Asymptotic Behavior of a Critical Fluid Model for a Processor Sharing Queue via Relative Entropy

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Joint work with Ruth J. Williams
Processor Sharing (PS) Queue
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- Each job in system simultaneously served at rate $\frac{1}{\# \text{ jobs in the system}}$. 
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- Idealized model for computer time-sharing algorithms introduced by Kleinrock in ‘60’s.
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  \]
- Idealized model for computer time-sharing algorithms introduced by Kleinrock in ‘60’s.
- Until early 2000’s, only analyzed under restrictive distributional assumptions.
GI/GI/1 PS Queue
• **Initial condition:**
  
  # jobs in the system a time $0$, each with strictly positive residual service time
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Arrivals:  
rate $\alpha$ delayed renewal process
**GI/GI/1 PS Queue**

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  - # jobs in the system at time 0, each with strictly positive residual service time

- **Arrivals:**
  - rate $\alpha$ delayed renewal process

- **Service times:**
  - strictly positive, i.i.d. with distribution $\nu$
GI/GI/1 PS Queue

$\alpha$

$\nu$
Residual service times:
For each job in the system at time $t$, the residual service time at time $t$ is the amount of processing time remaining at $t$. 
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Infinite dimensional system:
Must track all residual service times.
For each Borel set $A \subset [0, \infty)$,

$$\langle 1_A, Z(t) \rangle = \# \text{ jobs in the system at time } t$$

with residual service time in $A$. 
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$Z(\cdot)$ is an r.c.l.l. stochastic process taking values in the set of finite, nonnegative Borel measures $M$ on $[0, \infty)$. 
State Descriptor for GI/GI/1 PS Queue

For each Borel set $A \subset [0, \infty)$,

$$\langle 1_A, \mathcal{Z}(t) \rangle = \# \text{ jobs in the system at time } t$$

with residual service time in $A$.

$\mathcal{Z}(\cdot)$ is an r.c.l.l. stochastic process taking values in the set of finite, nonnegative Borel measures $\mathcal{M}$ on $[0, \infty)$. $\mathcal{M}$ endowed with the topology of weak convergence is a Polish space metrizable by the Prokhorov metric $d$. 
Observe that

\[ Q(t) \equiv \langle 1, \mathcal{Z}(t) \rangle = \# \text{ jobs in system at time } t, \]

\[ W(t) \equiv \langle \chi, \mathcal{Z}(t) \rangle = \text{ immediate workload at time } t, \]

where \( \chi(x) = x, \ x \in [0, \infty) \).
Survey

Yashkov ’87 (mostly with restrictive distributional assumptions)
PS Queues Literature

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More Recent (mostly with general distributions)

Baccelli & Towsley ‘90 (correlation of sojourn times)

Grishechkin ‘94 (heavy traffic steady-state asymptotics)

Jean-Marie & Robert ‘94 (transient, overloaded queue)

Chen, Kella & Weiss ‘97 (fluid limits for queue length)

Lambert, Simatos & Zwart ‘13 (diffusion limits via regeneration,M/G/1)
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Limit Theorems via a Modular Approach

Gromoll, Puha & Williams ‘02 (fluid limit)
Puha & Williams ‘04 (analysis of critical fluid model solutions)
Gromoll ‘04 (diffusion limit)
Outline

1. Critical Fluid Model Solution (CMFS)
   a) Definition
   b) Existence & Uniqueness
   c) Invariant States
2. Statement of the Main Result in PW ‘16
3. Proof Strategy via Relative Entropy Arguments
4. Statement of Main Technical Result in PW ‘16
5. Proof of the Main Technical Result
Critical Fluid Model (GPW ‘02)

**Model inputs:** critical data \((\alpha, \nu)\)

\(\alpha \in (0, \infty)\) is the arrival rate of fluid

\(\nu\) is a Borel probability measure on \([0, \infty)\) by which the fluid is distributed as it enters the system such that

\[ \nu(\{0\}) = 0 \quad \text{and} \quad \rho = \alpha \langle \chi, \nu \rangle = 1. \]
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\[\nu(\{0\}) = 0\quad \text{and} \quad \rho = \alpha \langle \chi, \nu \rangle = 1.\]

Initial Condition: \(\xi \in \mathcal{M}\)

