GROUP KNOWLEDGE AND MATHEMATICAL COLLABORATION

Fenner Stanley Tanswell

Joshua Habgood-Coote

Big Proof, Edinburgh 29/05/2019

Plan

- Introduction
- The Four Colour Theorem and a Reckoning for Proof
- The Classification Theorem
- New Questions
- A Social Epistemology Approach

INTRODUCTION

Social Epistemology of Mathematics

- Jeremy yesterday set out a proposal for philosophers to consider issues of reliability, knowledge and understanding in mathematics, especially in relation to “big proofs”.
- So that’s what we’ll be doing!
- We are especially interested in how social features (such as those in Michael’s talk) interact with the mathematics itself and the epistemology of mathematics.
- We think there are useful tools from social epistemology which should be applied to think about mathematical collaboration and communities.

What is social epistemology?

- "Social epistemology seeks to [investigate] the epistemic effects of social interactions and social systems." (SEP)
- What does it mean for a group to believe something? To be justified in their belief? To know something?
- E.g. the committee believes candidate X is the best person for the job.
- E.g. between us we know the way to Amarillo.
- E.g. the Los Alamos team know how to build an atomic bomb. (Mackenzie & Spinardi 1995)

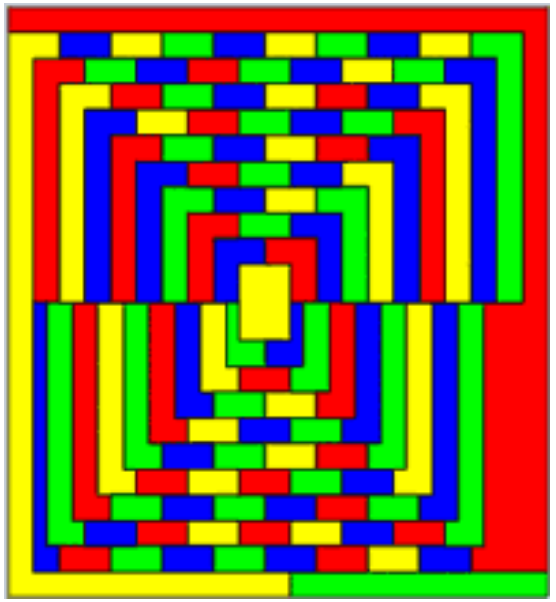
A reckoning for proofs

- The original computer-aided proof of the Four Colour Theorem led to a major reckoning for the concept of proof, as in Tymozcko (1979).
- However, features of large-scale collaborations, like the Classification of Finite Simple Groups, have the potential to similarly challenge our conceptions of what proofs are.
- Alma Steingart's 2012 paper on the Classification Theorem documents many shocking features of the proofs, and we believe it requires a proper philosophical response and reflection.
- In this paper, we will present some of these features of collaborative mathematics which challenge our conceptions of proof and mathematical knowledge, and consider how social epistemology can help.

THE FOUR COLOUR THEOREM

Four Colors Suffice

- (Four Colour Theorem) Any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color.



History

- Established by Appel, Haken & Koch (Appel & Haken 1977; Appel, Haken & Koch 1977) with the crucial aid of a computer to check the reducibility of each member of the unavoidable set, which forms the crux of the proof.
- This proof is also philosophically infamous precisely because of the computational component, which has generated a literature of discussions as to whether this still counts as a proof in the traditional sense, whether mathematical knowledge can be gained empirically and concerning the role of computers in mathematics more generally (e.g. Tymoczko 1979; Detlefsen 2008; McEvoy 2008; etc.).
- A history of the proof's development can also be found in (MacKenzie 2001, ch. 4).

A reckoning for proof

"I will suggest, however, that, if we accept the 4CT as a theorem, we are committed to changing the sense of 'theorem', or, more to the point, to changing the sense of the underlying concept of `proof'. " (Tymoczko 1979)

"such use of computers in mathematics, as in the 4CT, introduces empirical experiments into mathematics. Whether or not we choose to regard the 4CT as proved, we must admit that the current proof is no traditional proof, no a priori deduction of a statement from premises." (Tymoczko 1979)

Surveyability

- One necessary feature of traditional proofs for Tymoczko is *surveyability.*
- A proof is surveyable if it can be checked, reviewed, comprehended and verified by a rational agent.

"The proof relates the mathematical known to the mathematical knower, and the surveyability of the proof enables it to be comprehended by the pure power of the intellect-surveyed by the mind's eye, as it were. Because of surveyability, mathematical theorems are credited by some philosophers with a kind of certainty unobtainable in the other sciences. Mathematical theorems are known a priori." (Tymoczko 1979)

Trust and Testimony

- The use of a computer is also at first glance oracular: telling you something is true without being able to tell you *why.*
- Such an appeal is more testimonial than deductive, so you have to assess how much to trust the source.
- Questions about reliability and how to assess it are central to what has been discusses at Big Proof so far, and there are ways to increase our trust in the computational component.

