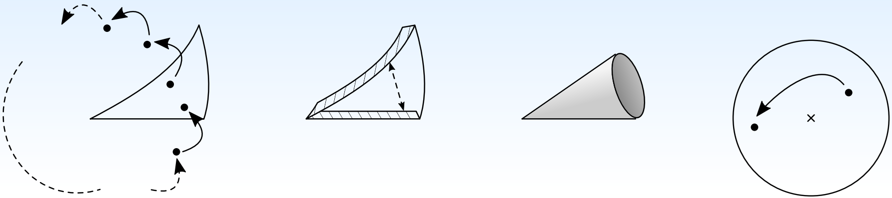
> *Given a holomorphic map with an indifferent fixed point of given rotation number:* $f(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$*, how small can its Siegel disk be?*

A normalization condition is required, all such condition are more or less equivalent to asking f to be *univalent* on the unit disk.

First return maps

[See applet]

Renormalization



Let $\rho(f) \in [0, 1)$ denote the representative modulo $\mathbb{Z}$ of the rotation number of f. Then

$$\rho(\mathcal{R}f) = \mathrm{Frac}\,\frac{1}{\rho(f)}$$

The enemy

Given α, what are the worst maps?

The enemy

Given α, what are the worst maps?

Yoccoz:

> *The other fixed points must avoid a disk centered on* 0 *of radius* $\approx \pi\alpha$*. On smaller disk of the same order, we have good estimates to perform sector renormalization.*

The enemy

Given α, what are the worst maps?

Yoccoz:

> *The other fixed points must avoid a disk centered on 0 of radius $\approx \pi\alpha$. On smaller disk of the same order, we have good estimates to perform sector renormalization.*
>
> *There are maps with a fixed point $\approx -i\pi\alpha$. This point and 0 are the only other fixed point nearby and the dynamics looks like a Möbius rotation.*

The enemy

Given α, what are the worst maps?

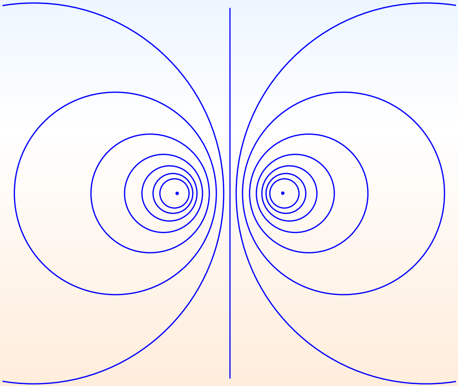
Yoccoz:

> *The other fixed points must avoid a disk centered on* 0 *of radius* $\approx \pi\alpha$*. On smaller disk of the same order, we have good estimates to perform sector renormalization.*
>
> *There are maps with a fixed point* $\approx -i\pi\alpha$*. This point and* 0 *are the only other fixed point nearby and the dynamics looks like a Möbius rotation.*

Yoccoz and Perez-Marco coined the term *nonlinearity* for this fixed point and by extension any identified cause preventing linearization to extend far away from the origin.

Möbius maps

$z \mapsto \frac{e^{2\pi i\alpha} z}{1-z}$



Eggbeater

Möbius maps

Sector renomalization is extremely simple for Möbius maps:

Möbius maps

Sector renomalization is extremely simple for Möbius maps:

Such a map is conjugate to the rotation by an explicit and simple Möbius conjugacy.

Möbius maps

Sector renomalization is extremely simple for Möbius maps:

Such a map is conjugate to the rotation by an explicit and simple Möbius conjugacy.

The renormalization change of variable associated to the map $z \mapsto e^{2\pi i\alpha} z$, $\alpha \in (0, 1)$ takes the form $z \mapsto z^{1/\alpha}$ (it is linear in log-coordinates).

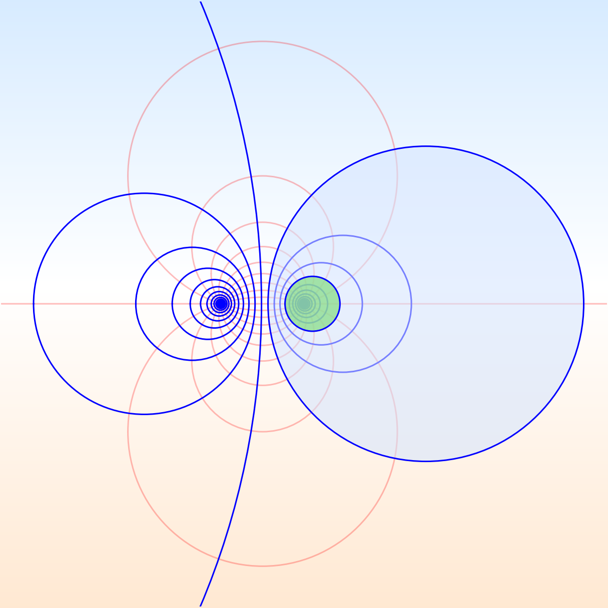
Möbius maps

Sector renomalization is extremely simple for Möbius maps:

Such a map is conjugate to the rotation by an explicit and simple Möbius conjugacy.