\(\xi\) is a finite, nonnegative Borel measure on \([0, \infty)\) that gives the initial distribution of fluid
A Fluid Model Solution for the critical data \((\alpha, \nu)\) and initial condition \(\xi \in M\) is a function \(\zeta : [0, \infty) \to M\) with \(\zeta(0) = \xi\) that is continuous, does not charge the origin, and
A Fluid Model Solution for the critical data $(\alpha, \nu)$ and initial condition $\xi \in \mathcal{M}$ is a function $\zeta : [0, \infty) \to \mathcal{M}$ with $\zeta(0) = \xi$ that is continuous, does not charge the origin, and

for all $g \in C^1_b$ with $g(0) = 0$ and $g'(0) = 0$, satisfies

$$\langle g, \zeta(t) \rangle = \langle g, \xi \rangle + \alpha t \langle g, \nu \rangle - \int_0^t \frac{\langle g', \zeta(u) \rangle}{\langle 1, \zeta(u) \rangle} du,$$

for $0 \leq t < t^* = \inf\{u : \langle 1, \zeta(u) \rangle = 0\}$, and

$$\zeta(t) = 0, \quad \text{for } t \geq t^*.$$
Existence and Uniqueness of CFMS

Let $K$ be the set of continuous measures in $M$:

$$K = \{ \eta \in M : \eta(\{x\}) = 0 \text{ for all } x \in [0, \infty) \}.$$
Existence and Uniqueness of CFMS

Let $K$ be the set of continuous measures in $M$:

$$K = \{ \eta \in M : \eta(\{x\}) = 0 \text{ for all } x \in [0, \infty) \}.$$ 

**Theorem (GPW '02).**

*Given critical data $(\alpha, \nu)$ and $\xi \in K$, there exists a unique fluid model solution $\zeta^\xi$ for the data $(\alpha, \nu)$ such that $\zeta^\xi(0) = \xi$.***
Invariant States for CFMS

**Definition.** Given critical data \((\alpha, \nu)\), \(\xi \in \mathbb{K}\) is an invariant state if \(\zeta^\xi(t) = \xi\) for all \(t \geq 0\).
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**Definition.** Given critical data $(\alpha, \nu)$, $\xi \in K$ is an invariant state if $\zeta^\xi(t) = \xi$ for all $t \geq 0$.

**Definition.** Given $\eta \in M$ such that $0 < \langle \chi, \eta \rangle < \infty$, the associated excess life probability measure $\eta_e$ is the probability measure with density $f_e$ given by

$$f_e(x) = \frac{\langle 1_{(x,\infty)}, \eta \rangle}{\langle \chi, \eta \rangle}, \quad \text{for } x \in [0, \infty).$$
Invariant States for CFMS

**Definition.** Given critical data $(\alpha, \nu)$, $\xi \in K$ is an invariant state if $\xi^\xi(t) = \xi$ for all $t \geq 0$.

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**Theorem (PW ‘04).** The set of invariant states $I$ for critical data $(\alpha, \nu)$ is given by

$$I = \{ c\nu_e : c \in [0, \infty) \}.$$
Main Result

Given critical data \((\alpha, \nu)\) and \(u, l > 0\), let

\[
M_{u,l} = \{ \eta \in M : l \leq \langle \chi, \eta \rangle \text{ and } \langle 1_{(x, \infty)}, \eta \rangle \leq u \langle 1_{(x, \infty)}, \nu_e \rangle \text{ for all } x \in [0, \infty) \},
\]

and set \(K_{u,l} = K \cap M_{u,l}\).
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and set \(K_{u,l} = K \cap M_{u,l}\).

**Theorem 3.1 (PW ‘16).** Let \((\alpha, \nu)\) be critical data such that \(\langle \chi^2, \nu \rangle < \infty\) and \(u, l > 0\). Then

\[
\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\xi(t), I) = 0.
\]
Relative Entropy

For absolutely continuous Borel probability measures $\eta$ and $\gamma$ on $\mathbb{R}_+$ with densities $f$ and $g$,

$$E(\eta, \gamma) = \int_{0}^{\infty} f(x) \ln \left( \frac{f(x)}{g(x)} \right) \, dx.$$  

By convention, $0 \ln 0 = 0$ and $y \ln(y/0) = \infty$ for $y > 0$. 
Relative Entropy

For absolutely continuous Borel probability measures \( \eta \) and \( \gamma \) on \( \mathbb{R}_+ \) with densities \( f \) and \( g \),

\[
\mathcal{E}(\eta, \gamma) = \int_0^\infty f(x) \ln \left( \frac{f(x)}{g(x)} \right) \, dx.
\]

By convention, \( 0 \ln 0 = 0 \) and \( y \ln(y/0) = \infty \) for \( y > 0 \).

Relative entropy is not a metric, but

1. \( \mathcal{E}(\eta, \gamma) = 0 \) if and only if \( \eta = \gamma \), and

2. \( d(\eta, \gamma) \leq \sqrt{\frac{\mathcal{E}(\eta, \gamma)}{2}} \).
Recall $I = \{ c\nu_e : c \in [0, \infty) \}$.
Recall $\mathcal{I} = \{c\nu_e : c \in [0, \infty)\}$.

Since $\nu_e$ is absolutely continuous, all invariant states are absolutely continuous.
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The value $\zeta(t)$ of a fluid model solution $\zeta$ at time $t$ is not necessarily absolutely continuous.
Recall \( \mathbf{I} = \{c\nu_e : c \in [0, \infty)\} \).

