SHIFTED POISSON STRUCTURES IN REPRESENTATION THEORY

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Abstract. These are notes from lectures given at the summer school on geometric representation theory and low-dimensional topology in Edinburgh in June 2019. These notes explain how shifted Poisson structures naturally appear in geometric representation theory.

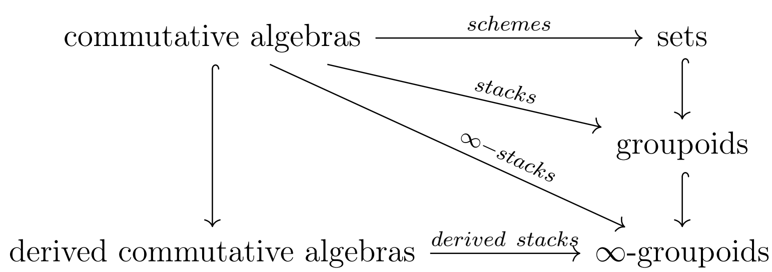
Introduction

What do we study? Let X be a scheme over a field k. Recall that it defines a functor

$$X\colon \mathrm{CAlg} \longrightarrow \mathrm{Set}$$

given by $X(R) = \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, X)$.

We have different kinds of generalized spaces:



In these lectures we will concentrate on derived stacks. The goals are twofold: introduce the language of derived algebraic (and symplectic) geometry and explain appearances of shifted Poisson/symplectic structures in geometric representation theory.

Why stacks? The basic object in geometric representation theory is an algebraic variety X with a G-action. One may reformulate this data as follows. Consider the quotient stack X/G. The projection $X \to \mathrm{pt}$ induces a map $X/G \to \mathrm{B}G = \mathrm{pt}/G$. Conversely, suppose Y is a stack with a map $Y \to \mathrm{B}G$. Then we may form $X = Y \times_{\mathrm{B}G} \mathrm{pt}$. This carries an action of the group $\Omega \mathrm{B}G = \mathrm{pt} \times_{\mathrm{B}G} \mathrm{pt} \cong G$. So, spaces with a G-action are the same as spaces over $\mathrm{B}G$. The latter perspective turns out to be useful as will be illustrated in these lectures.

Why shifted Poisson structures? Recall that a Poisson structure (a 0-shifted Poisson structure) on a variety X allows one to consider a deformation quantization of X which can either mean an associative deformation of $\mathcal{O}(X)$, the algebra of global functions on X, or a deformation of the category $\mathrm{QCoh}(X)$, the category of quasi-coherent sheaves. Note that $\mathrm{QCoh}(X)$ is a symmetric monoidal category, but we lose the monoidal structure after the deformation. If we think of $\mathrm{QCoh}(X)$ as a categorification of $\mathcal{O}(X)$, then a categorified deformation quantization should deform $\mathrm{QCoh}(X)$ as a monoidal category (but not necessarily braided). It turns out that the correct geometric data on X for a categorified deformation quantization is a 1-shifted Poisson structure.

Why derived stacks? To define symplectic and Poisson structures on a space, we first need to define a (co)tangent bundle. Recall that the tangent bundle has something to do with infinitesimal deformations of a space. For singular spaces or stacks the tangent bundle itself is not interesting, so instead we will need the tangent *complex*. The latter captures not just plain deformations, but also *derived* deformations, so as test objects we will need derived commutative algebras.

References.

- Some familiarity with ordinary symplectic and Poisson structures will be assumed, see e.g. [Can01] and [LPV13].
- A knowledge of the basics of quantum groups may be helpful to understand some applications of shifted Poisson structures in these lectures, see e.g. [CP94] and [ES02].
- Throughout the notes we will freely use the language of ∞-categories. We refer to [Lur09] and [Lur16] as foundational texts which use quasi-categories as models for ∞-categories and to [Gro10] as an introduction to the theory. This language is indispensable in dealing with higher homotopical structures and descent questions.
- We will work in the framework of derived algebraic geometry, see [TV08] and [GR17, Part I] for details and [Cal14], [Toë14] and [Toë09] for an introduction. We work over a base field k of characteristic zero.
- Excellent introductions to shifted symplectic geometry are [Cal14] and [Cal18].
- We will not define shifted Poisson structures in these notes, but we refer to [PV18], [Saf17a] and [Pri18] for detailed definitions.

1. Derived algebraic geometry

1.1. **Prestacks.** The basic object we will consider is not a set, but an ∞-groupoid (modeled by a simplicial set or a topological space) which we will call a *space*. Correspondingly, instead of categories, we will work with ∞-categories: for $x, y \in \mathcal{C}$ (some ∞-category), we have $\mathrm{Map}_{\mathcal{C}}(x, y)$ which is an ∞-groupoid. Moreover, for an ∞-category $\mathcal{C}$ we have its homotopy category $\mathrm{ho}\mathcal{C}$ such that $\mathrm{Hom}_{\mathrm{ho}\mathcal{C}}(x, y) = \pi_0\mathrm{Map}_{\mathcal{C}}(x, y)$. So, let $\mathcal{S}$ be the ∞-category of ∞-groupoids.

Let $\mathrm{CAlg}^{\leq 0}$ be the ∞-category of commutative dg algebras concentrated in non-positive cohomological degrees (connective cdgas). In this ∞-category quasi-isomorphic cdgas are considered equivalent (i.e. isomorphic).

Definition 1.1. A ***derived prestack*** is a functor $X\colon \mathrm{CAlg}^{\leq 0} \to \mathcal{S}$. The ∞-category of such functors is denoted by dPSt.

Suppose X is a scheme. A global function on X is the same as a function on each affine open subset $U \subset X$ which are compatible on overlaps. Now, since $U = \operatorname{Spec} R$ is affine, a function is just an element of R. This can be formalized in the following way. Consider the category $\mathrm{AffSch}_{/X}$ whose objects are affine schemes U mapping to X. We have a functor $\mathcal{O}\colon \mathrm{AffSch}_{/X}^{\mathrm{op}} \to \mathrm{CAlg}$ given by $U = \operatorname{Spec} R \mapsto R$. Then the algebra of global functions on X is $\lim_{\mathrm{AffSch}_{/X}} \mathcal{O}$.

This definition of global functions can be immediately generalized to derived prestacks.

Definition 1.2. Let X be a derived prestack. The ***algebra of functions on*** X is

$$\mathcal{O}(X) = \lim_{R\in \mathrm{CAlg}^{\leq 0}, f\in X(R)} R,$$

where we take the limit in the ∞-category of all commutative dg algebras.

In a similar way, we can define the category of quasi-coherent sheaves on a scheme. For an affine scheme $U = \operatorname{Spec} R$ the category of quasi-coherent sheaves if just the category of R-modules and a quasi-coherent sheaf on a scheme X is a quasi-coherent sheaf on each open affine subscheme together with an isomorphism on the double overlaps and a coherence condition on this isomorphism on triple overlaps.

If $R \in \mathrm{CAlg}^{\leq 0}$ is a connective cdga, we consider the ∞-category Mod_R of (unbounded) dg R-modules. The homotopy category of Mod_R is equivalent to the unbounded derived category of R-modules (which can be presented in terms of homotopically injective modules following Spaltenstein). For a morphism of connective cdgas $g\colon R \to S$ we get an induced functor $g^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ given by induction, i.e. $M \mapsto M \otimes_R S$.

Remark 1.3. Here and in the future the relative tensor product is taken in the ∞-category of R-modules. On the level of homotopy category this is is derived tensor product $M \otimes^{\mathbb{L}}_R S$.

Definition 1.4. Let X be a derived prestack. The ***∞-category of quasi-coherent sheaves on*** X is

$$\mathrm{QCoh}(X) = \lim_{R\in \mathrm{CAlg}^{\leq 0}, f\in X(R)} \mathrm{Mod}_R,$$

where we take the limit in the ∞-category of ∞-categories.

In other words, we can think of a quasi-coherent sheaf $\mathcal{F}$ on X as the following data:

- For any connective cdga R and an element $f \in X(R)$ we have an R-module $\mathcal{F}_f$.
- For a pair of connective cdgas R, S together with a morphism $g\colon R \to S$, two elements $f_R \in X(R), f_S \in X(S)$ such that $X(g)(f_R) \cong f_S$ we have an isomorphism of S-modules $g^*\mathcal{F}_{f_R} \cong \mathcal{F}_{f_S}$.
- Further coherences.

Remark 1.5.

(1) For a map $g\colon R \to S$ we have $g^*(R) \cong S$. Thus, we have a canonical object $\mathcal{O}_X \in \mathrm{QCoh}(X)$ which on an object $(R, f \in X(R))$ is given by $R \in \mathrm{Mod}_R$. We will think of $\mathcal{O}_X$ as the structure sheaf of X.

(2) For a map $f\colon X \to Y$ of derived prestacks by functoriality of the limit we obtain the pullback functor $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$.

(3) Mod_R is a symmetric monoidal ∞-category with monoidal structure given by the relative tensor product over R. Moreover, for a map of connective cdgas $g\colon R \to S$ the functor $g^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is symmetric monoidal. Thus, $\mathrm{QCoh}(X)$ inherits a natural symmetric monoidal structure with $\mathcal{O}_X$ the unit object and such that $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is a symmetric monoidal functor for every map $f\colon X \to Y$ of derived prestacks.

(4) The global sections functor $\Gamma(X, -)\colon \mathrm{QCoh}(X) \to \mathrm{Mod}_k$ is defined to be

$$\Gamma(X, -) = \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, -).$$

Since $\mathrm{QCoh}(X)$ is a symmetric monoidal ∞-category, we can talk about *dualizable* objects of $\mathrm{QCoh}(X)$. These are objects $V \in \mathrm{QCoh}(X)$ for which there is a dual object $V^\vee \in \mathrm{QCoh}(X)$, the evaluation and coevaluation maps $\mathrm{ev}\colon V \otimes V^\vee \to 1$ and $\mathrm{coev}\colon 1 \to V^\vee \otimes V$ which satisfy the usual identities.

Definition 1.6. Let X be a derived prestack. A ***perfect complex*** on X is a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ which is dualizable. We denote by $\mathrm{Perf}(X) \subset \mathrm{QCoh}(X)$ the full subcategory of perfect complexes.

