An introduction to
totally disconnected locally compact groups

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Abstract

These are notes of the expository lectures that will be delivered for the *LMS Autumn Algebra School*, 21-25 September 2020. The aim is to introduce beginning Ph.D. students to the theory of totally disconnected locally compact groups. The exposition then includes basic properties of topological groups, a proof of van Dantzig's theorem and some of the most popular examples of totally disconnected locally compact groups.

The focus is on totally disconnected locally compact groups that satisfy some finiteness conditions with more emphasis on *compact generation*: for compactly generated totally disconnected locally compact groups, the notion of *Cayley-Abels graph* permits to deal with the topological group as a geometric object.

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1 Introduction

The class of locally compact groups generalises discrete and Lie groups. Locally compact groups came to light in the first half of 20th century, and since then they have played a central role among topological groups. The 20th century witnessed intense activity on the structure theory of many algebraic objects, e.g., simple finite groups, fields, division algebras, rings with ACC or DCC. What about the general structure of locally compact groups? A basic strategy to understand the structure of a locally compact group G is to split it into smaller factor groups: let G_0 be the largest connected subset of G containing the identity element, which is a closed subgroup (see Proposition 2.1.2) and produce the short exact sequence

$$1 \to G_0 \to G \to G/G_0 \to 1,$$

where G_0 is a **connected** locally compact group and G/G_0 is a **totally disconnected** locally compact group (i.e., the connected components of G/G_0 are reduced to singletons). In other words, G is an extension of its connected component G_0 by the totally disconnected piece G/G_0; for example, G could be (semi)direct product of G_0 and G/G_0. It follows that, at least in theory, questions about the structure of locally compact groups may be dealt with by treating separately the cases where G is connected and where G is totally disconnected and then combining the two answers.

On one hand, with the solution of Hilbert's fifth problem, our understanding of connected locally compact groups has significantly increased: they can be approximated by Lie groups (see Corollary 2.3.4). Therefore, the contemporary structure problem on locally compact groups concerns the class of totally disconnected locally compact groups.

The first part of these notes is devoted to the definition of the main characters of the mini-course, i.e., topological groups that are locally compact and totally disconnected. In particular, it includes basic properties of topological groups, a proof of van Dantzig's theorem [32], which is the classical theorem on the structure of totally disconnected locally compact groups, and several examples.

The investigation of the class of totally disconnected locally compact groups can be made more manageable by dividing the infinity of objects under investigation into classes of types with "similar structure". For example, in the second part, we focus on totally disconnected locally compact groups satisfying some finiteness conditions.

The most common finiteness condition for totally disconnected locally compact groups is *compact generation*, i.e., the topological group is algebraically generated by a compact subset. Compact generation naturally generalises the notion of finite generation that has been widely (and fruitfully) used in group theory to study abstract groups. Since every totally disconnected locally compact group is a directed union of compactly generated open subgroups (see Fact 4.1.2), one can reduce to the case of groups that are compactly generated without losing to much information (at least from a local perspective).

All compactly generated totally disconnected locally compact groups fall in the class of automorphism groups of locally finite connected graphs (see § 3.3). Indeed, Abels [1] proved that, by using van Dantzig's theorem, we can always construct a locally finite connected graph on which the group acts vertex transitively with compact open

vertex stabilisers, the so-called *Cayley-Abels graph* (see § 4.2). What is more striking is that the Cayley-Abels graph of a compactly generated totally disconnected locally compact group is unique up to quasi-isometry. Therefore, as for finitely generated groups, we can produce geometric group invariants[1] and study a compactly generated totally disconnected locally compact group as a geometric object.

Indeed, totally disconnected locally compact groups are now viewed as simultaneously geometric groups and topological groups so that the interaction between the local structure (i.e., the topological one) and the large-scale structure (i.e., the geometric one) becomes also relevant for the general theory. Since profinite groups are trivial as geometric groups and discrete groups are trivial as topological groups, it is not surprising that the profinite groups and the discrete groups constitute the atomic pieces in the theory of totally disconnected locally compact groups.

The final part of the fourth chapter attempts to introduce the reader to those finiteness conditions for totally disconnected locally compact groups that generalise compact generation in higher dimension. Since these notes are meant to be on the non-specialist level, we only provide definitions (without details) and references (where the details are), letting the reader decide how deep dig into the subject.

The conclusive part of these notes briefly introduces the main ingredients of the theory of scale for totally disconnected locally compact groups. The seminal work of George Willis [36, 37] was a fundamental breakthrough in the theory of totally disconnected locally compact groups after several years of stillness. Willis' theory made a systematic study of totally disconnected locally compact groups feasible, giving then start to the research interest we now benefit from.

Pre-requisites and Notation: These notes aim to help non-specialists and beginning Ph.D. students moving the first steps towards the theory of totally disconnected locally compact groups. Therefore, this text is (supposed to be) accessible to people not familiar with the topic and I have tried to keep it essentially self-contained (most of the results are proved in these notes) but, as with any advanced topic, there are limits and some of the results are stated without proofs. In such a case, references - where the reader will be able to find complements and proofs of the corresponding results - are provided.

The interested reader can also have a look at a few conference proceedings that collect part of the progress made with the general theory of totally disconnected locally compact groups and include some open problems; see [12, 41].

The reader is supposed to have mastered linear algebra, fundamental topological notions (topologies, continuity, neighbourhood basis, compactness, etc...) and basic notions from group theory.

We denote by

- $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, ...\}$,

- $\mathbb{Z}$ the ring of rational integers,

- $\mathbb{R}$ the field of real numbers,

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[1]That are properties invariant up to quasi-isometriy. The number of ends, growth and hyperbolicity are examples of such geometric invariants.

- $\mathbb{R}_+$ the subset of non-negative real numbers,

- $\mathbb{R}_+^\times$ the group of positive real numbers,

- $\mathbb{C}$ the field of complex numbers.

- If R is a commutative ring with unit, $R^\times$ stands for its multiplicative group of units.

  For example, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.

2 Preliminaries on topological groups

A complete and detailed introduction to the theory of topological groups can be found in several textbooks; for example, [19]. For convenience, all topological spaces appearing below are assumed to satisfy the Hausdorff separation axiom.

2.1 Warming-up

Definition 2.1.1 (Topological group)**.** A **topological group** is a group $(G, \cdot)$ which is also a topological space such that the following maps are continuous:

- the group operation

$$ \_ \cdot \_ \colon G \times G \to G, \quad (x, y) \mapsto x \cdot y, \quad \forall x, y \in G $$

  where $G \times G$ is endowed with the product topology;

- the inversion map

$$ \_^{-1} \colon G \to G, \quad x \mapsto x^{-1}, \quad \forall x \in G. $$

If the underlying group G is cyclic (resp., abelian, nilpotent, etc.), the topological group G is also called cyclic (resp. abelian, nilpotent, etc.). The additive notation $(G, +)$ can be used to describe some topological groups but only in the case when the topological group is abelian.

Remark 2.1.1. Clearly, **topological rings** and **topological fields**[2] can be defined in an analogous way.

Every group G can be viewed as a topological group if given the discrete topology. In such a case, G is called a **discrete group** and, since these notes concern topological groups, we will often refer to (abstract) groups as discrete groups.

Exercise 2.1.1. Let G be a topological group. Prove the following properties:

1. If a subgroup $H \leq G$ is open, then it is also closed. (Hint: the complement of H is union of cosets)

2. A subgroup containing an open set is automatically open.

3. A connected subset C is contained in the intersection of all clopen[3] subsets of G.

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[2]By "field" we always mean "commutative field"
[3]A set which is both closed and open

4. If G is connected, then the only open subgroup of G is G itself.

5. Every quotient map of a topological group is open.

6. For $H \leq G$ normal, the quotient group G/H is discrete iff H is open.

Other examples of a topological group are provided by the additive group $(\mathbb{R}, +)$ of the reals equipped with the usual topology, its subgroups $\mathbb{Z}$ and $\mathbb{Q}$ (with the subspace topology) and its quotient $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ (with the quotient topology). This extends to all powers $(\mathbb{R}^d, +)$, and so $(\mathbb{C}, +)$, because it can be easily proved that products of topological groups are again topological groups. Moreover, if R is a topological ring, then the ring $M(n, R)$ of all $n \times n$ matrices with entries in R is a topological ring if endowed with the product topology of $R^{n \times n}$.

Exercise 2.1.2. Let R be a commutative topological ring such that inversion is continuous on the set of invertible elements (for example, if R is a topological field). Prove that

1. the group $(GL(n, R), \cdot)$ of all invertible $n \times n$ matrices with entries in R is a topological group (notice that $GL(n, R)$ is a subset of $M(n, R)$ but not a subgroup. Hint: Cramer's rule can be used to prove that inversion is continuous);

2. the set $SL(n, R) = \{M \in GL(n, R) \mid \det(M) = 1\}$ is closed in $GL(n, R)$.

A topology τ on the group G such that the space (G, τ) is a topological group is called a **group topology on** G. Obviously, a topology τ on G is a group topology if, and only if, the map

$$G \times G \to G, \quad (x, y) \mapsto xy^{-1},$$

is continuous for all $x, y \in G$. Notice that, for every $g \in G$, the **left translation** $x \mapsto gx$, the **right translation** $x \mapsto xg$, as well as the **conjugation** $x \mapsto gxg^{-1}$ are continuous (in other words, every topological group is a homogeneous topological space).

Exercise 2.1.3. Prove the assertions above.

As a consequence, the topology of G is determined by a neighbourhood basis[4] at the identity 1_G: a family $\{U_\alpha\}_{\alpha \in I}$ of arbitrarily small neighbourhoods of 1_G determines the family $\{gU_\alpha\}_{\alpha \in I}$ of arbitrarily small neighbourhoods of any other group element g.

Exercise 2.1.4. The group G is discrete iff the point 1_G is isolated, i.e., the singleton $\{1_G\}$ is open.