Depending on folk theorems

(3.3) *Let K be a configuration appearing in a triangulation T, and let S be the free completion of K. Then there is a projection ϕ of S into T such that $\phi(x) = x$ for all $x \in V(G(K)) \cup E(G(K)) \cup F(G(K))$.*

This is a "folklore" theorem, and we omit its proof, which is straightforward but lengthy. A function ϕ as in (3.3) is called a *corresponding projection.*

From Robertson, Sanders, Seymour & Thomas (1997) "The Four Colour Theorem", Journal of Combinatorial Theory 70, pp. 2-44.

See Rittberg, Tanswell & van Bendegem (2018) "Epistemic Injustice in Mathematics" *Synthese* (online first)

THE CLASSIFICATION THEOREM

History

- The Classification Theorem states that all finite simple groups are found in the taxonomy: they are either an alternating group, a cyclic group of prime order, of Lie type, or one of the 26 “sporadic” groups.
- The reason this was so challenging to prove was both in finding all of the groups in the first place, and in proving that the taxonomy was complete.

S|S|S

Social Studies of Science
42(2) 185–213

DOI: 10.1177/0306312712436547
sss.sagepub.com
SAGE

A group theory of group theory: Collaborative mathematics and the 'uninvention' of a 1000-page proof

Alma Steingart
Program in History, Anthropology, and Science, Technology, and Society, MIT, Cambridge, MA, USA

Abstract

Over a period of more than 30 years, more than 100 mathematicians worked on a project to classify mathematical objects known as finite simple groups. The Classification, when officially declared completed in 1981, ranged between 300 and 500 articles and ran somewhere between 5,000 and 10,000 journal pages. Mathematicians have hailed the project as one of the greatest mathematical achievements of the 20th century, and it surpasses, both in scale and scope, any other mathematical proof of the 20th century. The history of the Classification points to the importance of face-to-face interaction and close teaching relationships in the production and transformation of theoretical knowledge. The techniques and methods that governed much of

The Enormous Theorem

"Over a period of more than 30 years, more than 100 mathematicians worked on a project to classify mathematical objects known as finite simple groups. The Classification, when officially declared completed in 1981, ranged between 300 and 500 articles and ran somewhere between 5,000 and 10,000 journal pages." (Steingart 2012, p. 185)

"The Odd Order theorem not only introduced many of the techniques used in the Classification effort. Running to 255 pages, it also became a benchmark for the length of later Classification proofs. As the literature in finite simple group theory grew during the 1960s and 1970s, 100-page proofs became increasingly common, culminating in Aschbacher and Smith's hulking 1000-page monograph." (Steingart 2012, p. 197)

Decentralised collaboration

"The decentralized nature of the Classification effort made the close-knit network that grew around it crucial to the dissemination of knowledge, adjudication of proofs, and circulation of shared norms and values. The proof of the Classification was at once both a material artifact and a crystallization of one community's shared practices, values, histories, and expertise." (Steingart 2012, p. 187)

QUESTIONS

- How do we identify what is and isn't part of "the" proof of the CFSG?
- What is the role of the written proofs in our knowledge of the CFSG?
- What is the role of the mathematical literature in chronicling mathematical knowledge?
- Can the CFSG be know *a priori*?
- To what extent can the mathematical knowledge be obtained by any individual, given that they will be relying on testimony?
- Does mathematical knowledge require higher-order knowledge?
- To what extent does the CSFG rely on tacit and implicit knowledge of the finite simple groups community?
- Can the theorem be uninvented? (Steingart/ Mackenzie)
- Can a proof tolerate mistakes? Should we?

How do we identify what is and isn't part of the proof of the CFSG?

"Because the proof was long and scattered across multiple journals, as well as preprints, dissertations, and other **'gray literature',** very few mathematicians could claim a comprehensive grasp of it." (Steingart 2012, p. 189)

"In this case, however, the **volume and complexity of *published* work** could not be navigated by reading alone. Here, tacit knowledge and personal communication were not glossed over in publications, but instead were indispensable components of how mathematicians were able to approach, apprehend, and evaluate **an unwieldy body of literature**." (Steingart 2012, p. 198)

Hard to even be sure what counts as part of the proof at all! Formalisation helps here.

Can the CFSG be know a priori?

"In the early 1980s**, since no one individual could read and carefully check the thousands of pages that constituted the proof**, trust in the community's expertise was just as important to the status of the proof as were the numerous papers scattered across the mathematical literature." (Steingart, p. 204)

The proof is not even close to surveyable in Tymoczko's sense, simply because of the limitations on any individual. Is it therefore not a "traditional" proof either according to Tymoczko's argument.

But surveyability is also seemingly central to our a priori knowledge of the theorem. No individual has access to this.

Can the theorem be uninvented?

- Donald Mackenzie and Graham Spinardi (1995) "Tacit Knowledge, Weapons Design, and the Uninvention of Nuclear Weapons"

"The papers that constituted it could still be found scattered throughout the mathematical literature, but no one other than the dwindling community of group theorists would know how to find them or how to piece them together. This should not have been a pressing problem for the Classification, because, unlike nuclear weapons, the theorem's future use should not require the reconstruction of its proof. Nonetheless, the use of the theorem did require a faith in the proof's existence." (Steingart 2012, p. 204)

Formalisation helps with the confidence of existence, but not with the possibility of losing the implicit know-how.