Example 1.7. Suppose $A \in \mathrm{CAlg}^{\leq 0}$ is a connective cdga. Then

$$\operatorname{Spec} A\colon \mathrm{CAlg}^{\leq 0} \longrightarrow \mathcal{S}$$

given by $R \mapsto \mathrm{Map}_{\mathrm{CAlg}^{\leq 0}}(A, R)$ is a derived prestack. Such derived prestacks are called ***derived affine schemes***. We have $\mathcal{O}(\operatorname{Spec} A) = A$, since the corresponding indexing category in the limit (i.e. the category of $R \in \mathrm{CAlg}^{\leq 0}$ and $f \in (\operatorname{Spec} A)(R)$) has a finite object given by A and the identity map $\mathrm{id} \in (\operatorname{Spec} A)(A)$. In a similar way, $\mathrm{QCoh}(\operatorname{Spec} A) \cong \mathrm{Mod}_A$.

By the Yoneda Lemma we may identify $\mathrm{Map}_{\mathrm{dPSt}}(\operatorname{Spec} A, X) \cong X(A)$ and for a map $f\colon \operatorname{Spec} A \to X$ the functor $f^*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(\operatorname{Spec} A) \cong \mathrm{Mod}_A$ sends $\mathcal{F}$ to $f^*\mathcal{F} = \mathcal{F}_f$.

Example 1.8. Let $X\colon \mathrm{CAlg}^{\leq 0} \to \mathcal{S}$ be a derived prestack. We define its truncation $t_0(X)\colon \mathrm{CAlg} \to \mathcal{S}$ (an ∞-prestack) to be

$$t_0(X)(R) = X(R).$$

This defines a functor

$$t_0\colon \mathrm{dPSt} \longrightarrow \mathrm{PSt}$$

from derived prestacks to ∞-prestacks. For instance, $t_0 \operatorname{Spec} R = \operatorname{Spec} \mathrm{H}^0(R)$.

The functor t_0 has a fully faithful right adjoint $i\colon \mathrm{PSt} \to \mathrm{dPSt}$ which allows us to consider ∞-prestacks as derived prestacks. For instance, we may consider ordinary schemes as derived prestacks. From now on the functor i will be implicit. The image of an affine scheme $\operatorname{Spec} R$ (where R is an ungraded commutative algebra) is the derived affine scheme $\operatorname{Spec} R$ (where R is considered as a trivially graded cdga).

For X an ordinary scheme $\mathcal{O}(X)$ is the commutative dg algebra whose cohomology is $\mathrm{H}^\bullet(X, \mathcal{O}_X)$, the sheaf cohomology of the structure sheaf of X. $\mathrm{hoQCoh}(X)$ is the unbounded derived category of X.

Example 1.9. Suppose X and Y are derived prestacks. The ***mapping prestack*** $\underline{\mathrm{Map}}(X, Y)$ is defined to be

$$\underline{\mathrm{Map}}(X, Y)(R) = \mathrm{Map}_{\mathrm{dPSt}}(X \times \operatorname{Spec} R, Y).$$

Example 1.10. Let G be an affine algebraic group acting on an affine scheme X. The action defines a simplicial affine scheme

$$X^\bullet = \Big(X \leftleftarrows X \times G \,\substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow}\, X \times G \times G \,\substack{\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow\\[-1ex]\leftarrow}\, \cdots \Big) \tag{1}$$

with the maps given by action and projection. We define $X/G \in \mathrm{dPSt}$ as the colimit of $X^\bullet$.

Remark 1.11. We may first compute the colimit of $X^\bullet$ in PSt, the ∞-category of ∞-prestacks, and then apply the embedding $i\colon \mathrm{PSt} \hookrightarrow \mathrm{dPSt}$ or apply i first to obtain a simplicial derived prestack $i(X^\bullet)$ and then compute its colimit in dPSt.

For instance, the ***classifying prestack*** $\mathrm{B}G$ is defined as

$$\mathrm{B}G = \mathrm{pt}/G.$$

By definition both $\mathcal{O}$ and QCoh send colimits of derived prestacks to limits which allows us to compute them. For instance, $\mathcal{O}([X/G])$ is the limit of the cosimplicial commutative dg algebra

$$\mathcal{O}(X) \rightrightarrows \mathcal{O}(X) \otimes \mathcal{O}(G) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{O}(X) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

Such a limit can be computed in terms of the total complex which is the complex computing group cohomology $\mathrm{H}^\bullet(G, \mathcal{O}(X))$. Similarly, $\mathrm{QCoh}([X/G])$ is the limit of the cosimplicial ∞-category

$$\mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(X \times G) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathrm{QCoh}(X \times G \times G) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

An object of this limit is a quasi-coherent sheaf $\mathcal{F}$ on X, an isomorphism $\mathrm{act}^*\mathcal{F} \cong p_1^*\mathcal{F}$ of sheaves on $X \times G$ and some further coherences. In other words, $\mathcal{F}$ is a G-equivariant quasi-coherent sheaf on X which suggests that $\mathrm{hoQCoh}([X/G])$ is the G-equivariant derived category of quasi-coherent sheaves on X (a precise proof is given by applying the Barr–Beck–Lurie theorem). For example,

$$\mathrm{QCoh}(\mathrm{B}G) \cong \mathrm{Rep}(G)$$

is the ∞-category of complexes of G-representations. Under this equivalence the pullback along $p\colon \mathrm{pt} \to \mathrm{B}G$ coincides with the forgetful functor $p^*\colon \mathrm{QCoh}(\mathrm{B}G) \cong \mathrm{Rep}(G) \to \mathrm{Mod}_k$. The structure sheaf $\mathcal{O}_{\mathrm{B}G} \in \mathrm{QCoh}(\mathrm{B}G) \cong \mathrm{Rep}(G)$ is the trivial representation $k \in \mathrm{Rep}(G)$.

Example 1.12. Given a derived prestack X we may construct a derived stack X^{st} by forcing étale descent. For instance, we may consider the classifying stack $\mathrm{B}^{st}G$. If C is an algebraic variety, we may define the ***moduli stack of G-bundles on*** C to be

$$\mathrm{Bun}_G(C) = \underline{\mathrm{Map}}(C, \mathrm{B}^{st}G).$$

We will often use the following property of quotient prestacks.

Lemma 1.13. *Suppose G is a group object in derived prestacks and X, Y, Z are derived prestacks with a G-action and G-equivariant maps $X \to Z$ and $Y \to Z$. Then there is an equivalence*

$$X/G \times_{Z/G} Y/G \cong (X \times_Z Y)/G.$$

Proof. Limits and colimits in functor ∞-categories are computed pointwise, so it is enough to prove the statement in $\mathcal{S}$. But limits distribute over sifted colimits (such as colimits of simplicial objects) in $\mathcal{S}$. ☐

Corollary 1.14. *Suppose G is a group object in derived prestacks. Then* $\mathrm{pt} \times_{\mathrm{B}G} \mathrm{pt} \cong G$.

1.2. **Cotangent complex.** Let $S = \operatorname{Spec} R$ be an affine scheme. Recall that Ω^1_S is the R-module representing derivations. In other words, for any R-module M we have an isomorphism

$$\mathrm{Hom}_R(\Omega^1_S, M) = \mathrm{Der}_k(R, M).$$

We may also identify derivations as follows. Define by $R[M]$ the square-zero extension of R by M, i.e. the vector space $R\oplus M$ with the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2+r_2m_1)$. Then we have an isomorphism

$$\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_{\mathrm{CAlg}_{/R}}(R, R[M]), \tag{2}$$

where $\mathrm{CAlg}_{/R}$ is the category of commutative algebras with a map to R.

In the derived setting this isomorphism will be taken as the *definition* of the module of derivations.

Definition 1.15. Let X be a derived prestack. The ***cotangent complex*** is an object $\mathbb{L}_X \in \mathrm{QCoh}(X)$ together with a natural isomorphism

$$\mathrm{Map}_{\mathrm{Mod}_R}(f^*\mathbb{L}_X, M) \cong \mathrm{Map}_{\mathrm{dPSt}_{S/}}(\operatorname{Spec} R[M], X)$$

for every map $f\colon S = \operatorname{Spec} R \to X$, where R is a connective cdga and $M \in \mathrm{Mod}^{\leq 0}_R$ a connective R-module. Here $\mathrm{dPSt}_{S/}$ is the ∞-category of derived prestacks with a map from S.

Note that this universal property determines the cotangent complex of X uniquely if it exists. The existence boils down to the following two properties:

- For every map $f\colon S = \operatorname{Spec} R \to X$ the functor $\mathrm{Mod}_R \to \mathcal{S}$ given by $M \mapsto \mathrm{Map}_{\mathrm{dPSt}_{S/}}(\operatorname{Spec} R[M], X)$ is representable (by an object we denote $\mathbb{L}_{X,f} \in \mathrm{Mod}_R$).
- Consider a map $g\colon S_2 = \operatorname{Spec} R_2 \to S_1 = \operatorname{Spec} R_1$ of derived affine schemes and a connective R_2-module M. Denote by g_*M the R_1-module obtained by restricting scalars along $R_1 \to R_2$. We get a natural map $\operatorname{Spec} R_2[M] \to \operatorname{Spec} R_1[g_*M]$. Consider the composite

$$\begin{aligned}\mathrm{Map}_{R_2}(g^*\mathbb{L}_{X,f}, M) &\cong \mathrm{Map}_{R_1}(\mathbb{L}_{X,f}, g_*M)\\ &\cong \mathrm{Map}_{\mathrm{dPSt}_{S_1/}}(\operatorname{Spec} R_1[g_*M], X)\\ &\to \mathrm{Map}_{\mathrm{dPSt}_{S_2/}}(\operatorname{Spec} R_2[M], X)\\ &\cong \mathrm{Map}_{R_2}(\mathbb{L}_{X,f\circ g}, M).\end{aligned}$$

  By the Yoneda Lemma we get a morphism $\mathbb{L}_{X,f\circ g} \to g^*\mathbb{L}_{X,f}$ of R_2-modules and we assume it is an isomorphism.

Assuming the above two properties on X, the cotangent complex $\mathbb{L}_X$ is defined so that $f^*\mathbb{L}_X = \mathbb{L}_{X,f}$ for $f\colon \operatorname{Spec} R \to X$.

Remark 1.16. If we consider X as an ∞-prestack, we only allow R and M to be concentrated in degree 0. Therefore, the universal property only determines the zeroth cohomology of the cotangent complex $\mathbb{L}_X$, i.e. the cotangent sheaf Ω^1_X.

Example 1.17. Let X and Y be derived prestacks and consider the mapping prestack $\underline{\mathrm{Map}}(X, Y)$. Let $f\colon X \times \operatorname{Spec} R \to Y$ defining an R-point of $\underline{\mathrm{Map}}(X, Y)$. Suppose $\Gamma(X, f^*\mathbb{T}_Y)$ is a perfect

R-module for any R and f. Then $\underline{\mathrm{Map}}(X, Y)$ admits a perfect cotangent complex such that $f^*\mathbb{T}_{\underline{\mathrm{Map}}(X,Y)} \cong \Gamma(X, f^*\mathbb{T}_Y)$.