Example 2.1.1. One can define group topologies on G by declaring well-behaved collections of subsets to be the neighbourhood basis at 1_G (see [7]):

- the **pro-finite topology** is determined by the family of all normal subgroups of finite index of G;

- the **pro-p topology** is determined, for a prime p, by all normal subgroups of G of finite index that is a power of p;

---

[4]A family $\mathcal{B}$ of neighbourhoods of the point x is said to be a *basis of neighborhoods of x* if for every neighbourhood U of x there exists $V \in \mathcal{B}$ contained in U.

- the p-**adic topology** is determined, for a prime p, by the family $\{U_n\}_{n\in\mathbb{N}}$ of normal subgroups of G, where U_n is generated by the powers $\{g^{p^n} \mid g \in G\}$.

Example 2.1.2 (Absolute values on fields)**.** An **absolute value** on a field $\mathbb{K}$ is a function $|\_|\colon \mathbb{K} \to \mathbb{R}$ satisfying

(av.1) $|x| \geq 0$ for every $x \in \mathbb{K}$, and $|x| = 0$ if and only if $x = 0$,

(av.2) $|xy| = |x||y|$, for all $x, y \in \mathbb{K}$,

(av.3) $|x + y| \leq |x| + |y|$, for all $x, y \in \mathbb{K}$.

The absolute value $|\_|$ is said to be **non-archimedean** if it satisfies the stronger condition

(av.3') $|x + y| \leq \max\{|x|, |y|\}$, for all $x, y \in \mathbb{K}$,

otherwise it is **archimedean**. For example, the usual absolute value on $\mathbb{R}$ is an archimedean absolute value.

It is clear that $d(x, y) = |x - y|$ gives $\mathbb{K}$ a structure of metric space, and the topology for which the balls

$$\{x \in \mathbb{K} \mid |x| < \varepsilon\}, \quad \varepsilon > 0,$$

form a basis of neighbourhoods of 0 is a field topology.

Exercise 2.1.5. Let $\mathbb{K}$ be a field equipped with the absolute value $|\_|$. The field topology defined on $\mathbb{K}$ by $|\_|$ is discrete iff $|x| = 1$ for all $x \neq 0$.

Example 2.1.3 (The field of p-adic numbers)**.** For a given prime p, the p-**adic absolute value** on $\mathbb{Q}$ is

$$|x|_p = p^{-n}, \quad x \in \mathbb{Q},$$

where n is the unique integer such that $x = p^n\left(\frac{a}{b}\right)$ and neither of the integers a and b is divisible by p (with the convention, $|0|_p = 0$). It is an example of non-archimedean absolute value. The p-**adic metric** $d_p(x, y) = |x - y|_p$ induces a field topology on $\mathbb{Q}$ which is called p-**adic topology**. As it happens with the topology inherits from $\mathbb{R}$, the metric space $(\mathbb{Q}, d_p)$ is not complete (i.e., not every Cauchy sequence converges in $(\mathbb{Q}, d_p)$). Let $\mathbb{Q}_p$ denote the completion of $\mathbb{Q}$ with respect to the p-adic metric. In particular, $\mathbb{Q}_p$ is a field of characteristic zero that is called the **field of** p-**adic numbers**. The metric d_p (and so the p-adic topology) can be extended to $\mathbb{Q}_p$ to obtain a topological field (see [24, § 12.3.4]).

The identity component of G. Let X be a topological space. Recall that it is defined an equivalence relation $\sim$ on X as follows: $x \sim y$ if there exists a connected subspace $C \subseteq X$ such that $x, y \in C$. Each equivalence class is a maximal connected subspace which is called a connected component of X.

Definition 2.1.2. For a topological group G, the connected component of the space G containing the identity 1_G is called **identity component**, and it is denoted by G_0. Clearly, G_0 is the union of all connected subspaces of G containing 1_G, and the topological group G is connected if, and only if, $G = G_0$. A topological group is said to be **totally disconnected** if the identity 1_G is its own connected component, that is, $G_0 = \{1_G\}$.

Notice that, for every $g \in G$, the set $gG_0 = G_0g$ is nothing but the connected component containing g because continuous maps preserve connectedness and translations are continuous. As a consequence, one has the following important fact.

Proposition 2.1.2. *Given a topological group G, the connected component G_0 is a closed normal subgroup.*

Proof. The subspace G_0 is closed since the closure of a connected subspace is still connected and so one has: $1_G \in \overline{G_0} \subseteq G_0$, i.e., $\overline{G_0} = G_0$.

For every $x \in G_0$, the translate $x^{-1}G_0 \ni 1_G$ is connected. It follows that $x^{-1} \in x^{-1}G_0 \subseteq G_0$ and G_0 is closed under taking inverses. On the other hand, for all $x, y \in G_0$, one has $xy \in xG_0$ but xG_0 is the connected component of x which is in the same connected component as 1_G (because $x \sim 1_G$), i.e., $xy \in xG_0 = G_0$.

Finally, G_0 is characteristic (and in particular normal) because continuous homomorphisms preserve connectedness. $\qquad\square$

Remark 2.1.3. The identity component needs not to be open: for a totally disconnected group G, $\{1_G\}$ is the identity component which is open iff G is discrete.

Due to the maximal connected nature of G_0, one has the following.

Proposition 2.1.4. *Let G be a topological group and let G_0 be the identity component. Then G/G_0 is a totally disconnected group.*

Proof. Let $\pi \colon G \to L := G/G_0$ be the canonical quotient homomorphism. We claim that $\pi^{-1}(L_0) \supseteq G_0$ is connected and, hence, $\pi^{-1}(L_0) = G_0$ which finally implies that $L = G/G_0$ is totally disconnected. To prove the claim, suppose we can decompose $\pi^{-1}(L_0)$ into a disjoint union $C_1 \cup C_2$ of non-empty closed subsets of G. Since G_0 is connected, for every $g \in \pi^{-1}(L_0)$, we have either $gG_0 \subseteq C_1$ or $gG_0 \subseteq C_2$. It follows that C_1 and C_2 are union of G_0-cosets. Passing to the quotient, L_0 is the disjoint union of non-empty closed subsets, contradiction. $\qquad\square$

2.2 Profinite groups

For a complete introduction to the realm of profinite groups the reader is referred to [25, 40] and [12, Chapter 3]. We recall here the terminology which is necessary for the definition of a profinite group and provide a few easy examples.

A **directed poset** $(I, \preceq)$ is a set I with a binary relation $\preceq$ satisfying:

1. $i \preceq i$, for $i \in I$;

2. $i \preceq j$ and $j \preceq k$ imply $i \preceq k$, for $i, j, k \in I$;

3. $i \preceq j$ and $j \preceq i$ imply $i = j$, for $i, j \in I$; and

4. if $i, j \in I$, there exists some $k \in I$ such that $i \preceq k$ and $j \preceq k$.

An **inverse system of topological groups over** I consists of a family $\{G_i \mid i \in I\}$ of topological groups together with continuous group morphisms $\varphi_{ij} \colon G_i \to G_j$, defined whenever $j \preceq i$, such that the diagrams

$$
\begin{array}{ccc}
G_i & \xrightarrow{\ \varphi_{ik}\ } & G_k \\
& \searrow{\scriptstyle \varphi_{ij}} \quad \nearrow{\scriptstyle \varphi_{jk}} & \\
& G_j &
\end{array}
$$

commutes whenever $k \preceq j \preceq i$. In addition, we assume that φ_{ii} is the identity morphism for every $i \in I$. A projective system of topological groups is said to be surjective if every morphism φ_{ij} is surjective. A family of continuous group morphisms $\varphi_i \colon G \to G_i$ is said to be **compatible** with the inverse system (G_i, φ_{ij}, I) if, for every $i \preceq j$, the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\ \varphi_i\ } & G_i \\
& \varphi_j \searrow & \nearrow \varphi_{ji} \\
& G_j &
\end{array}
$$

commutes. A topological group G together with a compatible family of continuous morphisms $\varphi_i \colon G \to G_i$, $(i \in I)$ is an **inverse limit** of the inverse system (G_i, φ_{ij}, I) if the following universal property is satisfied: for every topological group $\tilde{G}$ together with a compatible family of continuous group morphisms $(\psi_i, i \in I)$, there exists a continuous group morphism $\psi \colon \tilde{G} \to G$ such that the diagram

$$
\begin{array}{ccc}
\tilde{G} & \dashrightarrow{\ \psi\ } & G \\
& \psi_i \searrow & \swarrow \varphi_i \\
& G_i &
\end{array}
$$

commutes for every $i \in I$. In such a case, we denote the inverse limit by

$$
G = \varprojlim_{i \in I} (G_i, \varphi_{ij}),
$$

and call the maps $\varphi_i \colon G \to G_i$ *projection morphisms*. If the family φ_{ij} is clear from the context, we use simply $G = \varprojlim_{i \in I} G_i$.

Proposition 2.2.1 ([25, Proposition 1.1.1]). *Let (G_i, φ_{ij}, I) be an inverse system of topological groups over a directed poset I. Then the following hold:*

(a) *There exists an inverse limit of the inverse system (G_i, φ_{ij}, I);*

(b) *This limit is unique in the following sense: if (G, φ_i) and (H, ψ_i) are two limits of (G_i, φ_{ij}, I), then there is a unique topological isomorphism $\phi \colon G \to H$ such that $\psi_i \phi = \varphi_i$ for each $i \in I$.*

In particular, the inverse limit (G, φ_i) can be constructed as follows:

- G is the subgroup of the direct product $\prod_{i \in I} G_i$ of topological groups consisting of those tuples $(g_i)_{i \in I}$ that satisfy the condition $\varphi_{ij}(g_i) = g_j$ if $j \preceq i$;

- the morphisms $\varphi_i \colon G \to G_i$ are the restriction of the canonical projection

$$
\prod_{i \in I} G_i \to G_i;
$$

- the group topology on G is the subspace topology inherited by the product topology of $\prod_{i \in I} G_i$.

