Stability of Dirac masses for simple alignment processes

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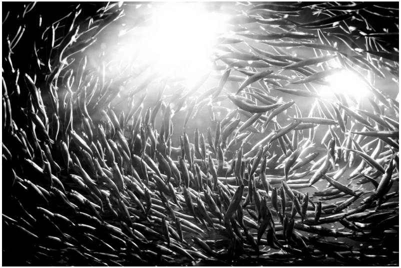
Workshop: "Collective dynamics and self-organization in biological sciences"
International Center for Mathematical Sciences, Edinburgh,
April 30th – May 4th, 2018

Context: alignment of self-propelled particles



- Unit speed, local interactions without leader
- Competition between alignment and noise
- Emergence of patterns

Focus on the alignment process only : no space !

What is alignment *without leader* ?

Framework : N unit vectors $v_1, \dots, v_N$ in $\mathbb{S}$ (unit sphere of $\mathbb{R}^d$).
Evolution process of v_i : defined in the same way for all indices i.

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Tool : empirical measure

$$f^N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{v_i(t)}.$$

Choice of a model : given a probability measure $f(t)$ on $\mathbb{S}$, build a process of evolution of a unit vector $v(t)$.
Apply this model with the measure $f^N(t)$, for all $v_i(t)$ (coupling).

What type of models ?

Time dynamics :

- Jump-based (discrete in time, synchronous/asynchronous)

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Which alignment mechanism ?

- "Compute" a target direction and use it in the dynamics
- Get an alignment mechanism with another agent, and combine all these mechanisms

Intrinsic versus Extrinsic noise

- Extrinsic noise : the agent with direction v has no direct access to the empirical measure f^N, rather to $f^N \circledast K_v$:

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- A combination of both ?

Mean-field limits

When N is large, behaviour of f^N ?

- Most cases : process giving evolution of $v(t)$ with respect to a “background” probability measure $f_0 \rightsquigarrow$ linear evolution of the law f of the distribution of v in time.

$$\partial_t f = \mathcal{L}_{f_0} f.$$

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- Note that some noise is already removed here (law of large numbers).

Some examples

- Async. jumps on target average direction (intrinsic noise):

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- Jumps at the position of another agent (extrinsic), with higher rate if agent is far:

$$\partial_t f = \int_{\mathbb{S}} \lambda(v', v) f(v') [K_{v'} \circledast f](v) \mathrm{d}v' - \int_{\mathbb{S}} \lambda(v, v') f(v) [K_v \circledast f](v') \mathrm{d}v'.$$

...

Qualitative long-time behaviour

- Does the alignment win ? When ?
- If the noise wins, at which speed does f converge to the uniform distribution ?
- What are the asymptotic "aligned" states ?
- If the alignment wins, can we determine the rate of convergence to the family of aligned states ?
- Can we prove that a solution will converge to a given "aligned" state, and can we determine this state ? At which rate ?

Example: "binary kinetic Vicsek" [AF, Liu, 2012]

$$\partial_t f = -\nabla_v \cdot \left[f(v) \nabla_v \left(v \cdot \int_{\mathbb{S}} v' f(v') \mathrm{d}v' \right) \right] + \sigma \Delta_v f.$$

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- Convergence to one given state, but no clue about Ω... Except quantitative stability : if $\| f_0 - M_{\kappa\widetilde{\Omega}} \| < \delta$, then $|\Omega - \widetilde{\Omega}| \leqslant \varepsilon$.

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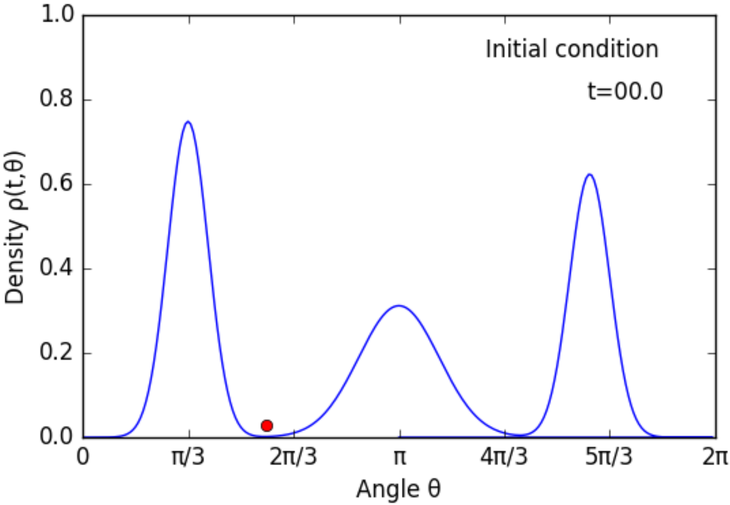
  Except quantitative stability : if $\|f_0 - M_{\kappa\widetilde{\Omega}}\| < \delta$, then $|\Omega - \widetilde{\Omega}| \leqslant \varepsilon$.
- Tool : dissipation of free energy (Lyapunov). Generalization with "extrinsic" noise [Degond, AF, Liu, 2015].