The renormalization change of variable associated to the map $z \mapsto e^{2\pi i\alpha} z$, $\alpha \in (0,1)$ takes the form $z \mapsto z^{1/\alpha}$ (it is linear in log-coordinates).

Important remark: renormalization for Möbius maps can be extended to a disk that is substantially bigger than the one for a generic map.



Bad maps

If the maps is not Möbius but still has a fixed point close to 0, the renormalization can also be extended to a big disk.

Bad maps

If the maps is not Möbius but still has a fixed point close to 0, the renormalization can also be extended to a big disk.

But are there maps which remain bad through infinitely many renormalizations?

Inverting renormalization

Yoccoz managed to build infinitely bad maps by reversing renormalization:

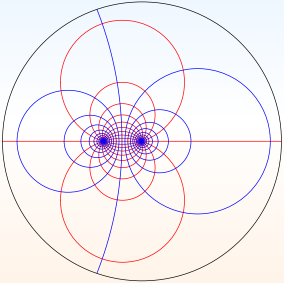
Inverting renormalization

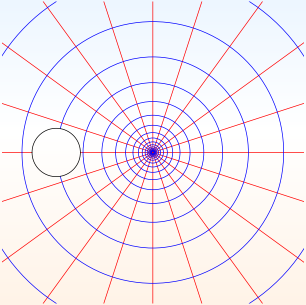
Yoccoz managed to build infinitely bad maps by reversing renormalization:

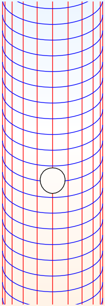
Starting from any germ f and given any $a_0 > 0$, he managed to build another germ g such that

- $\mathcal{R}g = f$
- g is bad

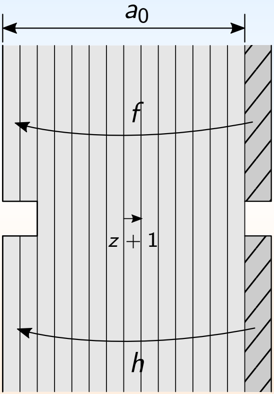
Yoccoz's construction







Yoccoz's construction



Intuition

Heuristics: *Quadratic maps are infinitely bad.*

Intuition

Heuristics: *Quadratic maps are infinitely bad.*

Backed by Yoccoz's result on the size of quadratic Siegel disks.

Intuition

Heuristics: *Quadratic maps are infinitely bad.*

Backed by Yoccoz's result on the size of quadratic Siegel disks.

Remark: Perez-Marco defined a class: *maps of quadratic type* are those whose power series expansion has a term in z^2 that is not too small. This approach has not been developed much.

Hedgehogs

Let $f(z) = e^{2\pi i\alpha}z + z^2$ with $\alpha \notin \mathbb{Q}$, and U a bounded topological disk such that $\overline{U}$ is contained in the domain of f, with C^1 boundary and on which f^{-1} has a well-defined branch defined in a neighborhood of $\overline{U}$. The hedgehog H (a.k.a. *Siegel compacta*) is the connected component containing 0 of the set of points whose orbit do not escape $\overline{U}$. It is compact, depends on U.

Hedgehogs

Let $f(z) = e^{2\pi i\alpha}z + z^2$ with $\alpha \notin \mathbb{Q}$, and U a bounded topological disk such that $\overline{U}$ is contained in the domain of f, with C^1 boundary and on which f^{-1} has a well-defined branch defined in a neighborhood of $\overline{U}$. The hedgehog H (a.k.a. *Siegel compacta*) is the connected component containing 0 of the set of points whose orbit do not escape $\overline{U}$. It is compact, depends on U.