Fact 2.2.2 ([25, Lemma 1.1.2]). *If (G_i, φ_{ij}, I) is an inverse system of topological groups, then $\varprojlim_{i \in I} G_i$ is a closed subgroup of the product $\prod_{i \in I} G_i$.*

Definition 2.2.1. A **profinite group** G is the inverse limit $\varprojlim_{i \in I} G_i$ of a surjective inverse system (G_i, φ_{ij}, I) of finite groups G_i, where each finite group G_i is assumed to have the discrete topology and the group topology on G is inherited from the product topology on $\prod_{i \in I} G_i$.

For a profinite group G, a neighbourhood basis at 1_G is given by the set

$$\{\ker(\varphi_i) \mid i \in I\},$$

where $\varphi_i \colon G \to G_i$ are the canonical projection homomorphisms.

Fact 2.2.3. *A profinite group G is compact and totally disconnected.*

Proof. It is an easy consequence of the fact that G is a closed subset of the product of finite groups. $\qquad\square$

Example 2.2.1.　● Let R be a profinite commutative ring with unit. Then the following groups (with topologies naturally induced from R) are profinite groups: $R^\times$ (the group of units of R), $GL(n, R)$ and $SL(n, R)$.

- Consider the natural numbers $I = \mathbb{N}$, with the usual partial ordering, and the group of integers $\mathbb{Z}$. Form the inverse system $\{\mathbb{Z}/n\mathbb{Z}, \phi_{nm}\}$, where the map $\phi_{nm} \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the natural projection for $m \leq n$. The inverse limit produces the profinite group

$$\widehat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{N}} \frac{\mathbb{Z}}{n\mathbb{Z}}$$

  which can be identified with the set of equivalence classes of tuples of integers

$$\{\overline{(x_1, x_2, x_3, \dots)} \mid x_n \in \mathbb{Z}, \forall n \in \mathbb{Z}, \text{ and } x_m = x_n \bmod m \text{ whenever } m|n\}.$$

  Note that $\widehat{\mathbb{Z}}$ naturally inherits a structure of profinite ring from the finite rings $\mathbb{Z}/n\mathbb{Z}$. The ring $\widehat{\mathbb{Z}}$ is called **profinite completion** of $\mathbb{Z}$.

- Let p be a prime and form the profinite group defined by the following inverse limit

$$\varprojlim_{n \in \mathbb{N}} \frac{\mathbb{Z}}{p^n\mathbb{Z}},$$

  over the system of canonical projections. It is called **pro-p completion** of $\mathbb{Z}$.

  The set of its elements can be identified with the set of all equivalence classes of sequences $(a_1, a_2, a_3, \dots)$ of natural numbers such that $a_m = a_n (\bmod p^m)$, whenever $m \leq n$.

Remark 2.2.4. The ring of p-adic integers $\mathbb{Z}_p$ is topologically isomorphic to $\varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$: it suffices to prove that $\mathbb{Z}_p$ is the inverse limit of its quotients $\mathbb{Z}_p/p^n\mathbb{Z}_p$ (where the family $\{p^n\mathbb{Z}_p\}_{n \in \mathbb{N}}$ is the neighbourhood basis at 0 in the group of p-adic integers) and that each $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is isomorphic to the finite group $\mathbb{Z}/p^n\mathbb{Z}$.

Exercise 2.2.1. Let $\{G_i \mid i \in I\}$ be a collection of finite groups. Is the direct product $\prod_{i \in I} G_i$ profinite?

Exercise 2.2.2. Consider the natural numbers $I = \mathbb{N}$, with the usual partial ordering, and form the constant inverse system $\{\mathbb{Z}, id\}$. Compute the inverse limit.

Exercise 2.2.3. Let G be a profinite group.

1. A closed normal subgroup $H \leq G$ is open if and only if it has finite index.

2. Every open subgroup H of G contains a subgroup H_G that is normal and open in G. (Hint: Let $H_G = \bigcap_{g \in G} gHg^{-1} \ldots$)

2.3 Locally compact groups

An arbitrary topological space X is locally compact if every point admits a compact neighbourhood (if in addition X is Hausdorff, then every point admits a fundamental system of compact neighbourhoods). A topological group G is thus **locally compact** if the identity 1_G admits a compact neighbourhood.

The additive group $(\mathbb{R}, +)$ with its usual topology is a locally compact, non-compact, abelian group. Clearly, the multiplicative group $(\mathbb{R}^\times, \cdot)$ is also locally compact (here $\mathbb{R}^\times$ carries the induced topology) and the same holds for the groups $(\mathbb{C}, +)$ and $(\mathbb{C}^\times, \cdot)$. On the other hand, $\mathbb{Q}$ is not locally compact with the topology inherited by $\mathbb{R}$ (see Proposition 2.3.2), therefore local compactness is not inherited by all the subgroups.

Different examples of locally compact groups are discrete groups and profinite groups. If $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle group, then Tychonov's theorem yields that every power $\mathbb{T}^I$ of $\mathbb{T}$ is again compact and, in particular, locally compact. This is actually the most general example of a compact abelian group: every compact abelian group is isomorphic to a closed subgroup of a power of $\mathbb{T}$ (via Pontryagin duality).

Example 2.3.1 (Non-discrete locally compact fields)**.** Non-discrete locally compact fields have been completely classified by van Dantzig [33]. A non-discrete locally compact field $\mathbb{K}$ is either Archimedean (see Example 2.1.2), and then isomorphic to either $\mathbb{R}$ or $\mathbb{C}$, or non-Archimedean, in which case it is defined to be a **local field**[5]. A non-discrete locally compact field is connected if and only if it is Archimedean. See [24, § 12.3.4] and references there.

Exercise 2.3.1. 1. A closed subgroup of a locally compact group is again locally compact (and the closure condition is necessary, see Proposition 2.3.2).

2. If R is a locally compact ring and n is a natural number, then $R^{n \times n}$ is a locally compact ring.

3. Every quotient of a locally compact group is locally compact.

4. The product of a finite family of locally compact groups is locally compact (for infinite products to be locally compact is necessary the condition "all but a finite number of factors are actually compact").

5. If $\mathbb{K}$ is a topological field, then $GL(n, \mathbb{K})$ is open in $\mathbb{K}^{n \times n}$. Consequently, $(GL(n, \mathbb{K}), \cdot)$ is locally compact exactly if $\mathbb{K}$ is.

6. If $\mathbb{K}$ is a locally compact field, then $(SL(n, \mathbb{K}), \cdot)$ is a locally compact group.

Proposition 2.3.1. *A locally compact countable group is discrete.*

Proof. Recall that a *Baire space* is a topological space with the property that for each countable collection of open dense sets $\{U_n\}_{n \in \mathbb{N}}$ their intersection $\bigcap_{n \in \mathbb{N}} U_n$ is dense.

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[5]Some authors define a local field to be any commutative non-discrete locally compact field

By the Baire category theorem, every locally compact group is a Baire space. Let G be a non-discrete locally compact group; in particular, each singleton $\{g\}$ is closed but not open in G. If G is countable, then

$$G = \{g_1, \ldots, g_n, \ldots\} = \bigcup_{n \in \mathbb{N}} (G \setminus \{g_n\})$$

but $\bigcap_{n \in \mathbb{N}} (G \setminus \{g_n\}) = \emptyset$, contraddiction. $\qquad\square$

Proposition 2.3.2 ([19, Theorem 5.11])**.** *If a subgroup H of a topological group G is locally compact, then it is closed.*

Since the identity component G_0 is a closed normal subgroup of the locally compact group G (see Proposition 2.1.2), one can form the quotient group G/G_0. By Exercise 2.3.1(3), G/G_0 is locally compact. Moreover, the locally compact group G/G_0 is totally disconnected by Proposition 2.1.4. Therefore, one has the short exact sequence

$$1 \to G_0 \to G \to G/G_0 \to 1 \tag{2.1}$$

and every locally compact group G is then an extension of a totally disconnected locally compact group G/G_0 by the connected locally compact group G_0.

Connected locally compact groups can be approximated by Lie groups as the following important theorem implies:

Theorem 2.3.3 (Gleason-Yamabe)**.** *Let G be a locally compact group. Then, for any open neighbourhood U of the identity, there exists an open subgroup G' of G and a compact normal subgroup K of G' in U such that G'/K is isomorphic to a Lie group.*

Corollary 2.3.4. *Every connected locally compact group is inverse limit of Lie groups.*

As Terrence Tao writes on his blog [29]: *this theorem asserts the "mesoscopic" structure of a locally compact group (after restricting to an open subgroup G' to remove the macroscopic structure, and quotienting out by K to remove the microscopic structure) is always of Lie type.*

We omit the proof of Gleason-Yamabe theorem but the reader is referred to [30] where an exposition on the celebrated solution of Hilbert's fifth problem can be found.

3 Totally disconnected locally compact groups

3.1 A proof of van Dantzig's theorem

By using arguments from [19], we prove van Dantzig's theorem which can be considered as the key theorem in the theory of totally disconnected locally compact groups. In fact, it shows that the topology of a totally disconnected locally compact group admits a well-behaved basis of identity neighbourhoods.

Theorem 3.1.1 (van Dantzig, 1936)**.** *Let G be a totally disconnected locally compact group. Then every neighbourhood of the identity contains a compact open subgroup.*

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Proof. Let G be a totally disconnected locally compact group. By Vedenissov's Theorem, there exists a compact open neighbourhood K of 1_G. For each $x \in K$, there are an open set $U_x \ni 1_G$ with $xU_x \subseteq K$ (because left translation by x is continuous at 1_G) and an open set $V_x \ni 1_G$ with $V_xV_x \subseteq U_x$ (because the multiplication $\mu\colon G \times G \to G$ is continuous at $(1_G, 1_G)$). Moreover, since also the inversion map is continuous, the open set V_x can be chosen to be symmetric. For K is compact, it suffices a finite number of elements of K, say $x_1, \ldots, x_n$, to have $K \subseteq x_1V_{x_1} \cup \ldots \cup x_nV_{x_n}$. Set $V = V_{x_1} \cap \ldots \cap V_{x_n}$ and notice that

$$KV \subseteq \left(\bigcup_{i=1}^{n} x_iV_{x_i}\right)V \subseteq \bigcup_{i=1}^{n} x_iV_{x_i}V_{x_i} \subseteq \bigcup_{i=1}^{n} x_iU_{x_i} \subseteq K.$$

The inclusion $V = 1_G V \subseteq KV \subseteq K$ implies that $VV \subseteq U$, $VVV \subseteq U$, etc. Since V is symmetric, the subgroup H generated by V is given by

$$H = \bigcup_{n \in \mathbb{N}} \underbrace{V \cdots V}_{n} \subseteq K.$$

By Exercise 2.1.1, H is open (and so closed) and, for $H \subseteq K$, H is also compact. $\quad\square$

Hence, every totally disconnected locally compact group contains arbitrarily small (compact) open subgroups. This is the opposite of what occurs in the connected case, where the only open subgroup is the whole group (see Exercise 2.1.1). This is also the opposite of what occurs in Lie Groups, which admit a neighbourhood of the identity that contains only the trivial subgroup.