For a large class of derived prestacks the cotangent complex exists. We will not define derived Artin stacks locally of finite presentation, but a recurring example in these notes will be the quotient (pre)stack X/G of a finite type scheme X by an affine algebraic group G.

Theorem 1.18. *Suppose X is a derived Artin stack. Then it admits a cotangent complex $\mathbb{L}_X$. If X is locally of finite presentation, then $\mathbb{L}_X$ is moreover a perfect complex.*

Definition 1.19. Suppose X is a derived prestack which admits a perfect cotangent complex $\mathbb{L}_X \in \mathrm{Perf}(X)$. The ***tangent complex*** is the dual object $\mathbb{T}_X = \mathbb{L}_X^\vee \in \mathrm{Perf}(X)$.

For a morphism $f \colon X \to Y$ of derived prestacks one may similarly define a relative cotangent complex $\mathbb{L}_{X/Y} \in \mathrm{QCoh}(X)$ by considering derived prestacks over Y. From the universal property of the cotangent complex one may construct the pullback map $f^*\mathbb{L}_Y \to \mathbb{L}_X$.

The computation of the cotangent complex is often facilitated by the following two statements.

Recall that if $\mathcal{C}$ is an ∞-category with a zero object $0 \in \mathcal{C}$ (an object which is both initial and final), then a fiber sequence $x \to y \to z$ in $\mathcal{C}$ is a pullback square

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z \end{array}$$

Theorem 1.20. *Suppose $f \colon X \to Y$ is a morphism of derived prestacks which admit cotangent complexes. Then there exists a relative cotangent complex $\mathbb{L}_{X/Y} \in \mathrm{QCoh}(X)$ and a fiber sequence*

$$f^*\mathbb{L}_Y \longrightarrow \mathbb{L}_X \longrightarrow \mathbb{L}_{X/Y}.$$

Theorem 1.21. *Suppose*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

is a pullback diagram of derived prestacks which admit cotangent complexes. Then $f^*\mathbb{L}_{Y/Z} \cong \mathbb{L}_{X/Z}$.

Example 1.22. Let X be a smooth scheme considered as a derived prestack (see example 1.8). Then $\mathbb{L}_X \in \mathrm{QCoh}(X)$ is equivalent to the vector bundle Ω^1_X concentrated in cohomological degree 0.

Example 1.23. Let $X = \mathrm{Spec}\, R$ be a derived affine scheme (more generally, a derived Deligne–Mumford stack). Then $\mathbb{L}_X \in \mathrm{QCoh}(X) \cong \mathrm{Mod}_R$ is connective, i.e. it is concentrated in cohomological degrees ≤ 0.

Example 1.24. Let G be an affine algebraic group and consider $X = \mathrm{B}G$ with the projection map $p\colon \mathrm{pt} \to \mathrm{B}G$. By corollary 1.14 we have a pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{B}G \end{array}$$

By theorem 1.21 we have $f^*\mathbb{L}_{\mathrm{pt}/\mathrm{B}G} \cong \mathbb{L}_{G/\mathrm{pt}}$, where $\mathbb{L}_{\mathrm{pt}/\mathrm{B}G} \in \mathrm{Mod}_k$ is simply a chain complex. But $\mathbb{L}_{G/\mathrm{pt}} \cong \mathfrak{g}^* \otimes \mathcal{O}_G$, so $\mathbb{L}_{\mathrm{pt}/\mathrm{B}G} \cong \mathfrak{g}^*$. By theorem 1.20 we have a fiber sequence

$$p^*\mathbb{L}_{\mathrm{B}G} \longrightarrow \mathbb{L}_{\mathrm{pt}} \longrightarrow \mathbb{L}_{\mathrm{pt}/\mathrm{B}G}.$$

Since $\mathbb{L}_{\mathrm{pt}} = 0$, this implies that $p^*\mathbb{L}_{\mathrm{B}G} \cong \mathfrak{g}^*[-1]$. Recall that $\mathrm{QCoh}(\mathrm{B}G) \cong \mathrm{Rep}(G)$ and $p^*\colon \mathrm{QCoh}(\mathrm{B}G) \to \mathrm{QCoh}(\mathrm{pt}) = \mathrm{Mod}_k$ is the forgetful functor. Thus, $\mathbb{L}_{\mathrm{B}G} \in \mathrm{QCoh}(\mathrm{B}G) \cong \mathrm{Rep}(G)$ is a G-representation whose underlying complex is $\mathfrak{g}^*[-1]$. With some more work one may show that this is in fact the shifted coadjoint representation.

Example 1.25. Let X be a derived prestack and $V \in \mathrm{QCoh}(X)$ a quasi-coherent sheaf. The ***total space of*** V is the derived prestack which sends $R \in \mathrm{CAlg}^{\leq 0}$ to the space of pairs of a point $f \in X(R)$ and an element s of the R-module f^*V. For instance, if X admits a cotangent complex, we define the ***cotangent stack*** T^*X to be the total space of $\mathbb{L}_X \in \mathrm{QCoh}(X)$ and the n***-shifted cotangent stack*** $\mathrm{T}^*[n]X$ to be the total space of $\mathbb{L}_X[n] \in \mathrm{QCoh}(X)$.

1.3. **Exercises.**

(1) Suppose G is an affine algebraic group and X an affine scheme equipped with a G-action.

(a) Identify

$$\lim\Big(\mathcal{O}(X) \rightrightarrows \mathcal{O}(X)\otimes\mathcal{O}(G) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{O}(X)\otimes\mathcal{O}(G)\otimes\mathcal{O}(G) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big)$$
$$\cong \mathrm{eq}\Big(\mathcal{O}(X) \rightrightarrows \mathcal{O}(X)\otimes\mathcal{O}(G) \Big)$$

where both limits are taken in the category of commutative algebras.

(b) Identify

$$\mathrm{eq}\Big(\mathcal{O}(X) \rightrightarrows \mathcal{O}(X)\otimes\mathcal{O}(G) \Big) \cong \mathcal{O}(X)^G$$

with the subset of G-invariants in $\mathcal{O}(X)$.

(c) Denote the three maps $X\times G\times G \to X\times G$ by

$$\begin{aligned} \mathrm{act}_1(x, g_1, g_2) &= (xg_1, g_2) \\ m_{23}(x, g_1, g_2) &= (x, g_1g_2) \\ p_{12}(x, g_1, g_2) &= (x, g_1). \end{aligned}$$

Consider the category $\mathrm{Desc}(X, G)$ whose objects are pairs $(\mathcal{F}, f)$ consisting of a quasi-coherent sheaf $\mathcal{F}$ on X and an isomorphism $f\colon \mathrm{act}^*\mathcal{F} \xrightarrow{\sim} p_1^*\mathcal{F}$ on $X\times G$ satisfying $p_{12}^*(f)\circ\mathrm{act}_1^*(f) = m_{23}^*(f)$. Show that $\mathrm{Desc}(X, G)$ is equivalent to the category of G-equivariant sheaves on X.

(d) (*) Consider a cosimplicial category

$$\mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(X\times G) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \mathrm{QCoh}(X\times G\times G) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

i.e. a pseudofunctor $\Delta \to \mathrm{Cat}$ from the category of simplices to the bicategory of categories. Show that the pseudolimit of this cosimplicial category is equivalent to $\mathrm{Desc}(X, G)$.

(2) Prove corollary 1.14.
(3) Construct the isomorphism (2).
(4) Suppose $f\colon X \to Y$ is a morphism of derived prestacks which admit cotangent complexes $\mathbb{L}_X$ and $\mathbb{L}_Y$ respectively. Use the universal property to construct the pullback map $f^*\mathbb{L}_X \to \mathbb{L}_Y$.
(5) Let G be an affine algebraic group and X a scheme with a G-action. Denote by $p\colon X \to X/G$ the natural projection map. Find $p^*\mathbb{L}_{X/G}$.

2. Shifted symplectic structures

In this section we define shifted symplectic structures following [Pan+13; Cal+17].

2.1. **Differential forms.** Recall that if $S = \operatorname{Spec} R$ is a derived affine scheme, it has a cotangent complex $\mathbb{L}_R \in \mathrm{Mod}_R^{\leq 0}$ which satisfies the universal property of definition 1.15. In particular,

$$\mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_R, \mathbb{L}_R) \cong \mathrm{Map}_{\mathrm{CAlg}_{/R}}(R, R[\mathbb{L}_R]).$$

Under this equivalence the identity map $\mathrm{id}\colon \mathbb{L}_R \to \mathbb{L}_R$ corresponds to a universal derivation $r \mapsto (r, \mathrm{d}_{\mathrm{dR}} r)$, where

$$\mathrm{d}_{\mathrm{dR}}\colon R \to \mathbb{L}_R$$

is the ***de Rham differential***.

We will now explain how to extend it to the algebra of differential forms.

Definition 2.1. A ***graded mixed cdga*** is a graded commutative dg algebra

$$A = \bigoplus_{n\in\mathbf{Z}} A(n)$$

equipped with a square-zero derivation $\epsilon\colon A(n) \to A(n+1)[1]$. Denote by $\mathrm{CAlg}^{gr,\epsilon}$ the ∞-category of graded mixed cdgas.

Note that a graded mixed cdga has two gradings:

- The cohomological grading (which we simply call the ***degree***).
- The external grading (which we call the ***weight***), so that elements in $A(n)$ have pure weight n.

It also has two differentials:

- The cohomological differential d of degree 1 and weight 0.
- The mixed structure ϵ of degree 1 and weight 1.

Definition 2.2. Let R be a connective cdga. Its ***de Rham algebra*** is a graded mixed cdga $\mathrm{DR}(R) \in \mathrm{CAlg}^{gr,\epsilon}$ such that for any graded mixed cdga $A \in \mathrm{CAlg}^{gr,\epsilon}$ we have a natural isomorphism

$$\mathrm{Map}_{\mathrm{CAlg}}(R, A(0)) \cong \mathrm{Map}_{\mathrm{CAlg}^{gr,\epsilon}}(\mathrm{DR}(R), A).$$

The universal property of the de Rham algebra looks similar to the universal property of the cotangent complex, so the following statement should come as no surprise.

Theorem 2.3. *Let R be a connective cdga. There is an equivalence of graded cdgas*

$$\mathrm{Sym}_R(\mathbb{L}_R[-1]) \cong \mathrm{DR}(R).$$

Under this equivalence the mixed structure $R \to \mathbb{L}_R$ corresponds to the de Rham differential d_{dR}.