Initial condition
t=00.0
Density ρ(t,θ)
1.0
0.8
0.6
0.4
0.2
0.0
0
π/3
2π/3
π
4π/3
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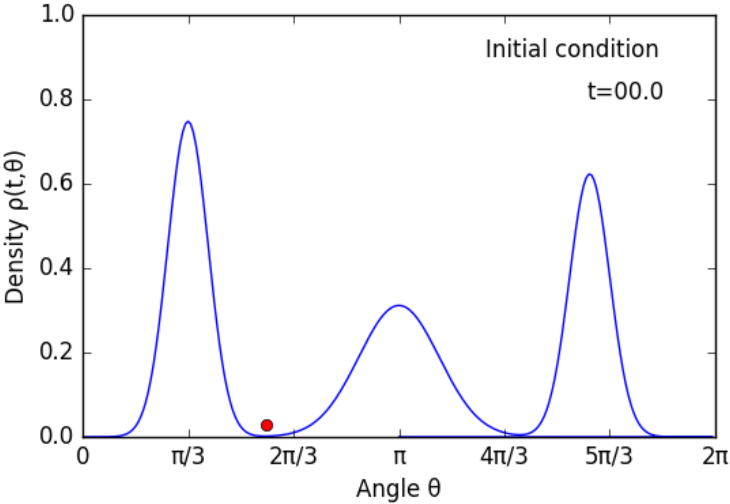
Angle θ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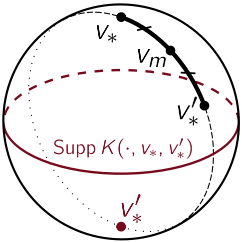
Midpoint model on the sphere

Kernel $K(v, v_*, v_*')$: probability density that a particle at position v_* interacting with another one at v_*' is found at v after collision.

$$\partial_t f_t(v) = \int_{\mathbb{S}\times\mathbb{S}} K(v, v_*, v_*')\,\mathrm{d}f_t(v_*)\mathrm{d}f_t(v_*') - f_t(v).$$



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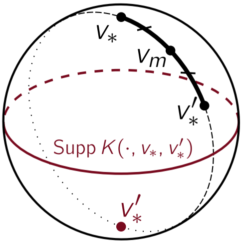

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Energy

$$E(f) = \int_{\mathbb{S}\times\mathbb{S}} d(v,u)^2 \,\mathrm{d}f(v)\,\mathrm{d}f(u).$$

Link “Energy – Wasserstein”

Useful Lemma – Markov inequalities

For $f \in \mathcal{P}(\mathbb{S})$, there exists $\bar{v} \in \mathbb{S}$ such that for all $v \in \mathbb{S}$:

$$W_2(f, \delta_{\bar{v}})^2 \leqslant E(f) \leqslant 4\, W_2(f, \delta_v)^2,$$

For such a $\bar{v}$ and for all $\kappa > 0$, we have

$$\int_{\{v\in\mathbb{S};\, d(v,\bar{v})\geqslant\kappa\}} \mathrm{d}f(v) \leqslant \frac{1}{\kappa^2} E(f),$$

$$\int_{\{v\in\mathbb{S};\, d(v,\bar{v})\geqslant\kappa\}} d(v,\bar{v})\, \mathrm{d}f(v) \leqslant \frac{1}{\kappa} E(f).$$

Evolution of the energy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(f) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(v_*, v_*', u)\,\mathrm{d}f(v_*)\,\mathrm{d}f(v_*')\,\mathrm{d}f(u).$$

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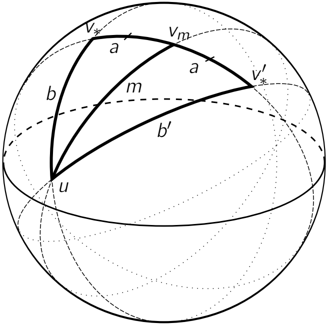