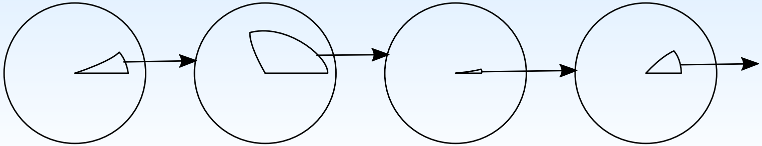
Properties (Perez Marco):

- H is forward and backward invariant.
- H is full (connected complement).
- The interior of H is the Siegel disk Δ.
- $H \cap \partial U \neq \varnothing$
- If $\overline{\Delta} \neq H$ then H is not locally connected.
- Varying U, the hedghehogs form a continuous and monotonous family of compact subsets of $\mathbb{C}$.

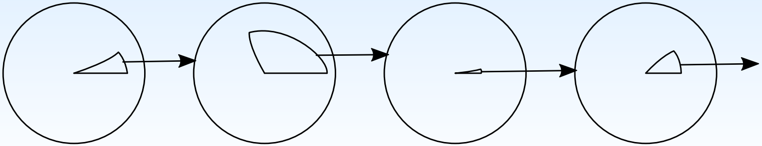
What do hedgehogs look like?

We have no picture of non-locally connected hedgehogs!

Heuristics

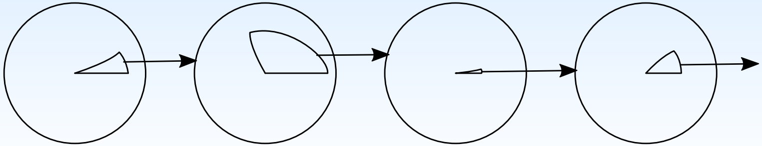


Heuristics

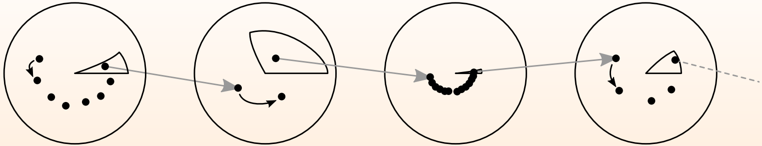


The hedgehog should be the set of points that "survive" infinitely many renormalization coordinate changes.

Heuristics

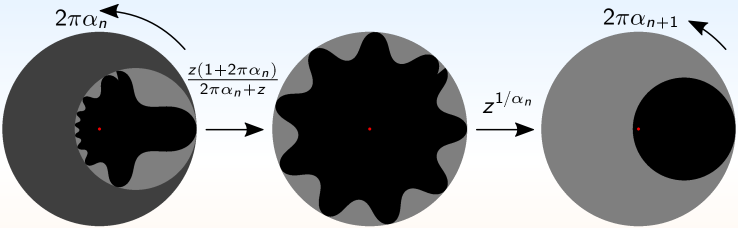


The hedgehog should be the set of points that "survive" infinitely many renormalization coordinate changes.



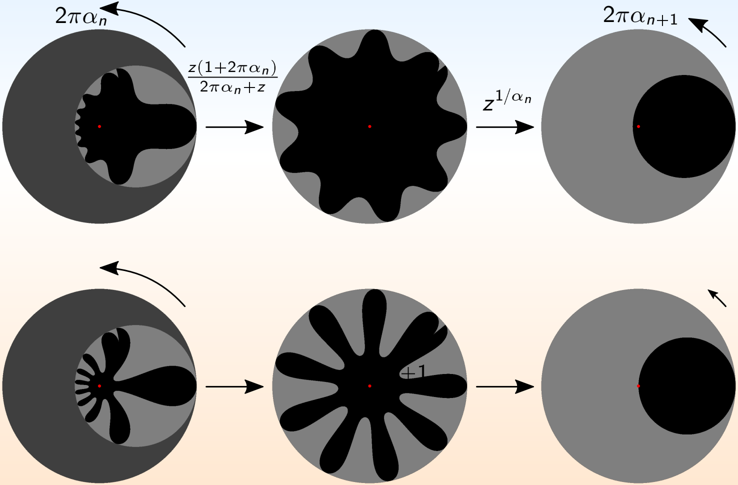
Toy model of hedgehogs of infinitely bad germs

As usual α_n denotes the rotation number of the n-th renormalization.



Toy model of hedgehogs of infinitely bad germs

As usual α_n denotes the rotation number of the n-th renormalization.



Toy models

If we work in logarithmic coordinates w, so $z = e^w$, the renormalization changes of coordinates (in the left-to-right direction in the previous diagram) take the form

$$\Phi_t(w) = \frac{1}{t} \log\left(\frac{1+2\pi t}{1+2\pi t e^{-w}}\right)$$

for w in the half plane $\operatorname{Re} w < 0$ and $t = \alpha_n$.