Remark 2.4. The map $\mathrm{Sym}_R(\mathbb{L}_R[-1]) \to \mathrm{DR}(R)$ is constructed as follows. Setting $A = \mathrm{DR}(R)$ in the universal property we obtain a morphism of cdgas $R \to \mathrm{DR}(R)(0)$. Post-composing it with the mixed structure we obtain a derivation $R \to \mathrm{DR}(R)(1)[1]$, i.e. a morphism of R-modules $\mathbb{L}_R[-1] \to \mathrm{DR}(R)(1)$. This uniquely extends to a morphism of graded cdgas $\mathrm{Sym}(\mathbb{L}_R[-1]) \to \mathrm{DR}(R)$.

From now on we simply denote the whole mixed structure on $\mathrm{DR}(R)$ by d_{dR}. We may extend the definition of the de Rham algebra from derived affine schemes to general derived prestacks by directly copying definitions 1.2 and 1.4.

Definition 2.5. Let X be a derived prestack. Its ***de Rham algebra*** is the graded mixed cdga

$$\mathrm{DR}(X) = \lim_{R \in \mathrm{CAlg}^{\leq 0}, f \in X(R)} \mathrm{DR}(R),$$

where the limit is taken in the ∞-category $\mathrm{CAlg}^{gr,\epsilon}$.

Taking the degree zero part we obtain the definition of $\mathcal{O}(X)$, so we have

$$\mathrm{DR}(X)(0) \cong \mathcal{O}(X) \cong \Gamma(X, \mathcal{O}_X).$$

Now suppose X admits a cotangent complex $\mathbb{L}_X \in \mathrm{QCoh}(X)$. Then we may consider the graded cdga $\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1]))$ (note that there is no a priori mixed structure on this graded cdga). For every map $f \colon \operatorname{Spec} R \to X$ from a derived affine scheme we have a pullback map

$$f^* \colon \Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1])) \longrightarrow \mathrm{Sym}_R(\mathbb{L}_R[-1]) \cong \mathrm{DR}(R)$$

by using the pullback map $f^*\mathbb{L}_X \to \mathbb{L}_R$. These pullback maps are compatible with maps of derived affine schemes, so by the universal property of the limit we obtain a map

$$\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1])) \longrightarrow \mathrm{DR}(X) \tag{3}$$

of graded cdgas.

Theorem 2.6 (PTVV)**.** *Suppose X is a derived Artin stack. Then the map* (3) *is an equivalence.*

Thus, for a nice class of derived prestacks we can think of the de Rham algebra as $\Gamma(X, \mathrm{Sym}(\mathbb{L}_X[-1]))$ equipped with the de Rham differential d_{dR}.

Definition 2.7. Let X be a derived prestack. A ***p-form of degree n*** on X is a d-closed element of $\mathrm{DR}(X)$ of weight p and degree $p+n$. A ***closed p-form of degree n*** on X is a collection of elements $\omega_p, \omega_{p+1}, \dots$ of $\mathrm{DR}(X)$, where ω_k has weight k and degree $p+n$, and such that

$$(\mathrm{d} + \mathrm{d}_{\mathrm{dR}})(\omega_p + \omega_{p+1} + \dots) = 0.$$

Remark 2.8. The equation $(\mathrm{d}+\mathrm{d}_{\mathrm{dR}})(\omega_p+\omega_{p+1}+\dots)=0$ splits according to weights as follows:

$$\begin{aligned}\mathrm{d}\omega_p &= 0\\ \mathrm{d}_{\mathrm{dR}}\omega_p+\mathrm{d}\omega_{p+1} &= 0\\ &\dots\end{aligned}$$

In other words, a closed p-form of degree n consists of a p-form ω_p of degree n and the data of ω_p being coherently d_{dR}-closed.

We denote by $\mathcal{A}^p(X,n)$ the space of p-forms of degree n and by $\mathcal{A}^{p,\mathrm{cl}}(X,n)$ the space of closed p-forms of degree n. For example, a path from α to β in $\mathcal{A}^p(X,n)$ is given by an element h of weight p and degree $p+n-1$ such that

$$\alpha-\beta=\mathrm{d}h.$$

Similarly, a path from $\alpha=\alpha_p+\dots$ to $\beta=\beta_p+\dots$ in $\mathcal{A}^{p,\mathrm{cl}}(X,n)$ is given by a formal power series $h=h_p+\dots$ such that

$$\alpha_k-\beta_k=\mathrm{d}h_k+\mathrm{d}_{\mathrm{dR}}h_{k-1}.$$

2.2. **Quotient stacks.** Let X be a smooth scheme and G an affine algebraic group acting on X. We will be interested in the de Rham algebra $\mathrm{DR}(X/G)$ of the quotient prestack X/G. By definition X/G is given by the colimit of the simplicial scheme (1) and $\mathrm{DR}(-)$ sends colimits to limits. Therefore, we may identify

$$\mathrm{DR}(X/G)\cong\lim\Big(\mathrm{DR}(X)\rightrightarrows\mathrm{DR}(X)\otimes\mathrm{DR}(G)\Rrightarrow\cdots\Big)$$

This complex is known as the ***Čech model*** of equivariant cohomology. We will now introduce a simpler model which is more convenient for computations. Denote by $a\colon\mathfrak{g}\to\Gamma(X,\mathrm{T}_X)$ the infinitesimal action map. Pick a basis $\{e_i\}$ of $\mathfrak{g}$ and let $\{e^i\}$ be the dual basis of $\mathfrak{g}^*$.

Definition 2.9. Let X be a smooth scheme and G an affine algebraic group acting on X. The ***Cartan model*** is the graded mixed cdga

$$\mathrm{DR}^{Cartan}(X/G)=(\mathrm{Sym}(\mathfrak{g}^*[-2])\otimes\mathrm{DR}(X))^G,$$

where $\mathfrak{g}^*[-2]$ is in weight 1 and the cohomological differential d is given by $e^i\iota_{a(e_i)}$

For the following statement, see [Beh04, Lemma 12].

Theorem 2.10. *Suppose X is a smooth scheme and G is a reductive algebraic group acting on X. Then we have an equivalence of graded mixed cdgas*

$$\mathrm{DR}(X/G)\cong\mathrm{DR}^{Cartan}(X/G).$$

Example 2.11. Let G be a reductive algebraic group and consider $\mathrm{DR}(\mathrm{B}G)$. By theorem 2.10 we have

$$\mathrm{DR}(\mathrm{B}G)\cong(\mathrm{Sym}(\mathfrak{g}^*[-2]))^G$$

with the trivial mixed structure and the cohomological differential. So, elements of weight p are concentrated in cohomological degree $2p$. Therefore, for $n<p$ the spaces $\mathcal{A}^{p,\mathrm{cl}}(\mathrm{B}G,n)$

and $\mathcal{A}^p(\mathrm{B}G, n)$ are contractible, i.e. the unique (closed) p-form of degree n is the zero form. For $n = p$ we have

$$\mathcal{A}^{p,\mathrm{cl}}(\mathrm{B}G, p) \cong \mathcal{A}^p(\mathrm{B}G, p) \cong \mathrm{Sym}^p(\mathfrak{g}^*)^G.$$

2.3. **Symplectic structures.** Let X be a smooth scheme of dimension $2d$. Recall that a symplectic structure on X is given by a closed two-form ω on X which is nondegenerate. Let us also recall that nondegeneracy can be phrased in the following two equivalent ways:

(1) $\omega^d \in \Gamma(X, \wedge^d(\mathrm{T}^*_X))$ is a nonvanishing section (i.e. a volume form).
(2) The map $\omega^\sharp\colon \mathrm{T}_X \to \mathrm{T}^*_X$ given by $v \mapsto \iota_v\omega$ is an isomorphism of vector bundles.

We have already defined the notion of a closed two-form on a derived prestack (definition 2.7). For a general derived prestack there may not be a number d such that $\wedge^d(\mathbb{L}_X)$ is a line bundle, so the first definition of nondegeneracy is problematic. However, the second definition of nondegeneracy extends immediately.

Let X be a derived Artin stack locally of finite presentation and $\omega \in \mathcal{A}^2(X, n)$ a two-form of degree n. By theorem 2.6 (and this is exactly the reason for the assumptions on X) we get that

$$\omega \in \Gamma(X, \mathrm{Sym}^2(\mathbb{L}_X[-1]))[n+2] \subset \Gamma(X, \mathbb{L}_X^{\otimes 2})[n] \cong \mathrm{Hom}(\mathbb{T}_X, \mathbb{L}_X[n]).$$

We denote the image of ω under this map by $\omega^\sharp\colon \mathbb{T}_X \to \mathbb{L}_X[n]$.

Definition 2.12. Let X be a derived Artin stack locally of finite presentation. An n-***shifted symplectic structure*** is a closed two-form $\omega \in \mathcal{A}^{2,\mathrm{cl}}(X, n)$ of degree n on X such that $\omega^\sharp\colon \mathbb{T}_X \to \mathbb{L}_X[n]$ is a quasi-isomorphism.

Example 2.13. Let X be a smooth scheme. Then $\mathbb{L}_X = \mathrm{T}^*_X$ is concentrated in degree 0, so we can only have a quasi-isomorphism $\mathrm{T}_X \cong \mathrm{T}^*_X[n]$ if $n = 0$. The space $\mathcal{A}^{2,\mathrm{cl}}(X, 0)$ parametrizes power series $\omega = \omega_2 + \dots$, where ω_2 if a two-form of degree 0. By degree reasons $\omega_k = 0$ for $k > 2$ and so $\mathcal{A}^{2,\mathrm{cl}}(X, 0)$ is isomorphic to the *set* of closed two-forms. The nondegeneracy condition on a 0-shifted symplectic structure is then manifestly the same as the second definition of nondegeneracy for ordinary symplectic structures. In this sense shifted symplectic structures provide a generalization of the classical notion of a symplectic structure.

Example 2.14. Let G be a reductive algebraic group and conisder $X = \mathrm{B}G$. We have $\mathbb{L}_{\mathrm{B}G} \cong \mathfrak{g}^*[-1]$ and $\mathbb{T}_{\mathrm{B}G} \cong \mathfrak{g}[1]$, so a quasi-isomorphism $\mathbb{T}_{\mathrm{B}G} \cong \mathbb{L}_{\mathrm{B}G}[n]$ is only possible for $n = 2$. Closed two-forms on $\mathrm{B}G$ of degree 2 are the same as elements $c \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$, i.e. G-invariant symmetric bilinear pairings on $\mathfrak{g}$. The nongeneracy condition on the shifted symplectic strucutre is that the map $\omega^\sharp\colon \mathrm{T}_{\mathrm{B}G} \to \mathbb{L}_{\mathrm{B}G}[2]$, i.e. the map $\mathfrak{g} \to \mathfrak{g}^*$ given by $v \mapsto c(v, -)$, is an isomorphism. In other words, the correspond closed two-form on $\mathrm{B}G$ of degree 2 is nondegenerate iff c is a *nondegenerate* pairing.