Configuration of Apollonius:

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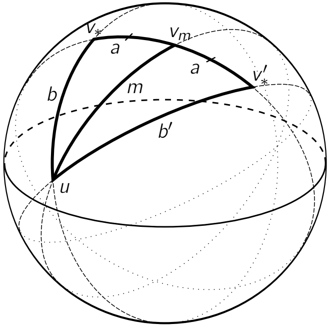

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Global and local (if distances all less than κ) estimates :

$$\alpha(v_*, v_*', u) \leqslant -\frac{1}{4}d(v_*, v_*')^2 + \begin{cases} 2\,d(v_*, v_*')\min\left(d(v_*, u),\ d(v_*', u)\right) \\ C_1\,\kappa^2\,d(v_*, v_*')^2. \end{cases}$$

Decreasing energy – Control on displacement

We set $\bar{\omega} := \{v \in \mathbb{S};\ d(v, \bar{v}) \leqslant \frac{1}{2}\kappa\}$, and we cut the triple integral in four parts following if v_*, v'_*, u is in $\bar{\omega}$ or not.

$$\frac{1}{2}\frac{d}{dt}E(f) + \frac{1}{4}E(f) \leqslant \underbrace{C\,\kappa^2\,E(f)}_{\text{Local lemma}} + \underbrace{12\frac{E(f)^{\frac{3}{2}}}{\kappa} + 24\frac{E(f)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

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Theorem: local stability of Dirac masses

There exists $C_1 > 0$ and $\eta > 0$ such that for all solution $f \in C(\mathbb{R}_+, \mathcal{P}(\mathbb{S}))$ with initial condition f_0 satisfying $W_2(f_0, \delta_{v_0}) < \eta$ for a $v_0 \in \mathbb{S}$, there exists $v_\infty \in \mathbb{S}$ such that

$$W_2(f_t, \delta_{v_\infty}) \leqslant C_1 W_2(f_0, \delta_{v_0})\, e^{-\frac{1}{4}t}.$$

Coupled nonlinear ordinary differential equations

Each velocity is attracted by the others, under the constraint that it stays on the sphere. Notation : $P_{v^\perp}u = u - (v \cdot u)v$ (orth. proj. if $v \in \mathbb{S}$, $u \in \mathbb{R}^n$).

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{N}\sum_{j=1}^{N} \nabla_{v_i}(v_i \cdot v_j) = P_{v_i^\perp} J \quad \text{where } J(t) = \frac{1}{N}\sum_{i=1}^{N} v_i.$$

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Special configurations : $J = 0$ (nothing moves...), or $v_i(0) = v_j(0)$ (then $v_i(t) = v_j(t)$ for all t).

Qualitative analysis, at most one at the back

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Repulsion of "back" particles

If $v_i(0) \neq v_j(0)$, we cannot have $v_i \cdot \Omega \to -1$ and $v_j \cdot \Omega \to -1$ (repulsion). Two possibilities :

- All particle are "front" : $v_i \cdot \Omega \to 1$.
- Up to renumbering, only v_N (or glued $v_k, \dots, v_N$) is "going to the back".

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Exponential estimate: $\|v_i - v_j\| = O(e^{-\lambda t})$ (for i, j "front").

Ω as a nearly conserved quantity

Theorem (if all particles are "front"):

There exists $\Omega_\infty \in \mathbb{S}$, and $a_i \in \{\Omega_\infty\}^\perp \subset \mathbb{R}^n$, for $1 \leqslant i \leqslant N$ such that $\sum_{i=1}^N a_i = 0$ and that, as $t \to +\infty$,

$$v_i(t) = (1 - |a_i|^2 e^{-2t})\,\Omega_\infty + e^{-t}\,a_i + O(e^{-3t}) \quad \text{for } 1 \leqslant i \leqslant N,$$

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Theorem (if v_N is the only "back" particle — convexity argument):

There exists $\Omega_\infty \in \mathbb{S}$, and $a_i \in \{\Omega_\infty\}^\perp \subset \mathbb{R}^n$, for $1 \leqslant i < N$ such that $\sum_{i=1}^{N-1} a_i = 0$ and that, as $t \to +\infty$,

$$v_i(t) = (1 - |a_i|^2 e^{-2\lambda t})\,\Omega_\infty + e^{-\lambda t}\,a_i + O(e^{-3\lambda t}) \quad \text{for } i \neq N,$$
$$v_N(t) = -\Omega_\infty + O(e^{-3\lambda t}), \quad \Omega(t) = \Omega_\infty + O(e^{-3\lambda t}).$$