3.2 Consequences of van Dantzig's theorem

Here we collect a few consequences of van Dantzig's theorem.

Proposition 3.2.1. *Given a locally compact group G, the connected component G_0 coincides with the intersection of all open subgroups of G.*

Proof. To prove that G_0 is contained in the intersection of all open subgroups of G, it suffices to notice that every open subgroup H of G is a clopen set (see Exercise 2.1.1(3)).

For the reverse inclusion, we show that, for every $x \in G \setminus G_0$, there is an open subgroup H_x not containing x. By Proposition 2.1.4, the group G/G_0 is totally disconnected and locally compact. Therefore, van Dantzig's theorem yields a neighbourhood basis at G_0 given by compact open subgroups of G/G_0. It follows that there is a compact open subgroup $K \subseteq G/G_0$ not containing xG_0 (because we can separate points). Given the quotient map $\pi_0\colon G \to G/G_0$, we set $H_x = \pi_0^{-1}(K)$. $\quad\square$

Corollary 3.2.2. *If a topological group G admits a neighbourhood basis $\mathcal{B}$ at 1_G consisting of compact open subgroups, then G is totally disconnected and locally compact.*

Proof. The only part that needs some work is the total disconnectedness of G. By Proposition 3.2.1, $G_0 = \bigcap\{U \mid U \in \mathcal{B}\}$. But $\bigcap\{U \mid U \in \mathcal{B}\} = \{1_G\}$ since we assume topological groups to be Hausdorff. $\quad\square$

In other words, van Dantzig's theorem characterises totally disconnected locally compact groups among topological (Hausdorff) groups: they are topological groups admitting a neighbourhood basis at 1_G formed by compact open subgroups.

The abundance of compact open subgroups will reveal itself to be the most fruitful property of this class of groups.

In general, total disconnectedness is not preserved under taking quotients. Thanks to van Dantzig's theorem this is not the case for locally compact groups.

Proposition 3.2.3. *The quotient of a TDLC-group by a closed normal subgroup is totally disconnected.*

Proof. Let G be a TDLC-group and let N be a closed normal subgroup of G. It follows from van Dantzig's theorem that the collection of all compact open subgroups of G form a neighbourhood basis at 1_G. Since quotient maps are open, the quotient G/N admits a neighbourhood basis at N formed by compact open subgroups, that are the quotients of all compact open subgroups of G. Thus G/N is totally disconnected by Corollary 3.2.2. □

Proposition 3.2.4. *Every locally compact group G contains an open subgroup H which is compact-by-connected.*

Proof. One applies van Dantzig's theorem to the TDLC-group G/G_0 and then pulls back the resulting compact open subgroup. □

Proposition 3.2.5. *A compact totally disconnected group is a projective limit of finite groups. In particular, a topological group is profinite if, and only if, it is compact and totally disconnected.*

Proof. Let G be a compact totally disconnected group. By van Dantzig's theorem, the set $\mathcal{O}$ of all compact open subgroups of G form a neighbourhood basis at 1_G. Since G is compact, every subgroup $H \in \mathcal{O}$ contains a subgroup which is both open and normal in G (see Exersice 2.2.3(2)). Thus, the family $\mathcal{NO} = \{H \in \mathcal{O} \mid H \trianglelefteq G\}$ is a neighbourhood basis at 1_G. Therefore, the morphism $G \to \prod_{H \in \mathcal{NO}} G/H$ of compact groups is injective and continuous, and provides a topological isomorphism from G to a closed subgroup of the above product of finite groups. □

3.3 Examples of totally disconnected locally compact groups

From now on, we shall use TDLC-group as shorthand. In the list below, it will become clear that examples of TDLC-groups can be produced by using van Dantzig's theorem.

- Discrete groups and profinite groups are rather trivial examples of TDLC-groups.

- (Local fields) Let $\mathbb{K}$ be a local field, i.e., a non-Archimedean locally compact field (see [24, § 12.3.4]). Then $\mathbb{K}$ has a unique maximal compact subring

$$\mathfrak{o}_{\mathbb{K}} = \{x \in \mathbb{K} \mid \{x^n \mid n \geq 1\} \text{ is relatively compact}\}$$

  and $\mathfrak{o}_{\mathbb{K}}$ has a unique maximal ideal

$$\mathfrak{p}_{\mathbb{K}} = \{x \in \mathbb{K} \mid \lim_{n \to \infty} x^n = 0\}.$$

Both $\mathfrak{o}_{\mathbb{K}}$ and $\mathfrak{p}_{\mathbb{K}}$ are compact and open in $\mathbb{K}$. The ideal $\mathfrak{p}_{\mathbb{K}}$ is principal in $\mathfrak{o}_{\mathbb{K}}$: there is $\pi \in \mathbb{K}$ such that $\mathfrak{p}_{\mathbb{K}}$ is the ideal (π) generated by π. The nested sequence of ideals

$$(\pi) \supset \cdots \supset (\pi^n) \supset (\pi^{n+1}) \supset \cdots$$

constitutes a basis of compact open subgroups at 0 in $\mathbb{K}$. Therefore, by van Dantzig's theorem, $\mathbb{K}$ is totally disconnected.

Local fields fall into two families, namely:

1. the fields of p-adic numbers, $\mathbb{Q}_p$ and their finite extensions, and

2. the fields of formal Laurent series, $\mathbb{F}_q((t))$, over some finite field $\mathbb{F}_q$.

Note that $\mathbb{Q}_p$ admits $\{p^n \mathbb{Z}_p\}_{n \in \mathbb{N}}$ as basis of compact open subgroups, where $\mathbb{Z}_p$ is the ring of p-adic integers, and $\mathbb{F}_q((t))$ has $\{t^n \mathbb{F}_q(t)\}_{n \in \mathbb{N}}$ as basis of compact open subgroups, where $\mathbb{F}_q(t)$ is the ring of formal Taylor series over $\mathbb{F}_q$.

- (Linear groups over local fields) Let $\mathbb{K}$ be a local field. The group $GL_n(\mathbb{K})$ with the topology inherited as a subset of $\mathbb{K}^{n^2}$ is TDLC.

- (Automorphism group of a connected locally-finite graph) All graphs will be assumed to be undirected. Therefore, a **graph** Γ is a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is a set and $E\Gamma$ is a collection of unordered distinct pairs of elements from $V\Gamma$. The elements of $V\Gamma$ are called **vertices** and the elements of $E\Gamma$ are called **edges**. We will need a bit of terminology for graphs: vertices v and u are said to be adjacent, if $\{v, u\}$ is an edge in Γ; a graph is **locally finite** if each vertex v have a finite number of adjacent vertices; a **path of length** n from v to u is a sequence $(v = v_0, v_1, \ldots, v_n = u)$ of vertices, such that v_i and v_{i+1} are adjacent for $i = 0, 1, \ldots, n-1$; a graph is **connected** if for any two vertices v and u there is a path from v to u in the graph. An **automorphism** of a graph Γ is a bijection $\varphi \colon V\Gamma \to V\Gamma$ such that $\{\varphi(v), \varphi(w)\} \in E\Gamma$ if and only if $\{v, w\} \in E\Gamma$. The collection of automorphisms forms a group under composition, and it is denoted by $\mathrm{Aut}(\Gamma)$.

Let Γ be a connected graph and endow $\mathrm{Aut}(\Gamma)$ with the compact-open topology via considering $V\Gamma$ to be a discrete space. Namely, a basis of this topology is given by the sets

$$\Sigma_{v,w} = \{g \in \mathrm{Aut}(\Gamma) \mid g(v_i) = w_i\},$$

where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ range over all finite[6] tuples of vertices of Γ. In particular, two automorphisms of Γ are "closed" to each other if they agree on "many" vertices.

Exercise 3.3.1. 1. $\mathrm{Aut}(\Gamma)$ with the compact-open topology is a topological group.

2. The compact-open topology of $\mathrm{Aut}(\Gamma)$ coincides with the pointwise convergence topology.

Remark 3.3.1. The compact-open topology on $\mathrm{Aut}(\Gamma)$ is often called **permutation topology**.

Let $G = \mathrm{Aut}(\Gamma)$. The identity element of G belongs to an open set $\Sigma_{a,b}$ iff $a = b$. Consequently, the compact-open topology on G has a neighbourhood basis at the

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[6]Notice that the length of the tuples is arbitrary

identity formed by the pointwise stabalizers of finite sets of vertices. Recall that, given a vertex $v \in \Gamma$, the set $G_{(v)} = \{g \in G \mid g(v) = v\}$ is a subgroup of G which is called the **stabilizer of** v. Given a finite set of vertices $\mathcal{V}$, the intersection $G_{(\mathcal{V})} = \bigcap_{v \in \mathcal{V}} G_{(v_i)}$ is the **poitwise stabilizer** of the set $\mathcal{V}$. Clearly, pointwise stabilizers of finite set of vertices are open subgroups of G.