Example 2.15. Let X be a derived Artin stack locally of finite presentation. Then by theorem 1.18 it admits a perfect cotangent complex $\mathbb{L}_X \in \mathrm{Perf}(X)$ whose dual is the tangent complex $\mathbb{T}_X \in \mathrm{Perf}(X)$. Let us consider the n-shifted cotangent stack $p\colon \mathrm{T}^*[n]X \to X$ (see example 1.25); it is also a derived Artin stack locally of finite presentation.

The pullback map $p^*\mathbb{L}_X \to \mathbb{L}_{\mathrm{T}^*[n]X}$ on the cotangent complex gives rise to a map

$$\Gamma(\mathrm{T}^*[n]X, p^*\mathbb{L}_X) \longrightarrow \Gamma(\mathrm{T}^*[n]X, \mathbb{L}_{\mathrm{T}^*[n]X}).$$

In turn, we may identify

$$\Gamma(\mathrm{T}^*[n]X, p^*\mathbb{L}_X) \cong \Gamma(X, \mathbb{L}_X \otimes \mathrm{Sym}(\mathbb{T}_X[-n]))$$

which has a canonical element $\mathrm{coev} \in \mathbb{L}_X \otimes \mathbb{T}_X$ (i.e. the canonical copairing) of degree n. Its image in $\Gamma(\mathrm{T}^*[n]X, \mathbb{L}_{\mathrm{T}^*[n]X})$ gives rise to a one-form λ on $\mathrm{T}^*[n]X$ of degree n known as the ***Liouville one-form*** (the tautological one-form on the shifted cotangent stack). Then $\omega = \mathrm{d}_{\mathrm{dR}}\lambda$ is a closed two-form on $\mathrm{T}^*[n]X$ of degree n. It is shown in [Cal19] that it is in fact nondegenerate, i.e. it defines an n-shifted symplectic structure.

Example 2.16. Let G be an affine algebraic group and consider the classifying prestack $\mathrm{B}G$. We have $\mathbb{L}_{\mathrm{B}G} \cong \mathfrak{g}^*[-1]$ (the shifted coadjoint representation), so $\mathrm{T}^*[1](\mathrm{B}G) \cong \mathfrak{g}^*/G$. We thus get a 1-shifted symplectic structure on $\mathfrak{g}^*/G$. We will relate the 1-shifted symplectic structure on $\mathfrak{g}^*/G$ to the Kirillov–Kostant–Souriau Poisson structure on $\mathfrak{g}^*$ in the next section.

2.4. **Lagrangian structures.** Let (X, ω) be a smooth symplectic scheme and $L \hookrightarrow X$ a smooth closed subscheme. Recall that L is an *isotropic* subscheme if $\omega|_L = 0$. L is called a *Lagrangian* subscheme if, in addition, $2\dim(L) = \dim(X)$. To generalize it to derived prestacks we will give an equivalent characterization of Lagrangian subschemes.

Suppose L is an isotropic subscheme. Since $\omega|_L = 0$, the composite

$$\mathrm{T}_L \longrightarrow (\mathrm{T}_X)|_L \xrightarrow{\omega^\sharp} (\mathrm{T}^*_X)|_L \longrightarrow \mathrm{T}^*_L$$

is zero. Let $\mathrm{N}_{L/X} = \mathrm{coker}(\mathrm{T}_L \to \mathrm{T}_X|_L)$ be the normal bundle. Then from the above observation we get a map $\mathrm{N}_{L/X} \to \mathrm{T}^*_L$. Nondegeneracy of ω implies that this map is injective. But an injective map of vector bundles is an isomorphism iff they have the same rank, i.f. iff $\dim(X) - \dim(L) = \dim(L)$. Thus, an isotropic subscheme $L \subset X$ is Lagrangian if $\mathrm{N}_{L/X} \to \mathrm{T}^*_L$ is an isomorphism. Equivalently, the sequence

$$0 \longrightarrow \mathrm{T}_L \longrightarrow \mathrm{T}_X|_L \longrightarrow \mathrm{T}^*_L \longrightarrow 0$$

is exact.

Definition 2.17. Let $f\colon L \to X$ be a morphism of derived prestacks where X is equipped with a closed two-form ω of degree n. An ***n-shifted isotropic structure on f*** is a nullhomotopy of $f^*\omega \in \mathcal{A}^{2,\mathrm{cl}}(L, n)$.

In other words, if we write $\omega = \omega_2 + \dots$, then an n-shifted isotropic structure on $f\colon L \to X$ is a power series $h = h_2 + \dots$ such that

$$\begin{aligned} f^*\omega_2 &= \mathrm{d}h_2 \\ f^*\omega_3 &= \mathrm{d}_{\mathrm{dR}}h_2 + \mathrm{d}h_1 \\ &\dots \end{aligned}$$

In particular, $f^*\omega_2 \in \mathcal{A}^2(L, n)$ is nullhomotopic. If we assume both L and X are derived Artin stacks locally of finite presentation, we get a nullhomotopy of the composite

$$\mathbb{T}_L \longrightarrow f^*\mathbb{T}_X \xrightarrow{\omega^\sharp} f^*\mathbb{L}_X[n] \longrightarrow \mathbb{L}_L[n].$$

Definition 2.18. Let $f\colon L \to X$ be a morphism of derived Artin stacks locally of finite presentation where X is equipped with an n-shifted symplectic structure ω. An ***n-shifted Lagrangian structure on*** f is an isotropic structure on f such that

$$\mathbb{T}_L \longrightarrow f^*\mathbb{T}_X \longrightarrow \mathbb{L}_L[n] \tag{4}$$

is a fiber sequence.

Example 2.19. Let (X, ω) be a smooth symplectic scheme considered as a 0-shifted symplectic scheme and $i\colon L \hookrightarrow X$ a smooth subscheme. Possible 0-shifted isotropic structures on i are given by a power series $h = h_2 + \dots$. But the degrees of h_k are all negative, so $h_k = 0$. Thus, i carries a 0-shifted isotropic structure iff $\omega|_L = 0$, i.e. L is an ordinary isotropic subscheme, in which case it is unique. As we have observed above, the 0-shifted isotropic structure is Lagrangian iff $\dim(L) = \dim(X)/2$.

Example 2.20. The point pt carries a unique n-shifted symplectic structure for any n: indeed, DR(pt) is zero in positive weights and $\mathbb{T}_{\mathrm{pt}} \cong \mathbb{L}_{\mathrm{pt}} \cong 0$. For any derived prestack X there is a unique map $p\colon X \to \mathrm{pt}$. An n-shifted isotropic structure on p is a nullhomotopy of p^*0 in $\mathcal{A}^{2,\mathrm{cl}}(X, n)$, i.e. a closed two-form h of degree $(n-1)$. The sequence (4) becomes

$$\mathbb{T}_X \longrightarrow 0 \longrightarrow \mathbb{L}_X[n]$$

which is a fiber sequence iff $h^\sharp\colon \mathbb{T}_X \to \mathbb{L}_X[n-1]$ is an equivalence, i.e. iff h defines an $(n-1)$-shifted symplectic structure on X.

An important example of a shifted Lagrangian structure is given by the following statement (see [Cal15]).

Let X be a smooth symplectic scheme equipped with a G-action which preserves the symplectic structure. Recall that a G-equivariant map $\mu\colon X \to \mathfrak{g}^*$ is called a ***moment map*** for the G-action on X if for every $v \in \mathfrak{g}$ we have

$$\iota_{a(v)}\omega = \mathrm{d}_{\mathrm{dR}}\mu(v),$$

where $a\colon \mathfrak{g} \to \Gamma(X, \mathrm{T}_X)$ is the infinitesimal action map.

Theorem 2.21. *Let G be a reductive algebraic group, X a smooth G-scheme and $\mu\colon X \to \mathfrak{g}^*$ a G-equivariant map which induces the map $[\mu]\colon X/G \to \mathfrak{g}^*/G$ after quotienting by G. Then the space of 1-shifted Lagrangian structure on $[\mu]$ is equivalent to the set of symplectic structures on X for which μ is a moment map.*

Remark 2.22. Recall that a derived prestack X with a G-action may be encoded in the projection $p\colon X/G \to \mathrm{B}G$. The data of a moment map is then a lift of $X/G \to \mathrm{B}G$ to a 1-shifted Lagrangian map $X/G \to \mathfrak{g}^*/G$.

An important construction with shifted Lagrangian structures is their intersection. Recall that for a diagram of sets $L_1 \to X \leftarrow L_2$ their intersection is the same as the fiber product $L_1 \times_X L_2$. In a similar way, if L_1, L_2 and X are schemes, their schematic intersection is given by the fiber product.

Theorem 2.23. *Suppose $L_1 \to X \leftarrow L_2$ is a diagram of derived prestacks where X is equipped with an n-shifted symplectic structure ω and $f_1\colon L_1 \to X$ and $f_2\colon L_2 \to X$ are equipped with n-shifted Lagrangian structures. Then $L_1 \times_X L_2$ carries a natural $(n-1)$-shifted symplectic structure.*

Proof. Consider a pullback diagram

$$\begin{array}{ccc} L_1 \times_X L_2 & \xrightarrow{g_1} & L_1 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} \\ L_2 & \xrightarrow{f_2} & X \end{array}$$

The n-shifted Lagrangian structure on f_1 gives a nullhomotopy $f_1^*\omega \sim 0$ in $\mathcal{A}^{2,\mathrm{cl}}(L_1, n)$ hence a nullhomotopy $g_1^* f_1^*\omega \sim 0$ in $\mathcal{A}^{2,\mathrm{cl}}(L_1 \times_X L_2, n)$. In a similar way, from L_2 we get a nullhomotopy $g_2^* f_2^*\omega \sim 0$ in the same space. But since the above diagram commutes, we also have a homotopy $g_1^* f_1^* \sim g_2^* f_2^*\omega$. Thus, we obtain a composite homotopy

$$0 \sim g_1^* f_1^*\omega \sim g_2^* f_2^*\omega \sim 0$$

in $\mathcal{A}^{2,\mathrm{cl}}(L_1 \times_X L_2, n)$. Such a homotopy from 0 to 0 in $\mathcal{A}^{2,\mathrm{cl}}(L_1 \times_X L_2, n)$ is given by an element $h = h_2 + \dots$ which satisfies $0 - 0 = (\mathrm{d} + \mathrm{d}_{\mathrm{dR}})h$, i.e. $h \in \mathcal{A}^{2,\mathrm{cl}}(L_1 \times_X L_2, n-1)$. The fact that h is nondegenerate follows from the nondegeneracy of f_1 and f_2 for which we refer to [Pan+13, Theorem 2.9]. □

Example 2.24. Recall from theorem 2.21 that if X is a symplectic scheme equipped with a G-action and a moment map $\mu\colon X \to \mathfrak{g}^*$, then $X/G \to \mathfrak{g}^*/G$ carries a 1-shifted Lagrangian structure. For instance, taking $X = \mathrm{pt}$ and $\mu\colon \mathrm{pt} \to \mathfrak{g}^*$ given by the inclusion of the origin, we get a 1-shifted Lagrangian structure on $\mathrm{B}G = \mathrm{pt}/G \to \mathfrak{g}^*/G$. Therefore, by theorem 2.23 we get a 0-shifted symplectic structure on

$$X/G \times_{\mathfrak{g}^*/G} \mathrm{B}G \cong (X \times_{\mathfrak{g}^*} \mathrm{pt})/G = \mu^{-1}(0)/G.$$

This space is known as the *symplectic reduction* of X by G.