Toy models

If we work in logarithmic coordinates w, so $z = e^w$, the renormalization changes of coordinates (in the left-to-right direction in the previous diagram) take the form

$$\Phi_t(w) = \frac{1}{t} \log\left(\frac{1+2\pi t}{1+2\pi t e^{-w}}\right)$$

for w in the half plane $\operatorname{Re} w < 0$ and $t = \alpha_n$.

When α_n is small this map tends to $\lim_{t\to 0} \Phi_t(w) = 1 - e^{-w}$.

Fixing the unmatching waves

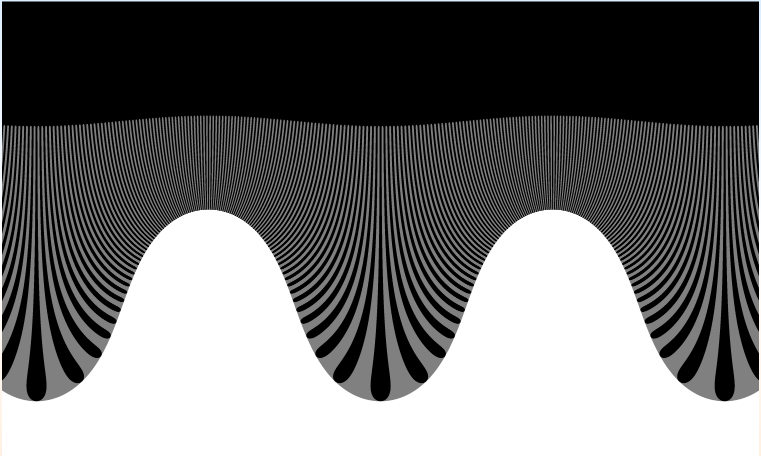
Let

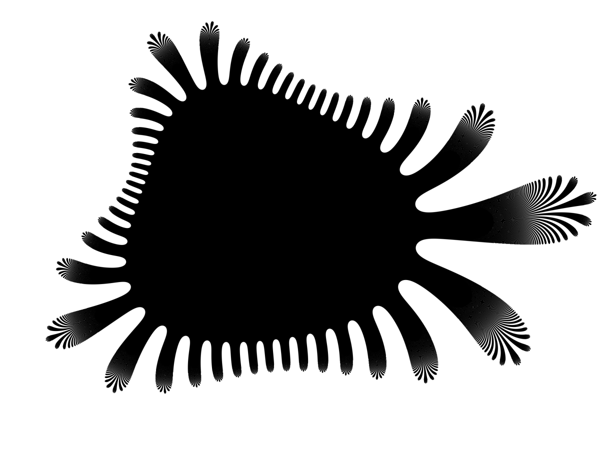
$$\Psi_t(z) = \Phi_t^{-1}(z).$$

For $\alpha \in [0,1]\backslash\mathbb{Q}$ let p_n/q_n be it approximants and $t_n = q_n/q_{n+1} \approx \alpha_n$. Then $\Psi_{t_n}(z + i2\pi q_{n+1}) = \Psi_{t_n}(z + i2\pi q_n)$. Let

$$\Psi^n := \Psi_{t_0} \circ \cdots \circ \Psi_{t_{n-1}}$$

$\tilde{K}^n = \Psi^n(\overline{\mathbb{H}})$. Then $\tilde{K}^n + i2\pi = \tilde{K}^n$. Let $K^n = \{0\} \cup \exp(\tilde{K}^n)$ and $K = \bigcap K^n$.





Toy models

Why?

- To draw pictures that should resemble hedgheogs of quadratic maps.
- As a testbed for theorems
- As an inspiration for properties of hedgehogs

Let Δ denote the interior of K, possibly empty.

– If Δ is empty then K is a Cantor bouquet (homeomorphic to the Lelek fan).

– Δ is non-empty iff the rotation number is a Brjuno number.

– If Δ is non-empty then $\partial\Delta$ is a Jordan curve.

– $K = \overline{\Delta}$ iff the rotation number is a Herman number.

– If $K \neq \overline{\Delta}$ and $\Delta \neq \emptyset$ then K is homeomorphic to a straight brush on a closed disk.

Hedgehogs of quadratic polynomials

Recent progresses are made thanks to the Inou-Shishikura renormalization, on the description of *mother hedghehog*, the closure of the critical orbit.

Hedgehogs of quadratic polynomials

Recent progresses are made thanks to the Inou-Shishikura renormalization, on the description of *mother hedghehog*, the closure of the critical orbit.

People working on that include Cheraghi and Shishikura.