Theorem 3.3.2. *Let Γ be a connected locally finite graph. Poinwise stabilizers of finite sets of vertices are compact in the compact-open topology. In particular, $G = \mathrm{Aut}(\Gamma)$ is a TDLC-group.*

Sketch of Proof. Let $v \in V\Gamma$. The proof consists of several steps:

Construction of the group morphism ϕ_v: for $k > 0$, set

$$S_{v,k} = \{w \in V\Gamma \mid \text{the shortest path connecting } v \text{ and } w \text{ has length at most } k\},$$

which is called the *k-sphere around v*. The stabiliser $G_{(v)}$ permutes the elements in each k-sphere, i.e., there exists a group homomorphism $\phi_{v,k} \colon G_{(v)} \to Sym(S_{v,k})$. Define

$$\phi_v \colon G_{(v)} \to \prod_{k>0} Sym(S_{v,k}), \quad \phi_v(g) := (\phi_{v,k}(g))_{k>0}, \quad \forall g \in G_{(v)}.$$

The topology on $\prod_{k>0} Sym(S_{v,k})$: Since Γ is locally finite, it follows that every k-sphere around v is finite. Give the finite groups $Sym(S_{v,k})$ the discrete topology and $\prod_{k>0} Sym(S_{v,k})$ the product topology. Therefore, $\prod_{k>0} Sym(S_{v,k})$ is a compact topological group;

Properties of ϕ_v: The group morphism ϕ_v is continuous, closed and injective. Therefore, ϕ_v is a topological group isomorphism onto its image; see [35] for details.

Conclusion: Since $G_{(v)}$ is isomorphic to a closed subgroup of a compact group, it is compact. It follows that every $G_{(\mathcal{V})}$ is compact because it is finite intersection of compact open subgroups of G. Thus, G is TDLC since it admits a neighbourhood basis at 1_G consisting of compact open subgroups.

$\square$

Remark 3.3.3. For connected locally finite graphs, Theorem 3.3.2 shows that $\mathrm{Aut}(\Gamma)$ is a TDLC-group but does not say anything on the "non-discreteness" of $\mathrm{Aut}(\Gamma)$. In the theory of TDLC-groups, it is notoriously a difficult problem to determine if a given TDLC-group is non-discrete.

- (Neretin group $\mathcal{N}_d$ of spheromorphism of a d-regular tree) Firstly, we need a bit of terminology on trees. A **tree** is a connected graph without nontrivial cycles, where for nontrivial cycle we mean a path $(v_0, \ldots, v_n)$ such that $n \geq 1$ and $v_0 = v_n$. A vertex $v \in VT$ of degree 1 (i.e., has a unique adjacent vertex) is called a **leaf**. For $d \in \mathbb{N}$, an **infinite d-regular tree** is an infinite tree whose vertices have degree $d + 1$. A **finite d-regular tree** is a finite tree, whose every vertex is either a leaf, or has degree $d + 1$, i.e., it is an **internal vertex** (see Figure 1).
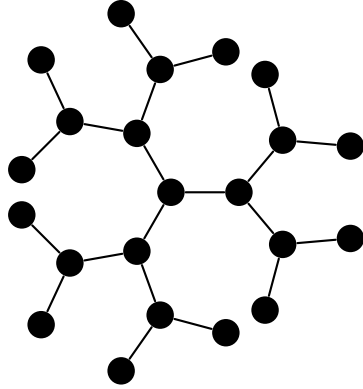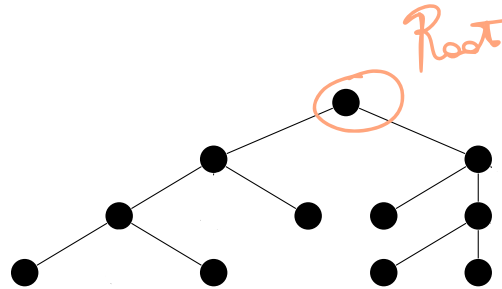
Figure 1: Finite 2-regular tree   Figure 2: Finite rooted 2-regular tree

A **rooted tree** is a tree with a distinguished vertex $o \in T$, called its **root**. For rooted trees, the definition of d-regularity is slightly modified: the root has degree d instead of $d+1$ (see Figure 2).

An important property of a tree T is given by the fact that there exists a unique path connecting two vertices v and u. A **ray** in T is defined to be an infinite path, i.e., a sequence $(v_0, v_1, \dots)$ of distinct vertices of T such that the consecutive ones are adjacent. Two rays are said to be asymptotic if, after removing some finite initial subsequences, they become equal. Namely, two rays are asymptotic if they have common tails. Equivalence classes of asymptotic rays in T are called the **ends** of T. All ends of T form the set ∂T which is called the **boundary** of T.
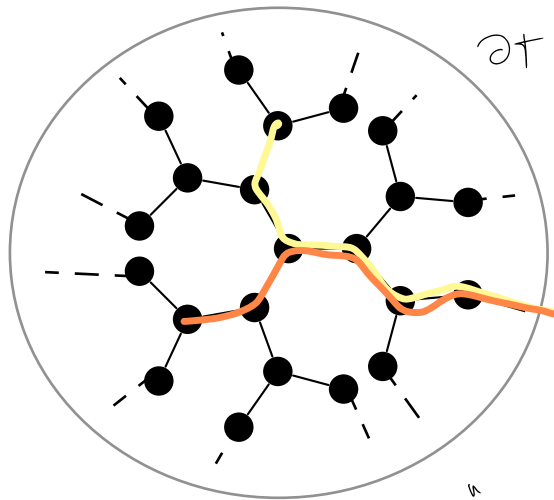

Figure 1: Asymptotic rays and the boundary

The group of spheromorphisms of a d-regular tree has been introduced by Neretin [23] by analogy with the diffeomorphism group of a circle. Roughly speaking, a **spheromorphism** of ∂T is a transformation induced in the boundary ∂T by a piecewise tree automorphism. Indeed, spheromorphisms are often called **almost automorphisms of** T.

Construction of Neretin's group $\mathcal{N}_d$:   Here we follow the construction used in [12, Chapter 8].

Let T be a d-regular tree. For every finite d-regular subtree $F \subseteq T$, denote by $T \setminus F$ the (no longer connected) graph obtained by removing from T all the edges and internal vertices of F. The connected components of $T \setminus F$ are rooted d-regular trees whose roots are the leaves of F. In particular, $T \setminus F$ is a rooted d-regular forest such that $\partial(T \setminus F)^7 = \partial T$.

$$T_2 \qquad\qquad T_2 \setminus F$$
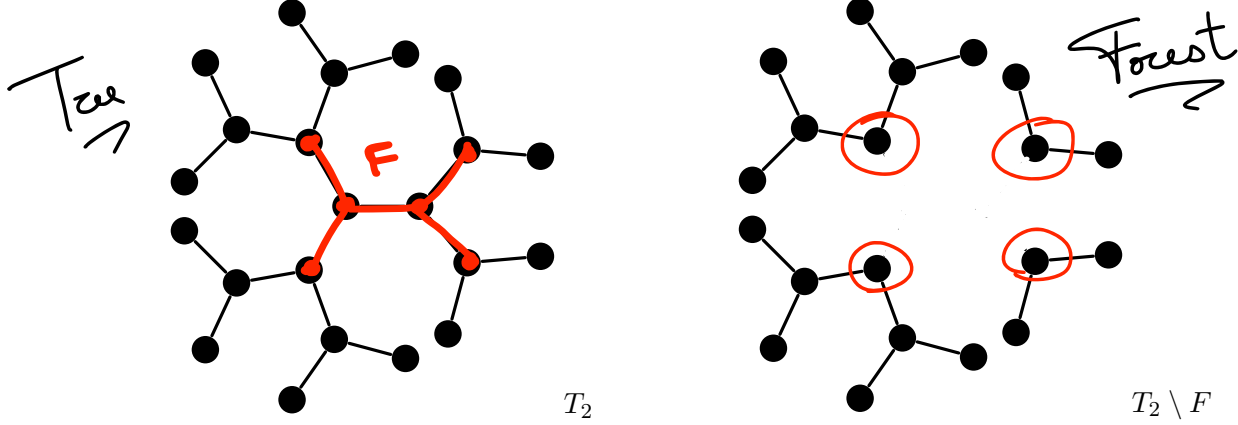
Let $F_1, F_2 \subseteq T$ be two finite d-regular subtrees. Each forest isomorphism $\phi \colon T \setminus F_1 \to T \setminus F_2$ induces a homeomorphism ϕ_* of ∂T, called *spheromorphism* of T. Clearly, different choices of subtrees F_1, F_2 can induce the same spheromorphism. Therefore, ϕ is just a representative of ϕ_*. This is important because, for each pair of spheromorphisms ϕ_* and ψ_* we may find representatives which are composable: we can always enlarge the finite trees that represent the speromorphisms in order to make the isomorphisms of the forests composable. This procedure shows that there exists also the spheromorphism $\psi_* \circ \phi_*$. Hence, the set $\mathcal{N}_d$ of all spheromorphisms of a d-regular tree is a group, which is called **Neretin's group**.

Fact 3.3.4. *The set of all spheromorphisms is a subgroup of the homeomorphism group of the boundary ∂T. Moreover, every automorphism ϕ of T induces an isomorphism $T \setminus F \to T \setminus \phi(F)$ of forests which is independent on the finite d-regular subtree F. Thus, $\mathrm{Aut}(T)$ can be regarded as a subgroup of $\mathcal{N}_d$.*

Remark 3.3.5. Let ϕ_* be a spheromorphism of T_d and suppose that ϕ_* admits a representative $\phi \colon T_d \setminus F \to T_d \setminus F$ that leaves the trees of $T_d \setminus F$ in place. Then ϕ can be extended to an automorphism of the tree T_d which belongs to the pointwise stabiliser of the finite tree F. As a consequence, ϕ_* belongs to the image of $\mathrm{Aut}(T_d)$ in $\mathcal{N}_d$.

In order to topologize the group $\mathcal{N}_d$, the first attempt is to endow $\mathrm{Homeo}(\partial T)$ with the compact-open topology and then give $\mathcal{N}_d$ the subspace topology. Unfortunately, the resulting topological group is not locally compact: $\mathcal{N}_d$ is not closed in $\mathrm{Homeo}(\partial T)$ with respect to the compact-open topology (see Proposition 2.3.2).