Example 2.25. Suppose $f\colon L \to X$ is a morphism of derived Artin stacks locally of finite presentation. The ***n*-shifted conormal stack** $\mathrm{N}^*[n](L/X) \to L$ is defined to be the total space of the perfect complex $\mathbb{L}_{L/X}[n-1] \in \mathrm{Perf}(L)$. We have a fiber sequence

$$f^*\mathbb{L}_X \longrightarrow \mathbb{L}_L \longrightarrow \mathbb{L}_{L/X}$$

which after rotation and shifting gives rise to a morphism $\mathbb{L}_{L/X}[n-1] \to f^*\mathbb{L}_X[n]$. Therefore, on the level of total spaces there is a natural morphism $\tilde{f}\colon \mathrm{N}^*[n](L/X) \longrightarrow \mathrm{T}^*[n]X$.

Proceeding as in example 2.15 we may obtain a nullhomotopy of $\tilde{f}^*\lambda$, the pullback of the Liuoville one-form. Therefore, $\tilde{f}$ carries a natural n-shifted isotropic structure. It is moreover shown in [Cal19] that it is in fact n-shifted Lagrangian.

An important idea going back to Weinstein [Wei82] is that Lagrangians can be considered as morphisms between symplectic manifolds. For a derived prestack X equipped with an n-shifted symplectic structure we denote by $\overline{X}$ the same derived prestack with the opposite shifted n-symplectic structure.

Definition 2.26. Let X and Y be n-shifted symplectic derived prestacks. An ***n*-shifted Lagrangian correspondence** $X \leftarrow L \to Y$ from X to Y is an n-shifted Lagrangian structure on $L \to \overline{X} \times Y$.

We can organize Lagrangian correspondences into an ∞-category:

- Its objects are derived prestacks equipped with an n-shifted symplectic structure.

- Morphisms from X to Y are given by n-shifted Lagrangian correspondences $X \leftarrow L \rightarrow Y$.
- A composition of two Lagrangian correspondences $X \leftarrow L_1 \rightarrow Y$ and $Y \leftarrow L_2 \rightarrow Z$ is given by the pullback

$$\begin{array}{ccccccc} & & & L_1 \times_Y L_2 & & & \\ & & \swarrow & & \searrow & & \\ & L_1 & & & & L_2 & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X & & & Y & & & Z \end{array}$$

and a variant of theorem 2.23 shows that $X \leftarrow L_1 \times_Y L_2 \rightarrow Z$ is an n-shifted Lagrangian correspondence.

Such an ∞-category has been constructed in [Hau18]. One can moreover extend it to a higher category by considering iterated Lagrangian correspondences. Namely, given X and Y equipped with n-shifted symplectic structures and two n-shifted Lagrangian correspondences $X \leftarrow L_1 \rightarrow Y$ and $X \leftarrow L_2 \rightarrow Y$ a 2-morphism is given by a correspondence $L_1 \leftarrow Z \rightarrow L_2$ such that $Z \rightarrow L_1 \times_{\overline{X}\times Y} L_2$ is equipped with an $(n-1)$-shifted Lagrangian structure. In this way one may construct an (∞, n)-category of Lagrangian correspondences. Such a construction will appear in the upcoming work [CHS19], see also [AB17] where the homotopy 2-category of n-shifted Lagrangian correspondences is constructed.

2.5. **AKSZ construction.** Let C be a smooth projective variety of dimension d and X a derived Artin stack locally of finite presentation. Recall that the mapping stack $\underline{\mathrm{Map}}(C, X)$ admits a cotangent complex, so that for $f \in \underline{\mathrm{Map}}(C, X)(R)$ we have

$$f^*\mathbb{T}_{\underline{\mathrm{Map}}(C,X)} \cong \Gamma(C, f^*\mathbb{T}_X).$$

In particular,

$$f^*\mathbb{L}_{\underline{\mathrm{Map}}(C,X)} \cong \Gamma(C, f^*\mathbb{T}_X)^\vee \cong \Gamma(C, f^*\mathbb{L}_X \otimes K_C[-d]), \tag{5}$$

where K_C is the canonical bundle of C and we have used the Serre duality in the last equivalence.

Now suppose C is Calabi-Yau, i.e. $K_C \cong \mathcal{O}_C$, and fix a one-form $\omega \in \Gamma(X, \mathbb{L}_X)[n]$ of degree n. Using the equivalence (5) we obtain a one-form on $\underline{\mathrm{Map}}(C, X)$ of degree $n-d$. This can be generalized to forms of higher weights.

Theorem 2.27. *Let C be a smooth projective variety of dimension d equipped with a trivialization $K_C \cong \mathcal{O}_C$ and X an n-shifted symplectic stack. Then $\underline{\mathrm{Map}}(C, X)$ admits a natural $(n-d)$-shifted symplectic structure.*

Remark 2.28. The above theorem was proven in [Pan+13, Theorem 2.5] following a differential-geometric construction given in [Ale+97].

A variant of theorem 2.27 also works for Lagrangian maps.

Theorem 2.29. *Let C and X be as before and $f\colon L \to X$ a map of derived prestacks equipped with an n-shifted Lagrangian structure. Then*

$$\underline{\mathrm{Map}}(C, L) \longrightarrow \underline{\mathrm{Map}}(C, X)$$

admits a natural $(n-d)$-shifted Lagrangian structure.

2.6. **Exercises.**

(1) Let G be a reductive algebraic group. Describe the spaces of differential forms $\mathcal{A}^{p,\mathrm{cl}}(\mathrm{B}G, n)$ and $\mathcal{A}^{p}(\mathrm{B}G, n)$ for $n > p$.
(2) (*) Let G be a not necessarily reductive algebraic group. Show that 2-shfited symplectic structures on $\mathrm{B}G$ coincide with nondegenerate G-invariant symmetric bilinear pairings on $\mathfrak{g}$.
(3) Let G be a reductive algebraic group. Write down a representative of the Liouville one-form λ in the Cartan model of equivariant cohomology $\mathrm{DR}^{Cartan}(\mathrm{B}G)$. Check explicitly that $\mathrm{d}_{\mathrm{dR}}\lambda$ is nondegenerate.
(4) Prove theorem 2.21.

3. Shifted Poisson structures in representation theory

3.1. **Shifted Poisson structures.** Let X be a smooth scheme. Consider the algebra of polyvector fields $\Gamma(X, \wedge^\bullet \mathrm{T}_X)$; we will call the grading on the exterior algebra as the ***weight*** grading by analogy with differential forms. This vector space has a Lie bracket (the ***Schouten bracket***) which has weight -1, i.e. the Lie bracket of a p-vector and a q-vector is a $(p+q-1)$-vector. Recall that a Poisson structure on X is a bivector $\pi \in \Gamma(X, \wedge^2 \mathrm{T}_X)$ such that $[\pi, \pi] = 0$.

Let us also recall that a symplectic structure on X is an example of a Poisson structure: given a nondegenerate Poisson structure, i.e. one which induces an isomorphism $\pi^\sharp\colon \mathrm{T}^*_X \to \mathrm{T}_X$, there is a unique symplectic structure ω such that $\omega^\sharp\colon \mathrm{T}_X \to \mathrm{T}^*_X$ is inverse to $\pi^\sharp$.

(Shifted) Poisson structures on stacks are defined in a similar way. We will only give a flavor of the definition of shifted Poisson structures without going into all the details (the precise definition is developed in [Cal+17] and [Pri17]). Let X be a derived Artin stack locally of finite presentation with the tangent complex $\mathbb{T}_X \in \mathrm{Perf}(X)$. Consider the algebra of *n-**shifted polyvector fields***

$$\mathrm{Pol}(X, n) = \Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-n-1])).$$

As for differential forms, it has two gradings:

- Weight grading such that $\mathbb{T}_X$ has weight 1.
- Internal cohomological grading.

Example 3.1. Let X be a smooth affine scheme and consider $n = 0$. Then we may identify $\mathrm{Sym}^n(\mathrm{T}_X[-1]) \cong \wedge^n \mathrm{T}_X[-n]$, so the two gradings coincide and the algebra $\mathrm{Pol}(X, 0)$ coincides with the usual algebra of polyvector fields.

One may define an analog of the Schouten bracket on $\mathrm{Pol}(X, n)$ which has weight -1 and degree $-n-1$.

Definition 3.2. Let X be a derived Artin stack locally of finite presentation. An ***n-shifted Poisson structure*** on X is a formal power series $\pi = \pi_2 + \pi_3 + \dots$, where π_k is an element of $\mathrm{Pol}(X, n)$ of weight k and degree $n+2$ such that π satisfies the Maurer–Cartan equation $\mathrm{d}\pi + \frac{1}{2}[\pi, \pi] = 0$.

Remark 3.3. We may split the Maurer–Cartan equation according to weights as follows:

$$\mathrm{d}\pi_2 = 0$$
$$\mathrm{d}\pi_3 + \frac{1}{2}[\pi_2, \pi_2] = 0$$
$$\dots$$

In other words, π_2 is a d-closed bivector, such that $[\pi_2, \pi_2]$ is homotopic to zero in a coherent way.