Aggregation equation on the sphere

PDE for the empirical distribution

Define $f(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{v_i(t)} \in \mathcal{P}(\mathbb{S})$ (empirical measure), then f is a weak solution of the following PDE:

$$\partial_t f + \nabla_v \cdot (f\, P_{v^\perp} J_f) = 0, \quad \text{where } J_f = \int_{\mathbb{S}} v \, \mathrm{d}f(v). \qquad (1)$$

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Theorem

Given a probability measure f_0, there exists a unique global (weak) solution to the aggregation equation given by (1).

Tools: optimal transport, or for this case harmonic analysis (Fourier for $n = 2$), which gives well-posedness in Sobolev spaces.

Properties of the model

Characteristics, if the function $J(t)$ is given

Define Φ_t as the flow of the one-particle ODE ($\frac{\mathrm{d}v}{\mathrm{d}t} = P_{v^\perp} J$) :

$$\frac{\mathrm{d}\Phi_t(v)}{\mathrm{d}t} = P_{\Phi_t(v)^\perp} J(t) \quad \text{with } \Phi_0(v) = v.$$

Then the solution to the (linear) equation $\partial_t f + \nabla_v \cdot (f\, P_{v^\perp} J) = 0$ is given by $f(t) = \Phi_t \# f_0$ (push-forward):

$$\int_{\mathbb{S}} \psi(v) \mathrm{d}(\Phi_t \# f_0)(v) = \int_{\mathbb{S}} \psi(\Phi_t(v)) \mathrm{d}f_0(v) \text{ for } \psi \in C^0(\mathbb{S}).$$

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We can perform computations exactly as for the particles:

$$\frac{\mathrm{d}}{\mathrm{d}t} J_f = \int_{\mathbb{S}} P_{v^\perp} J_f \mathrm{d}f(v) = \langle P_{v^\perp} \rangle_f J_f = M_f J_f.$$

Increase of $|J_f(t)|$, integrability of $J_f \cdot M_f J_f$. Moments C^∞.

Convergence of $\Omega(t)$

We have $\dot\Omega = \frac{\mathrm{d}\Omega}{\mathrm{d}t} = P_{\Omega^\perp} M_f \Omega$, which is L^2 in time, but L^1 ?
After a few computations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\dot\Omega| = |\dot\Omega|\big(1 - \Omega \cdot M_f\Omega - \langle (u \cdot P_{\Omega^\perp} v)^2 \rangle_f\big) \\ + 2|J|\langle (1-(v\cdot\Omega)^2) u \cdot P_{\Omega^\perp} v\rangle_f,$$

where $u = \frac{\dot\Omega}{|\dot\Omega|}$ (when it is well-defined, 0 otherwise).

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Lemma: Tail of a perturbed ODE (integrability)

If $\frac{\mathrm{d}x}{\mathrm{d}t} = x + g$ where x is bounded, $g \in L^1(\mathbb{R}_+)$,
then $x(t) \in L^1(\mathbb{R}_+)$.

Therefore we get that $|\dot\Omega|$ is integrable, and
then $\Omega(t) \to \Omega_\infty \in \mathbb{S}$.

Convergence of f

Proposition: unique back (same idea of "repulsion")

Suppose that $J(t)$ is given (continuous, $|J| \geqslant c > 0$), with $\Omega(t) = \frac{J(t)}{|J(t)|}$ converging to $\Omega_\infty \in \mathbb{S}$. Then there exists a unique $v_{\text{back}} \in \mathbb{S}$ such that the solution $v(t)$ of $\frac{\mathrm{d}v}{\mathrm{d}t} = P_{v^\perp} J(t)$ with $v(0) = v_{\text{back}}$ satisfies $v(t) \to -\Omega_\infty$ as $t \to +\infty$.

Conversely, if $v(0) \neq v_{\text{back}}$, then $v(t) \to \Omega_\infty$ as $t \to \infty$.

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Theorem

Convergence in Wasserstein distance to $m\delta_{-\Omega_\infty} + (1-m)\delta_{\Omega_\infty}$, where m is the mass of $\{v_b\}$ with respect to the measure f_0. In particular, if f_0 has no atoms, then $f \to \delta_{\Omega_\infty}$.

No rate ...

Thanks!