Instead of restricting a topology from a larger topological group, we could try to "copy and paste around" a topology of subgroup which is a topological group.

Lemma 3.3.6 ([12, Lemma 8.4, pg. 137]). *Suppose that an abstract group G contains a topological group H as a subgroup. If, for all open subsets $U \subseteq H$ and*

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[7]Since a forest is a disjoint union of trees the notion of bouandary can be easily extended to the context of forests

$g, g' \in G$, the intersection $gUg' \cap H$ is open in H, then G admits a unique group topology such that the inclusion $H \to G$ is continuous and open.

Theorem 3.3.7. *Neretin group $\mathcal{N}_d$ admits a unique group topology such that the natural embedding $\mathrm{Aut}(T_d) \to \mathcal{N}_d$ is continuous and open. With this topology, $\mathcal{N}_d$ is a TDLC-group.*

Proof. By the lemma above, one needs to show that for every open $U \subseteq \mathrm{Aut}(T_d)$ and all $\phi_*, \psi_* \in \mathcal{N}_d$, the subset $\mathrm{Aut}(T_d) \cap \phi_* U \psi_*$ is open in $\mathrm{Aut}(T_d)$. A sub-basis of the compact-open topology on $\mathrm{Aut}(T_d)$ is given by vertex stabilisers and therefore one has to show the claim only for the sets in the sub-basis.

To this end, let v be a vertex of Γ. Let S be a sufficiently large sphere centred in v such that the spheromorphisms ϕ_* and ψ_* admit representatives $\phi \colon T_d \backslash F_1 \to T_d \backslash S$ and $\psi \colon T_d \setminus S \to T_d \setminus F_2$. Denote by $G_{(S)}$ the pointwise stabilizer of S. Since $G_{(S)}$ is an open subgroup of $\mathrm{Aut}(T_d)$ contained in $G_{(v)}$, there exist finitely many elements $g_1, \dots, g_n \in G_{(v)}$ such that $G_{(v)} = \bigsqcup_{i=1}^n g_i G_{(S)}$. Therefore,

$$\psi_* G_{(v)} \phi_* = \bigsqcup_{i=1}^n \psi_* g_i G_{(S)} \phi_* = \bigsqcup_{i=1}^n \psi_* g_i \phi_* (\phi_*^{-1} G_{(S)} \phi_*).$$

By Remark 3.3.5, $\phi_*^{-1} G_{(S)} \phi_*$ coincides with the pointwise stabiliser of the finite tree F_1 and, therefore, it is contained in the image of $\mathrm{Aut}(T_d)$ in $\mathcal{N}_d$. It then follows that $\psi_* G_{(v)} \phi_* \cap \mathrm{Aut}(T_d)$ is open in $\mathrm{Aut}(T_d)$ (it is union of translates of the open subgroup $G_{(F_1)}$). $\qquad\square$

- (Topological semi-direct products) Let G and H be topological groups. Suppose that G acts on H continuously, i.e., there is a group action of G on H such that the map $\alpha \colon G \times H \to H$ defined by the action is continuous. The **topological semi-direct product** is the abstract semi-direct product $H \rtimes G$ endowed with the product topology.

  Proposition 3.3.8. *Let G and H be TDLC-groups such that G acts continuously on H. The topological semi-direct product $H \rtimes G$ is a TDLC-group.*

- (Powers of topological groups) Let G be a locally compact group. For I infinite, the power G^I fails to be locally compact as soon as G is non-compact. To deal with this issue, given a compact open subgroup U in G, one defines the **semi-restricted power**

$$G^{I,U} = \{(g_i)_{i \in I} \in G^I \mid g_i \in U \text{ for all but finitely many } i \in I\}.$$

There is a unique group topology on $G^{I,U}$ that makes the embedding of U^I a topological isomorphism onto an open subgroup. Moreover, such a group topology is locally compact; see [16, Proposition 2.4]. TDLC-groups are full of compact open subgroups, and therefore they are amenable to this construction; in particular, semi-restricted powers of TDLC-groups are again TDLC-groups.

Remark 3.3.9. The semi-restricted power is a special case of the restricted product, which is also available in the context of TDLC-groups. Notice that in literature the term "(semi)restricted" is often replaced by the term "local".

4 Finiteness properties for TDLC-groups

4.1 Compact generation and presentation

There are several finiteness conditions that a TDLC-group can satisfy. At this early stage, we are interested in two finiteness conditions that naturally generalise the notions of finite generation and finite presentation in the context of locally compact groups.

Definition 4.1.1. A locally compact group G is said to be

(CG) **compactly generated** if it has a compact generating set S.

(CP) **compactly presented** if it has a presentation $\langle S \mid R \rangle$ as an abstract group with the generating set S compact in G and the relators in R of bounded length.

It is straightforward that being compactly presented implies being compactly generated. The converse is not true (see [17, Example 8.A.28]). The notion of compact presentation was introduced in 1964 by Kneser but it has received little attention (compared to compact generation) until recently. For example, in [15, § 5.8] it is provided an equivalent definition of compact presentation based on van Dantzig's theorem and the notion of fundamental group of finite graphs.

Remark 4.1.1. The finiteness conditions above are both equivalent to a metric condition as shown in [17].

Example 4.1.1. 1. Every profinite group is trivially compactly generated and compactly presented.

2. Every compactly generated abelian TDLC-group is topologically isomorphic to $\mathbb{Z}^n \times K$, where $n \in \mathbb{N}$ and K is a compact abelian group; see [18, Theorem 12.5.5]. In particular, it is compactly presented.

3. The field of p-adic numbers $\mathbb{Q}_p$ is not a compactly generated because it is the ascending union of nested compact open subgroups, i.e., $\mathbb{Q}_p = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p$.

4. The automorphism group of a d-regular tree is compactly generated; see Corollary 4.2.2.

5. The special linear group $\mathrm{SL}_2(\mathbb{Q}_p)$ is compactly generated. Indeed, by Ihara's Theorem, we can decompose $\mathrm{SL}_2\mathbb{Q}_p$ into the amalgamated free product

$$\mathrm{SL}_2\mathbb{Q}_p \cong \mathrm{SL}_2\mathbb{Z}_p *_I \mathrm{SL}_2\mathbb{Z}_p,$$

where I is a compact open subgroup.

6. Neretin's groups $\mathcal{N}_d$ are compactly presented (see [10, 21]).

Fact 4.1.2. *Every locally compact group is directed union of compactly generated open subgroups.*

Proof. It suffices to notice that, for any $g \in G$ and any compact open neighbourhood V_g of g, the subgroup $\bigcup_{n>0}(V_g \cup V_g^{-1})^n$ is open in G and compactly generated. $\square$

4.2 The Cayley-Abels graphs

Recall that a **group G acts on a graph** Γ if the set of vertices $V\Gamma$ is a G-set and, for every $g \in G$, $\{gv, gw\} \in E\Gamma$ if and only if $\{v, w\} \in E\Gamma$. The group G acts **vertex-transitively** on Γ if $V\Gamma$ is a transitive G-set. Given a vertex $v \in V\Gamma$, the set $G_{(v)} = \{g \in G \mid gv = v\}$ is the **vertex stabiliser** of v in G.

Definition 4.2.1. For a TDLC-group G, a locally finite connected graph Γ on which G acts vertex-transitively with compact open vertex stabilizers is called a **Cayley-Abels graph of G.**

Exercise 4.2.1. Let Γ be a Cayley-Abels graph of G. Let $V\Gamma$ be endowed with the discrete topology. Prove that the map $G \times V\Gamma \to V\Gamma$ is continuous; that is, a TDLC-group G always acts continuously on its Cayley-Abels graphs. Moreover, prove that, as far as G is non-discrete, the action of G on Γ is never free[8].

Proposition 4.2.1. *Let G be a TDLC-group. If G has a Cayley-Abels graph, then G is compactly generated.*

Proof. Let Γ be a Cayley-Abels graph of G and $v \in V\Gamma$. Since Γ is locally finite, we can list all neighbours of v by $v_1, \ldots, v_n$. Since G acts on Γ vertex-transitively, there are $g_1, \ldots, g_n \in G$ such that $v_i = g_i v$ for all $i = 1, \ldots, n$. We claim that, for every $g \in G$, there is $h \in \langle g_1, \ldots, g_n \rangle$ such that $gv = hv$. This implies that $h^{-1}g \in G_{(v)}$, which is compact and open by hypothesis. In other words, $G = \langle G_{(v)}, g_1, \ldots, g_n \rangle$ and this concludes the proof.

Let us prove the claim: for every $g \in G$ there is a path in Γ connecting v and gv because Γ is connected. We proceed by induction on the length of the path. For $k = 0$ there is nothing to prove. Suppose the hypothesis for k and prove it for $k + 1$. For Γ is vertex-transitive, a path of length $k + 1$ connecting v and gv is given by a $(k + 1)$-tuple $(v, \gamma_1 v, \ldots, \gamma_k v, gv)$ with $\gamma_1, \ldots, \gamma_k \in G$. By the inductive hypothesis, there is $h \in \langle g_1, \ldots, g_n \rangle$ such that $\gamma_k v = hv$. Therefore, the group element h^{-1} maps the edge $\{\gamma_k v, gv\}$ to the edge $\{v, h^{-1}gv\}$. In other words, $h^{-1}gv$ is adjacent to v, i.e., $h^{-1}gv = g_j v$ for some $j \in \{1, \ldots, n\}$ and the claim holds. $\square$

Corollary 4.2.2. *For every d-regular tree $\mathcal{T}_d$, $\mathrm{Aut}(\mathcal{T}_d)$ is compactly generated.*

Proof. Let T_d be a regular tree. Then $G = \mathrm{Aut}(T_d)$ with the compact-open topology is a TDLC-group for which T_d is a Cayley-Abels graph. $\square$

Now, we do the converse: we start with a compactly generated TDLC-group G and we construct a (family of) Cayley-Abels graph(s) of G. In particular, we show that, for every compact open subgroup U of G, there is a Cayley-Abels graph admitting U as stabiliser of some vertex.