Instead of using a precise definition, in these notes we will only use some properties that they have:

- If X is a smooth scheme, a 0-shifted Poisson structure on X is the same as an ordinary Poisson structure.
- If X is a derived prestack, an n-shifted Poisson structure on X has an underlying d-closed bivector $\pi \in \Gamma(X, \mathrm{Sym}^2(\mathbb{T}_X[-n-1]))[n+2]$ which induces a chain map $\pi^\sharp \colon \mathbb{L}_X \to \mathbb{T}_X[-n]$.
- If X has an n-shifted symplectic structure ω, it also has an n-shifted Poisson structure π such that $\pi^\sharp \colon \mathbb{L}_X \to \mathbb{T}_X[-n]$ is inverse to $\omega^\sharp \colon \mathbb{T}_X \to \mathbb{L}_X[n]$.
- If $f \colon L \to X$ has an n-shifted Lagrangian structure, there is an $(n-1)$-shifted Poisson structure on L. The underlying bivector can be extracted as follows. Recall that an n-shifted Lagrangian structure gives rise to a fiber sequence (4)
$$\mathbb{T}_L \longrightarrow f^*\mathbb{T}_X \longrightarrow \mathbb{L}_L[n].$$
The connecting homomorphism gives rise to a map $\pi^\sharp \colon \mathbb{L}_L \to \mathbb{T}_L[1-n]$ which is the underlying bivector of the $(n-1)$-shifted Poisson structure. The symplectic leaves of the $(n-1)$-shifted Poisson structure on L are given by the fibers of $L \to X$.
  We can relate this property to the previous property as follows. Recall from example 2.20 that an n-shifted Lagrangian structure on $p \colon X \to \mathrm{pt}$ is the same as an $(n-1)$-shifted symplectic structure on X. The underlying $(n-1)$-shifted Poisson structure on this Lagrangian is then inverse to the $(n-1)$-shifted symplectic structure on X.

We will mostly consider examples of 0-shifted Poisson structures on smooth schemes, so these properties will be enough to determine everything.

Example 3.4. Let us begin with the simplest example where we use these properties. Let G be an affine algebraic group and recall from example 2.15 that $\mathfrak{g}^*/G \cong \mathrm{T}^*[1](\mathrm{B}G)$ carries a 1-shifted symplectic structure. Taking the 1-shifted conormal stack of the projection $\mathrm{pt} \to \mathrm{B}G$ by example 2.25 we obtain a 1-shifted Lagrangian structure on $\mathrm{N}^*[1](\mathrm{pt}/\mathrm{B}G) \to \mathrm{T}^*[1](\mathrm{B}G)$ which can be identified with the projection $p \colon \mathfrak{g}^* \to \mathfrak{g}^*/G$. Therefore, there is an underlying Poisson structure on $\mathfrak{g}^*$ which we are going to compute.

Recall that $\mathrm{QCoh}(\mathfrak{g}^*/G)$ is equivalent to the ∞-category of complexes of G-equivariant quasi-coherent sheaves on $\mathfrak{g}^*$. We may identify the tangent complex of $\mathfrak{g}^*/G$ with the two-term complex

$$\mathbb{T}_{\mathfrak{g}^*/G} = (\mathfrak{g}\otimes \mathcal{O}_{\mathfrak{g}^*} \longrightarrow \mathfrak{g}^*\otimes \mathcal{O}_{\mathfrak{g}^*}).$$

The differential sends $v\in\mathfrak{g}$ at a point $x\in\mathfrak{g}^*$ to $\mathrm{coad}_v(x)$, the coadjoint action of v on x.

For a vector space V let us denote by $\underline{V} = V\otimes\mathcal{O}_{\mathfrak{g}^*}$ the corresponding trivial vector bundle on $\mathfrak{g}^*$. The fiber sequence (4) associated to the 1-shifted Lagrangian $\mathfrak{g}^*\to\mathfrak{g}^*/G$ is

$$\underline{\mathfrak{g}}^* \longrightarrow (\underline{\mathfrak{g}}\to\underline{\mathfrak{g}}^*) \longrightarrow \underline{\mathfrak{g}}[1],$$

where both maps are the obvious projections.

The connecting homomorphism is computed as follows. Fix an element $v\in\mathfrak{g}$ lying over a point $x\in\mathfrak{g}^*$. It can be lifted to the element $(v,0)\in(\underline{\mathfrak{g}}[1]\oplus\underline{\mathfrak{g}}^*)$. We have $\mathrm{d}(v,0) = (0,\mathrm{coad}_v(x))$ which comes from the element $\mathrm{coad}_v(x)\in\mathfrak{g}^*$. Thus,

$$\pi^\sharp\colon \mathfrak{g}\otimes\mathcal{O}_{\mathfrak{g}^*}\longrightarrow \mathfrak{g}^*\otimes\mathcal{O}_{\mathfrak{g}^*}$$

is given by $\pi^\sharp_x(v) = \mathrm{coad}_v(x)$.

Recall that $\mathfrak{g}^*$ has the Kirillov–Kostant–Souriau Poisson structure which is uniquely specified on linear functions on $\mathfrak{g}^*$, i.e. on $\mathfrak{g}$, to be the Lie bracket. It is not difficult to see that the Poisson structure we have computed above is the same as the Kirillov–Kostant–Souriau one. Moreover, from the general construction we know that the symplectic leaves of the Poisson structure on $\mathfrak{g}^*$ are given by the fibers of $\mathfrak{g}^*\to\mathfrak{g}^*/G$ as expected.

3.2. **Springer resolution.** Let G be a complex semisimple group. We refer to [CG10, Chapter 3] for some ideas explained here (see also [BN13] and [Saf17b] for the derived perspective).

Definition 3.5. The ***Grothendieck–Springer resolution*** $\tilde{\mathfrak{g}}$ is the variety parametrizing Borel subgroups $B\subset G$ equipped with an element $x\in\mathrm{Lie}(B)$.

We have a natural G-action on $\tilde{\mathfrak{g}}$ given by $g(B,x) = (gBg^{-1},\mathrm{Ad}_g(x))$ and there is a natural G-equivariant projection $\tilde{\mathfrak{g}}\to\mathfrak{g}$ given by $(B,x)\mapsto x$.

Let $\mathcal{N}\subset\mathfrak{g}$ be the nilpotent cone, i.e. the subvariety of nilpotent elements of $\mathfrak{g}$.

Definition 3.6. The ***Springer resolution*** $\tilde{\mathcal{N}}\to\mathcal{N}$ is the variety parametrizing Borel subgroups $B\subset G$ with unipotent radical $N\subset B$ and an element $x\in\mathrm{Lie}(N)$.

Let us fix a Borel subgroup $B\subset G$ with unipotent radical $N\subset B$ and let $H = B/[B,B]$ be the maximal torus. We get a natural correspondence capturing parabolic induction

$$\mathrm{B}G \longleftarrow \mathrm{B}B \longrightarrow \mathrm{B}H$$

Lemma 3.7. *The 1-shifted conormal stack of* $\mathrm{B}B\to\mathrm{B}G\times\mathrm{B}H$ *is equivalent to* $\tilde{\mathfrak{g}}/G$.

Proof. We have an exact sequence of B-representations

$$0\longrightarrow\mathfrak{b}\longrightarrow\mathfrak{g}\oplus\mathfrak{h}\longrightarrow\mathfrak{b}^*\longrightarrow 0,$$

so $\mathrm{N}^*[1](\mathrm{B}B/\mathrm{B}G\times\mathrm{B}H)\cong\mathfrak{b}/B$. But we may identify $\tilde{\mathfrak{g}}\cong G\times_B\mathfrak{b}$ which gives the result. ☐

Remark 3.8. In a similar way, the 1-shifted conormal stack of $\mathrm{B}B \to \mathrm{B}G$ is equivalent to $\tilde{\mathcal{N}}/G$.

Thus, taking the 1-shifted conormal stack we obtain a 1-shifted Lagrangian correspondence

$$\begin{array}{ccc} & \tilde{\mathfrak{g}}/G & \\ \swarrow & & \searrow \\ \mathfrak{g}^*/G & & \mathfrak{h}^*/H \end{array} \tag{6}$$

This correspondence allows one to turn 1-shifted Lagrangians in $\mathfrak{g}^*/G$ into 1-shifted Lagrangians in $\mathfrak{h}^*/H$ by sending $L \to \mathfrak{g}^*/G$ to

$$L \times_{\mathfrak{g}^*/G} \tilde{\mathfrak{g}}/G \longrightarrow \mathfrak{h}^*/H.$$

Recall that by theorem 2.21 Lagrangians in $\mathfrak{g}^*/G$ are the same as Hamiltonian G-varieties. The above procedure of turning a Hamiltonian G-variety into a Hamiltonian H-variety is known as symplectic implosion [DKS13].

Composing the correspondence (6) with its opposite we obtain a correspondence

$$\begin{array}{ccc} & (\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})/G & \\ \swarrow & & \searrow \\ \mathfrak{h}^*/H & & \mathfrak{h}^*/H \end{array}$$

Its image in $\mathfrak{h}^* \times \mathfrak{h}^*$ is given by the graph of the Weyl group action. In particular, restricting to $(0,0) \in \mathfrak{h}^* \times \mathfrak{h}^*$ (equivalently, taking the Hamiltonian reduction by H), we obtain a 0-shifted symplectic stack St/G, where $\mathrm{St} = \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$ is known as the ***Steinberg variety***.

3.3. **Manin triples.** In this section we interpret Manin triples in terms of shifted symplectic structures.

Definition 3.9. A ***Manin triple*** is a triple (D, G, G^*), where D is an algebraic group, $\mathfrak{d} = \mathrm{Lie}(D)$ is equipped with a nondegenerate D-invariant symmetric bilinear pairing $(-,-)$, $G, G^* \subset D$ are subgroups such that $\mathfrak{g} = \mathrm{Lie}(G), \mathfrak{g}^* = \mathrm{Lie}(G^*) \subset \mathfrak{d}$ are Lagrangian and $\mathfrak{g} \cap \mathfrak{g}^* = 0$.

Remark 3.10. Usually a Manin triple is defined as a triple of Lie algebras as above.

Note that the complementarity condition implies that the map $\mathrm{Lie}(G^*) \to \mathrm{Lie}(G)^*$ given by $v \in \mathrm{Lie}(G^*) \mapsto (w \in \mathrm{Lie}(G)^* \mapsto (w, v))$ is an isomorphism. In other words, $\mathrm{Lie}(G^*)$ is canonically dual to $\mathrm{Lie}(G)$ which explains the notation.