Errors in proofs

"Indeed, **many of the papers in simple groups are known to contain a considerable number of 'local' errors …** . Most of these errors, when uncovered, can be fixed up 'on the spot'. But many of the arguments are ad hoc, so how can one be certain that the 'sieve' has not let slip a configuration leading to yet another simple group? **The prevailing opinion among finite group theorists is that the overall proof is accurate** and that with so many individuals working on simple groups during the past fifteen years, often from such differing perspectives, every significant configuration has loomed into view sufficiently often and could not have been overlooked." (Gorenstein, 1982: 7–8)

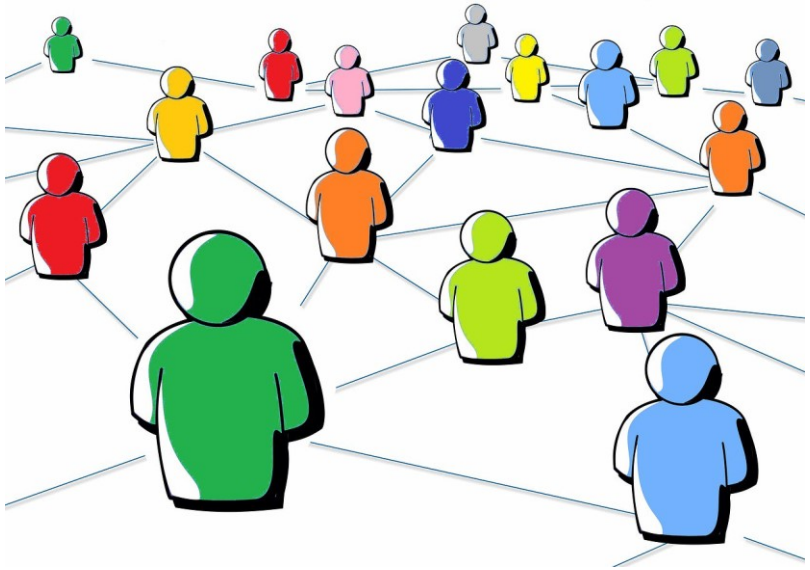
Reading and assessing proofs

- In her study of maths peer-review practices, Line Andersen (2017) describes how mathematicians (often) don't check all proofs but instead use expertise to judge where errors might appear and to check those places carefully.
- As came up in Silvia's talk earlier, this is not infallible.
- But there is an obvious tension between acknowledging the fallibility at this scale, and thinking the proof is accurate overall.
- I also suspect there is a conflation between whether the proof is correct and whether all of the groups have been found.

SOCIAL EPISTEMOLOGY

Idea

- At the individual level, we need to make judgments of trust that errors are all local and easily fixable, and that the overall proof is correct.
- However, we propose that the interesting level of analysis is not at the individual level, but at the level of the *group/community/collective*.



Collective Justification

De Ridder argues that there are situations where scientific justification can only and irreducibly be had by a collective:

"While team members will typically share their results and conclusions with each other through testimony, they typically do not share all the original justification that underpins these results and conclusions. This is not just more efficient but simply inevitable, because it is impossible for a single individual to possess the relevant expertise, knowledge, and skills needed to understand and evaluate all epistemically relevant aspects of the entire inquiry." (de Ridder 2014, p. 46)

The Classification Theorem

- We claim that a similar case can be made for the epistemic status of the proof of the Classification Theorem.
- The CFSG depends on a large number of people working together.
- It also required multiple people with different specialities and ways of thinking.

"While each theoretician came to the collaboration with his own toolbox of techniques and methods, Thompson's remark suggests that the product of their joint work was greater than the sum of their prior skills and knowledge." (Steingart 2012, p. 194)

Collective Knowledge

- The claim then is that the a priori knowledge of the proof of the CFSG is possessed at a collective level.
- It can be said that the Classification Theorem is a part of the body of mathematical knowledge.
- Any individual mathematician can have access to parts of this a priori but must rely on others to know the main theorem.
- This points to the complex relationship between the epistemology of groups and that of individuals.

Social Machines of Mathematics

- Ursula's project is on the *Social Machines of Mathematics.*
- Social machines combine groups of human agents, computers, archives and equipment into single problem-solving bodies. (Berners-Lee 1999)
- The group-level perspective extends rather easily to thinking about the epistemic status of social machines of mathematics.
- This has direct impacts on the status of computer mathematics and proof-checking.

Surveyability

- Recall Tymoczko: A proof is surveyable if it can be checked, reviewed, comprehended and verified by a rational agent.
- Social machines solve problems collectively, but it leaves open what the epistemic status of what is produced.
- Can a social machine be a rational agent in this sense?
- Would that mean that the CFSG is surveyable after all? What about the 4CT?

Conclusions

- The Classification of Finite Simple Groups presents us with fundamental questions about proofs in mathematics.
- Steingart's analysis of the history of the proof shows that there are many social and communal factors that interact with the mathematics.
- We believe that the answer to some of the questions is to see the relevant epistemic unit of analysis as the mathematical collective, rather than the sum of individuals.