Let U be a compact open subgroup of G. For every symmetric subset $S = S^{-1} \subseteq G \setminus U$ define the graph $\Gamma_{U,S}^G$ such that

$$V\Gamma_{U,S}^G = \{gU \mid g \in G\}, \quad \text{and} \quad E\Gamma_{U,S}^G = \{\{gU, gsU\} \mid g \in G, s \in S\}.$$

Clearly, G acts transitively on the set of vertices of $\Gamma_{U,S}^G$.

---

[8]A groups acts *freely* on a set if point stabilisers are trivial.

Proposition 4.2.3. *With the above notation, the following hold:*

1. *if S is a finite set, then $\Gamma_{U,S}^G$ is locally finite;*

2. *Γ is connected if, and only if, $G = \langle S, O \rangle$.*

In particular, if G is compactly generated, then there exists a Cayley-Abels graph of G and there is a vertex v such that $G_{(v)}$ is compact and open.

Proof. 1. Since the action is vertex-transitive, it suffices to prove that the vertex U has finitely many neighbours if S is finite. Since $U = xU$, for every $x \in U$, one has that the set $\{xU, xsU\}$ is an edge for every $s \in S$, and therefore the set of all neighbours of U coincides with $\{xsU \mid x \in U, s \in S\}$. In order to determine the cardinality of such a set, one must count the number of left cosets of U that are necessary to cover each double coset UsU. But this number is finite because U is open and the double coset UsU is compact (since U is compact). Therefore, if S is finite, the graph $\Gamma_{U,S}^G$ is locally finite.

2. Suppose the graph $\Gamma_{U,S}^G$ is connected. For every $g \in G$, there is a path $p = (v_0, \ldots, v_n)$ connecting the vertex U to the vertex gU. In particular, the vertices of the path can be written as $v_0 = U, v_1 = u_1 s_1 U, \ldots, v_n = u_1 s_1 \cdots u_n s_n U$, where each $u_i \in U$ and each $s_j \in S$. Since $u_1 s_1 \cdots u_n s_n U = v_n = gU$, it follows that g belongs to the subgroup generated by $U \cup S$.

Conversely, suppose that $G = \langle U, S \rangle$. Let gU and hU be any two vertices of the graph. The group element $g^{-1}h$ can be then written as a word $u_1 s_1 \cdots u_n s_n u_{n+1}$ such that each $u_i \in U$ and each $s_j \in S$. Thus, the sequence of vertices

$$(gU, gu_1 s_1 U, gu_1 s_1 u_2 s_2 U, \cdots, gu_1 s_1 \cdots u_n s_n u_{n+1} U = hU)$$

is a path in Γ connecting gU and hU. $\qquad\square$

Remark 4.2.4. The first (technical) construction of the Cayley-Abels graph of a TDLC-group G is due to Abels [1]. A less technical approach to Cayley-Abels graphs was provided in [20], where the Cayley-Abels graphs were at the time called *rough Cayley graphs.* Today the widely accepted nomenclature is "Cayley-Abels graph".

Proposition 4.2.5. *Let G be a TDLC-group and Γ a Cayley-Abels graph of G. If $\mathrm{Aut}(\Gamma)$ is given the compact-open topology, then the group homomorphism $\psi \colon G \to \mathrm{Aut}(\Gamma)$ defined by the action of G on Γ is continuous, the kernel of ψ is compact and the image of ψ is closed.*

Proof. A basis of the compact-open topology of $\mathrm{Aut}(\Gamma)$ is given by the family $\mathcal{S}$ of pointwise stabilisers in $\mathrm{Aut}(\Gamma)$ of finite sets of vertices. The pre-image of each of these sets is the intersection of finitely many open subgroups of G, that are the stabilisers in G of the single vertices in the finite set. Since the stabiliser $G_{(v)}$ is open for every $v \in V\Gamma$, the pre-image of each set in $\mathcal{S}$ is open, i.e., ψ is continuous.

The kernel of ψ is then closed in G (because it is pre-image of the closed set $\{1\}$). Since $\ker(\psi) \subseteq G_{(v)}$, $\forall v \in V\Gamma$, we deduce that it is compact because $G_{(v)}$ is so.

Finally, in order to prove that the image $\psi(G)$ is closed, it suffices to prove that $\psi(G) \cap H$ is closed for every $H \in \mathcal{S}$. Since every element H of $\mathcal{S}$ is the intersection of finitely many vertex stabilisers in $\mathrm{Aut}(\Gamma)$, we only need to prove that $\psi(G) \cap H$ is closed whenever H is the stabiliser in $\mathrm{Aut}(\Gamma)$ of a single vertex v. In such a case, $\psi(G) \cap H$ coincides with $\psi(G_{(v)})$ which is compact because ψ is continuous and $G_{(v)}$ is compact. In particular, $\psi(G) \cap H$ is closed. $\qquad\square$

The representation $\psi\colon G \to \mathrm{Aut}(\Gamma)$ is called the **Cayley-Abels representation of G.** Hence, there is an equivalence between compactly generated TDLC-groups and closed subgroups of automorphism groups of connected locally finite graphs.

Remark 4.2.6. Actually, one could say even more on the image $\psi(G)$ of the Cayley-Abels representation: $\psi(G)$ is a cocompact subgroup of $\mathrm{Aut}(\Gamma)$, see [35, Lemma 3.12].

Exercise 4.2.2. Suppose G is a compactly generated TDLC-group and Γ is a Cayley-Abels graph of G. Show that a closed subgroup $K \leq G$ is compact if and only if, for all $v \in V\Gamma$, the set $Kv := \{kv \mid k \in K\}$ is finite.

The geometric structure of compactly generated TDLC-groups:

Definition 4.2.2 (Gromov). Two metric spaces (X, d_X) and (Y, d_Y) are said to be **quasi-isometric** if there is a map $\phi\colon X \to Y$ and constants $a \geq 1$ and $b \geq 0$ such that, for all $x_1, x_2 \in X$, one has

$$\frac{1}{a} d_X(x_1, x_2) - \frac{b}{a} \leq d_Y(\phi(x_1), \phi(x_2)) \leq a d_X(x_1, x_2) + ab,$$

and, for all $y \in Y$,
$$d_Y(y, \phi(X)) \leq b.$$

Such a map ϕ is called a **quasi-isometry**. Moreover, being quasi-isometric is an equivalence relation on the class of metric spaces.

Every connected graph Γ can be regarded as a metric space: two vertices v and w are points at distance 1 if, and only if, there is an edge connecting v and w. In other words, we endow the set $V\Gamma$ with the path-length metric $d_\Gamma\colon V\Gamma \times V\Gamma \to \mathbb{N}$ defined as follows:

$$d_\Gamma(v, w) = \min\{\text{length of } p \mid p \text{ path connecting } v \text{ and } w\}, \quad v, w \in V\Gamma.$$

Theorem 4.2.7 ([1],[20, Theorem 2]). *Let G be a compactly generated totally disconnected locally compact group. Any two Cayley-Abels graphs of G are quasi-isometric.*

This quasi-isometric invariance of Cayley-Abels graphs allows us to define *geometric invariants* of a compactly generated TDLC-group G by considering quasi-isometric invariants of a Cayley-Abels graph associated to G. For example, one can give the following definitions (that are long-known for discrete groups):

(Hyp) A compactly generated TDLC-group G is said to be **hyperbolic** if some (and hence any) Cayley-Abels graph of G is hyperbolic.

(Ends) The **number of ends** of a compactly generated TDLC-group is defined to be the number of ends of some (and hence any) Cayley-Abels graph of G.

The class of hyperbolic TDLC-groups is a rich source of compactly presented TDLC-groups; see [17]. Indeed, geometric invariants often reflect structural properties of the group (see [1, Struktursatz 5.7 and Korollar 5.8], [20, Theorem 1.3] and [13] for Stallings' decomposition theorem in the context of TDLC-groups).

4.3 Finiteness conditions in higher dimension

For discrete groups, finite generation is the first of two sequences of increasingly stronger properties: the homological finiteness conditions, the **types** $(\mathbf{FP}_n)_{n \in \mathbb{N}}$, and the homotopical finiteness conditions, the **types** $(\mathbf{F}_n)_{\in \mathbb{N}}$. We recall here the definitions but the reader is referred to [9, Chapther VIII] for details:

$\underline{(\mathrm{FP}_n \text{ over } R)}$ A discrete group G is **of type $\mathbf{FP}_n$ over** R $(0 \leq n < \infty)$ if there is a projective resolution $\{P_i\}$ of the trivial module R over $R[G]$ such that each projective $R[G]$-module P_i is finitely generated for $i \leq n$. (If $R = \mathbb{Z}$, the reference to the ring in the notation usually drops)

$\underline{(\mathrm{F}_n)}$ A discrete group G is **of type $\mathbf{F}_n$** $(0 \leq n < \infty)$ if there exists a contractible G-CW-complex with trivial cell stabilisers and such that G acts on the n-skeleton with finitely many orbits.

The conditions above are known to satisfy the following:

- A discrete group G is of type F_1 over R if, and only if, it is finitely generated if, and only if, it is of type FP_1 over R.

- A discrete group G is of type F_2 if and only if it is finitely presented but being of type FP_2 over R is strictly weaker than finite presentation; see [6].

- For each $n \geq 1$, a discrete group of type F_n is of type FP_n over R but the converse is not true (the converse becomes true if the group is suppose to be finitely presented).

- Being of type FP_n over R (resp. of type F_n) is a geometric property of the group.