Since $\mathfrak{g}^*$ has a Lie bracket, $\mathfrak{g}$ has a Lie cobracket. Moreover, it is a standard fact that it is compatible with the original Lie bracket on $\mathfrak{g}$, so that $\mathfrak{g}$ in fact becomes a Lie bialgebra.

Example 3.11. Let G be a complex semisimple group, $B_+, B_- \subset G$ are two opposite Borel subgroups and $H = B_+ \cap B_-$ the maximal torus. Denote by $p_\pm \colon B_\pm \to H$ the abelianization maps. Consider $D = G \times G$ with the pairing on its Lie algebra given by

$$((x_1, y_1), (x_2, y_2)) = (x_1, y_2)_{\mathfrak{g}} + (y_1, x_2)_{\mathfrak{g}},$$

for $x_i, y_i \in \mathfrak{g}$ and where we denote by $(-,-)_{\mathfrak{g}}$ the Killing form on $\mathfrak{g}$. Then the diagonal subgroup $G \subset D$ is clearly Lagrangian. We define the subgroup $G^* \subset D$ as

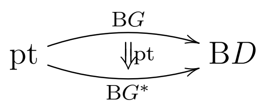
$$G^* = \{(b_+, b_-) \in B_+ \times B_- \mid p_+(b_+)p_-(b_-) = e\}.$$

This Manin triple defines the so-called standard Lie bialgebra structure on the semisimple Lie algebra $\mathfrak{g}$.

Recall from example 2.14 that the nondegenerate pairing on $\mathfrak{d}$ gives rise to a 2-shifted symplectic structure on $\mathrm{B}D$. Since $\mathfrak{g}, \mathfrak{g}^* \subset \mathfrak{d}$ are Lagrangians, we obtain natural 2-shifted Lagrangian structures on $\mathrm{B}G, \mathrm{B}G^* \to \mathrm{B}D$. Finally, complementarity condition may be encoded as follows. $\mathrm{B}G \times_{\mathrm{B}D} \mathrm{B}G^*$ is a Lagrangian intersection, so by theorem 2.23 it has a 1-shifted symplectic structure. Then the complementarity condition is equivalent to the fact that the natural projection $\mathrm{pt} \to \mathrm{B}G \times_{\mathrm{B}D} \mathrm{B}G^*$ is 1-shifted Lagrangian.

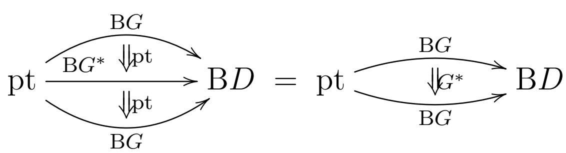
In terms of the 2-category of 2-shifted Lagrangian correspondences, we have encoded a Manin triple into a 2-morhpism

(7)

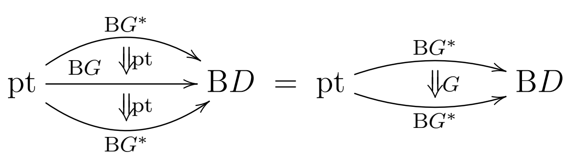


We may construct some new 2-morphisms in the following way.

(1) Let us compose the 2-morphism (7) with its opposite. We get

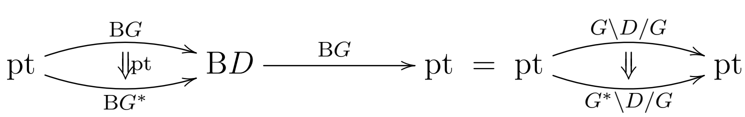


and



Thus, we obtain 1-shifted Lagrangian maps $G^* \to \mathrm{B}G \times_{\mathrm{B}D} \mathrm{B}G \cong G\backslash D/G$ and $G \to \mathrm{B}G^* \times_{\mathrm{B}D} \mathrm{B}G^* \cong G^*\backslash D/G^*$. For instance, we get a Poisson structure on G whose symplectic leaves are given parametrized by $G^*\backslash D/G^*$. From the construction one may moreover see that it is compatible with the group structure on G, i.e. it defines a Poisson-Lie structure on G. This is known as the ***Sklyanin Poisson structure***. The quantizations of G^* and G with respect to these Poisson structures is usually denoted by $\mathrm{U}_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$.

(2) We may whisker the 2-morphism (7) with $\mathrm{pt} \leftarrow \mathrm{B}G \to \mathrm{B}D$ or $\mathrm{pt} \leftarrow \mathrm{B}G^* \to \mathrm{B}D$. Then we get



and

$$\mathrm{pt} \underset{\mathrm{B}G^*}{\overset{\mathrm{B}G}{\underset{\Downarrow \mathrm{pt}}{\rightrightarrows}}} \mathrm{B}D \xrightarrow{\mathrm{B}G^*} \mathrm{pt} \;=\; \mathrm{pt} \underset{G^*\backslash D/G^*}{\overset{G\backslash D/G^*}{\underset{\Downarrow}{\rightrightarrows}}} \mathrm{pt}$$

So, we obtain 1-shifted Lagrangians $D/G \to G\backslash D/G \times G^*\backslash D/G$ and $D/G^* \to G\backslash D/G^* \times G^*\backslash D/G^*$. For instance, we get a Poisson structure on D/G known as the ***Semenov–Tian–Shansky*** Poisson structure whose symplectic leaves are parametrized by $G\backslash D/G \times G^*\backslash D/G$. The quantization of D/G with respect to this Poisson structure is given by the so-called reflection equation algebra [KS92].

Remark 3.12. In the case of the standard Manin triple $(G\times G, G, G^*)$ we obtain the Semenov–Tian–Shansky Poisson structure on $(G\times G)/G \cong G$. Note that it is incompatible with the multiplication on G.

3.4. **Feigin–Odesskii Poisson structures.** Let G be a complex semisimple group, $B\subset G$ a Borel subgroup and $H = B/[B,B]$ the maximal torus. The Killing form equips $\mathrm{B}G$ with a 2-shifted symplectic structure and the restriction of the Killing form to H equips $\mathrm{B}H$ with a 2-shifted symplectic structure. The exact sequence

$$0 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{g}^*\oplus\mathfrak{h}^* \longrightarrow \mathfrak{b}^* \longrightarrow 0$$

implies that we have a natural 2-shifted Lagrangian correspondence

$$\begin{array}{ccc} & \mathrm{B}B & \\ \swarrow & & \searrow \\ \mathrm{B}G & & \mathrm{B}H \end{array}$$

Now let E be an elliptic curve. Applying $\underline{\mathrm{Map}}(E,-)$ to the above correspondence by theorem 2.29 we obtain a 1-shifted Lagrangian correspondence

$$\begin{array}{ccc} & \mathrm{Bun}_B(E) & \\ \swarrow & & \searrow \\ \mathrm{Bun}_G(E) & & \mathrm{Bun}_H(E) \end{array}$$

Now fix an H-bundle $\mathcal{L}\to E$. Let $\mathrm{Bun}_B(E,\mathcal{L})$ be the fiber of $\mathrm{Bun}_B(E)$ at $\mathcal{L}\in\mathrm{Bun}_H(E)$. We obtain 1-shifted Lagrangian structures on the maps $\mathrm{Bun}_B(E)\to\mathrm{Bun}_G(E)\times\mathrm{Bun}_H(E)$ and $\mathrm{Bun}_B(E,\mathcal{L})\longrightarrow\mathrm{Bun}_G(E)$ and hence Poisson structures on $\mathrm{Bun}_B(E)$ and $\mathrm{Bun}_B(E,\mathcal{L})$. These Poisson structures are known as the ***Feigin–Odesskii Poisson structures***.

Remark 3.13. Explicitly, the Poisson structure on $\mathrm{Bun}_B(E)$ has the following description [FO98]. For a B-bundle $P\to E$ we have identifications

$$\mathbb{T}_{\mathrm{Bun}_B(E),P}\cong\Gamma(E,\mathrm{ad}P)[1],\qquad \mathbb{L}_{\mathrm{Bun}_B(E),P}\cong\Gamma(E,\mathrm{coad}P).$$

We have an exact sequence of B-representations

$$0 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{g}^*\oplus\mathfrak{h}^* \longrightarrow \mathfrak{b}^* \longrightarrow 0,$$

so the connecting homomorphism. In particular, the connecting homomorphism gives rise to a morphism $\mathfrak{b}^* \to \mathfrak{b}[1]$ in the derived category of B-representations. Therefore, we get a map

$$\Gamma(E, P \times_B \mathfrak{b}^*) \longrightarrow \Gamma(E, P \times_B \mathfrak{b}[1])$$

which is the Poisson bivector $\pi^\sharp\colon \mathbb{L}_{\mathrm{Bun}_B(E),P} \to \mathbb{T}_{\mathrm{Bun}_B(E),P}$.

Example 3.14. Let $G = \mathrm{PGL}_2$, so that an H-bundle is a line bundle $\mathcal{L} \to E$ and a B-bundle is an extension

$$0 \longrightarrow \mathcal{L} \longrightarrow V \longrightarrow \mathcal{O} \longrightarrow 0.$$

If $\deg(\mathcal{L}) \leq 0$, we may identify

$$\mathrm{Bun}_B(E, \mathcal{L}) \cong \mathrm{H}^0(E, \mathcal{L}^*)/\mathbf{G}_m.$$

In particular, removing the origin we obtain a Poisson structure on $\mathbf{P}(\mathrm{H}^0(E, \mathcal{L}^*))$. By the Riemann–Roch theorem we have $\dim \mathrm{H}^0(E, \mathcal{L}^*) = -\deg(\mathcal{L})$. In particular, for $\deg(\mathcal{L}) = -3$ we obtain a Poisson structure on $\mathbf{P}^2$ and for $\deg(\mathcal{L}) = -4$ we obtain a Poisson structure on $\mathbf{P}^3$. These two Poisson structures were previously defined by Sklyanin [Skl82] and their quantizations give rise to the 3- and 4-dimensional Sklyanin algebras.

3.5. **Exercises.**

(1) Let G be a complex semisimple group and consider the standard Manin triple $(G \times G, G, G^*)$. Find the symplectic leaves of G with respect to the Sklyanin Poisson structure with double Bruhat cells.
(2) Let G be a complex semisimple group, $B_+, B_- \subset G$ a pair of opposite Borel subgroups and $H = B_+ \cap B_-$. Let $N_+ \subset B_+$ be the unipotent radical. Find the symplectic leaves of the Semenov–Tian–Shansky Poisson structure on G/N_+ with respect to the Manin triple $(G \times H, B_+, B_-)$.
(3) Classify Poisson structures on $\mathbf{P}^2$ with a smooth degeneracy locus and relate them to the Sklyanin Poisson structure.

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