A first attempt at generalising this to the realm of locally compact groups is due to Abels and Tiemeyer [3]. They introduced **compactness properties** for locally compact groups - we avoid here the too much technical definitions - that are two sequences $(\mathbf{CP}_n)_{n \geq 0}$ and $(\mathbf{C}_n)_{n \geq 0}$ of increasingly stronger properties satisfying:

- for all $n \geq 1$, a discrete group is of type (CP_n) (resp. C_n) if and only if it is of type FP_n (resp. F_n);

- a locally compact group is of type C_1 if, and only if, it is compactly generated if, and only if, it is of type CP_1;

- a locally compact group is of type C_2 if and only if it is compactly presented but being of type CP_2 is strictly weaker than compact presentation;

- for each $n \geq 1$, a locally compact group of type C_n is also of type CP_n but the converse is not true;

- being of type CP_n (resp. C_n) is invariant "up to compactness": the compactness properties remain unchanged by passing to a cocompact[9] subgroup or by taking the quotient by a compact normal subgroup.

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[9]A closed subgroup H is *cocompact* if the quotient G/H, equivalently $H\backslash G$, equipped with the quotient topology is compact.

For the (more amenable) class of TDLC-groups, a different approach to finiteness conditions was recently introduced in [13].

$\underline{(\text{FP}_n)}$ A TDLC-group G is **of type FP_n** $(0 \le n < \infty)$ if there is a (projective) resolution $\{P_i\}$ of the trivial module $\mathbb{Q}$ over $\mathbb{Q}[G]$ such that each P_i is a permutation $\mathbb{Q}[G]$-module[10] with compact open stabilisers and finitely many orbits.

$\underline{(\text{F}_n)}$ A TDLC-group G is **of type** F_n $(0 \le n < \infty)$ if there exists a contractible G-CW-complex X with compact open cell stabilisers such that G acts on the n-skeleton of X with finitely many orbits.

These finiteness conditions for TDLC-groups satisfy the following properties:

- for all $n \ge 1$, a discrete group is of type FP_n in the category of TDLC-groups (resp. F_n) if and only if it is of type FP_n over $\mathbb{Q}$ (resp. F_n) in the classical notion;

- a TDLC-group is of type F_1 if, and only if, it is compactly generated if, and only if, it is of type FP_1 over $\mathbb{Q}$;

- a TDLC-group is of type F_2 if and only if it is compactly presented but being of type FP_2 is strictly weaker than compact presentation;

- for each $n \ge 1$, a TDLC-group of type F_n is also of type FP_n but the converse is not true (but the converse becomes true if the group is supposed to be compactly presented);

- being of type FP_n (resp. F_n) is a geometric property (see [14, Theorem 5.5]).

Remark 4.3.1. All the finiteness conditions above can be extended to the infinite degree: for example, a TDLC-group is said to be of type F_∞ if it is of type F_n for all n. Sauer and Thumann [ST15] showed that Neretin's groups are of type F_∞.

Remark 4.3.2. In [14], the authors showed that it is also possible to introduce two more sequences (**types KP_n**)$_{n \ge 0}$ and (**types K_n**)$_{n \ge 0}$ of increasingly stronger compactness properties.

Open Problem. Despite the abundance of finiteness properties that are available in the TDLC context, the theory of finiteness conditions for TDLC-groups is still much less developed than the one for discrete groups. Moreover, very little is known about the relation (if there exists one) among the properties of different sequences CP_n, FP_n and KP_n (resp, C_n, F_n and K_n).

Open Problem. It would be relevant to find an example of a TDLC-group of type FP_2 which is not compactly presented and it is "essentially" a TDLC-group (for example, it is not quasi-isometric to a discrete group). Unfortunately, the strategy developed in [6] does not seem to have a TDLC-analogue.

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[10]A *permutation $\mathbb{Q}[G]$-module* is a module $\mathbb{Q}[\Omega]$ freely $\mathbb{Q}$-generated by a G-set Ω.

5 Willis' theory of TDLC-groups

5.1 Scale function and tidy subgroups

Connected groups can be approximated by Lie groups and then Lie group techniques may be used to analyse the structure of connected groups and their automorphims. A canonical form for automorphisms of totally disconnected locally compact groups has been developed in [36, 37].

Let $\alpha \in \text{Aut}(G)$ and U be a compact open subgroup of G. Then

$$[\alpha(U) : U \cap \alpha(U)] < \infty$$

because $U \cap \alpha(U)$ is open whereas U is compact. Define the **scale of** α to be the value

$$s(\alpha) = \inf\{[\alpha(U) : U \cap \alpha(U)] \mid U \text{ compact open subgroup of } G\}. \qquad (5.1)$$

A subgroup U is **tidy** for α if the infimum is attained at U. Tidy subgroups for α always exists because actually $s(\alpha)$ is the minimum of a set of positive integers. Every tidy subgroup U can be expressed as the product of a subgroup where α expands and a subgroup where α shrinks:

$$\text{if } U_\pm := \bigcap_{k>0} \alpha^{\pm k}(U), \text{ then } U = U_+ U_-.$$

It follows from the definitions that U_+ and U_- are closed subgroups, and that $\alpha(U_+) \geq U_+$ and, similarly, $\alpha(U_-) \leq U_-$. Moreover, it can be shown that $s(\alpha)$ represents the factor by which α expands U_+, i.e., one has

$$s(\alpha) = [\alpha(U_+) : U_+].$$

A striking result in the theory of the scale is the existence of an algorithm, known as **tidying procedure**, for producing a tidy subgroup starting from an arbitrary compact open subgroup.

Definition 5.1.1. The **scale function** of a TDLC-group G is defined to be the map

$$s \colon G \to \mathbb{Z}^+, \quad x \mapsto s(\alpha_x), \quad \forall x \in G,$$

where α_x denotes the inner automorphism $y \mapsto xyx^{-1}$.

The scale function s is known to satisfy the following properties:

(s1) s is continuous if $\mathbb{Z}^+$ carries the discrete topology;

(s2) $s(x) = 1 = s(x^{-1})$ if and only if there is a compact open subgroup U such that $xUx^{-1} = U$;

(s3) $s(x^n) = s(x)^n$, for every $x \in G$ and $n \geq 0$;

(s4) $\Delta(x) = s(x)/s(x^{-1})$, where $\Delta \colon G \to \mathbb{Q}^+$ denotes the modular function on G;

(s5) $s(\alpha(x)) = s(x)$ for every $x \in G$ and $\alpha \in \text{Aut}(G)$.

The scale function encodes structural information of the group G. For a summary on the scale function (which in particular highlights the fact that tidy subgroups for automorphisms of TDLC-groups are analogues of the Jordan canonical form of linear transformations) the reader is referred to [38] and references there.

Remark 5.1.1. In last years, Willis' theory has been investigated from different points of views bringing new approaches to the scale function of a TDLC-group. For example, in [22], the author offers an interpretation of the fundamental ingredients of Willis' theory (that are tidy subgroups and scale function) in the new setting of permutation group theory. Another example is given by the work initiated in [5], where Willis' topological dynamics of automorphisms were reformulated in the long-known theory of topological entropy.

5.2   Comments on simple TDLC-groups

Simple groups play an important role in group theory as the "indecomposable factors". Therefore, it is significant that several types of simple groups have been completely classified; for instance, the simple finite groups and the simple connected Lie groups. Long-known classes of simple TDLC-groups are the class of simple Lie groups over local fields [8] and the class of automorphism groups of trees [31].

In the realm of simple TDLC-groups, it is necessary a distinction between topological simplicity (i.e., every closed normal subgroup is trivial) and abstract simplicity (i.e., the underlying abstract group is simple). Examples show that a topologically simple TDLC-group can fail to be abstractly simple, see [39], but no example is known of topologically simple compactly generated TDLC-group that fail to be abstractly simple. Among compactly generated TDLC-groups, Simon M. Smith [28] has shown that there are $2^{\aleph_0}$ non-isomorphic compactly generated abstractly simple TDLC-groups (only countably many such groups were known before). Nowadays, a classification of compactly generated topologically simple TDLC-groups is probably the best which can be hoped for.

Scale function: The theory of the scale produces invariants that could be important tools in the classification. For example,

- The set of values of the scale function: if G is compactly generated, the range of the scale function has only finitely many prime divisors, and so this set could distinguish between compactly generated simple TDLC-groups.

  Question. Let G be a compactly generated topologically simple TDLC-group. Is the scale $s\colon G \to \mathbb{Z}^+$ non-trivial?

- The flat-rank: a notion of rank for totally disconnected locally compact groups which is defined thanks to a natural distance on the space of compact open subgroups.

Local-to-Global principle: It relates the global properties of a compactly generated simple TDLC-group with the structural properties of its compact open subgroups. This approach was initiated

- in [39], where the author shows that, for compactly generated TDLC-groups, the simplicity imposes restrictions on the local structure of the group: if G is compactly generated and topologically simple, then no compact open subgroup of G is solvable;

- in [4], where the authors investigate which profinite groups embed as a compact open subgroup in a compactly generated topologically simple TDLC-group.

Decomposition theory: it includes methods for "breaking" a given TDLC-group into smaller, and often simple, pieces; see [11]. This approach has been successful for several classes of groups; for example, finite groups, profinite groups and algebraic groups. Therefore, one would hope to obtain analogous results for TDLC-groups.

General decomposition results [26, 27] on compactly generated second countable TDLC-groups have been obtained by using the theory of *elementary groups*: TDLC-groups that are exclusively built out of discrete and compact pieces [34].

Geometrisation: The existence of Cayley-Abels graphs allow the study of compactly generated TDLC-groups as geometric objects. Moreover, one can consider other types of geometric objects, e.g., *buildings*, that essentially determine the group that they are associated to. For example, semi-simple Lie groups over a local field act on *affine buildings*, and also on related *spherical buildings*. For groups of rank 1, e.g., $SL_2(\mathbb{Q}_p)$, the affine building is a tree and the spherical building is its boundary. Also Kac-Moody groups act on buildings and on the boundary of the building. In particular, closed subgroups of automorphism groups of buildings are a rich source of examples of TDLC-groups.

Since profinite groups are trivial as geometric objects, the geometric behaviour of compactly generated TDLC-groups is related to the geometric behaviour of discrete groups. There is an ongoing program of studying geometric properties of TDLC-groups by analogy with discrete groups. The aim is to understand to what extent long-known results on discrete groups find an analogue in the framework of TDLC-groups; see for example [20, 13, 14, 15, 3, 17].

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