Since \( \nu_e \) is absolutely continuous, all invariant states are absolutely continuous.

The value \( \zeta(t) \) of a fluid model solution \( \zeta \) at time \( t \) is not necessarily absolutely continuous.

Hence, it is possible that for all \( t \geq 0 \),

\[
\mathcal{E} \left( \frac{\zeta(t)}{\langle 1, \zeta(t) \rangle}, \nu_e \right) = \infty.
\]
Key Idea

Fix critical data \((\alpha, \nu)\) such that \(\langle \chi^2, \nu \rangle < \infty\).
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**Key Idea**

Fix critical data \((\alpha, \nu)\) such that \(\langle \chi^2, \nu \rangle < \infty\).

Recall that \(I = \{c\nu_e : c \in [0, \infty)\}\).

Note that, for \(c > 0\), \((c(\nu_e))_e = (\nu_e)_e\).
Fix critical data \((\alpha, \nu)\) such that \(\langle \chi^2, \nu \rangle < \infty\).

Recall that \(I = \{cv_e : c \in [0, \infty)\}\).

Note that, for \(c > 0\), \((c(\nu_e))_e = (\nu_e)_e\).

Given \(\eta \in M\) such that \(0 < \langle \chi, \eta \rangle < \infty\), let

\[H(\eta) = \mathcal{E}(\eta_e, (\nu_e)_e)\]
Fix critical data \((\alpha, \nu)\) such that \(\langle \chi^2, \nu \rangle < \infty\).

Recall that \(\mathbf{I} = \{c\nu_e : c \in [0, \infty)\}\).

Note that, for \(c > 0\), \((c\nu_e)_e = (\nu_e)_e\).

Given \(\eta \in \mathbf{M}\) such that \(0 < \langle \chi, \eta \rangle < \infty\), let

\[
H(\eta) = \mathcal{E}(\eta_e, (\nu_e)_e).
\]

Then, given \(\xi \in \mathbf{K}\) such that \(0 < \langle \chi, \xi \rangle < \infty\), let

\[
\mathcal{H}_\xi(t) = H(\zeta_\xi^\xi(t)) = \mathcal{E}(\zeta_\xi^\xi(t), (\nu_e)_e), \quad \text{for } t \geq 0.
\]
Strategy for Proving the Main Result

Show:

$\mathcal{H}_\xi(t) \to 0$ uniformly as $t \to \infty$ on $K_{u,l}$ for any $u, l > 0$. 
Strategy for Proving the Main Result

Show:
\( \mathcal{H}_\xi(t) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).

Immediate Conclusion:
\( d(\xi(t), (\nu_e)_e) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).
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\( \mathcal{H}_\xi(t) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).

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\( d(\zeta_e^\xi(t), (\nu_e)_e) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).

Desired Conclusion:
\( d(\zeta^\xi(t), I) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).
Strategy for Proving the Main Result

Show:
\( \mathcal{H}_\xi(t) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).

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Desired Conclusion:
\( d(\zeta^\xi(t), I) \to 0 \) uniformly as \( t \to \infty \) on \( K_{u,l} \) for any \( u, l > 0 \).

Final Step:
Show that the Desired Conclusion follows.
Theorem 3.2 (PW ‘16). Let $(\alpha, \nu)$ be critical data such that $\langle \chi^2, \nu \rangle < \infty$ and let $u, l > 0$. For each $\xi \in K_{u,l}$, $\mathcal{H}_\xi$ is nonincreasing. Furthermore,

$$\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} \mathcal{H}_\xi(t) = 0.$$ 

Recall $\mathcal{H}_\xi(t) = \mathcal{E}(\zeta_\xi^e(t), (\nu_e)_e)$ for $t \geq 0$ and $\xi \in K_{u,l}$. 
Absolute Continuity of $\mathcal{H}_\xi$

**Theorem 7.1** (PW ‘16). Let $(\alpha, \nu)$ be critical data such that $\langle \chi^2, \nu \rangle < \infty$ and let $u, l > 0$. For each $\xi \in \mathcal{K}_{u,l}$, there exists a continuous function $\kappa_\xi : [0, \infty) \to (-\infty, 0]$ such that for all $0 \leq s < t < \infty$,

$$\mathcal{H}_\xi(t) - \mathcal{H}_\xi(s) = \int_s^t \kappa_\xi(u)du,$$

and $\kappa_\xi(u) = 0$ if and only if $\xi^\xi(u) \in I$. 
Absolute Continuity of $\mathcal{H}_\xi$

**Theorem 7.1 (PW ‘16).** Let $(\alpha, \nu)$ be critical data such that $\langle \chi^2, \nu \rangle < \infty$ and let $u, l > 0$. For each $\xi \in \mathbf{K}_{u, l}$, there exists a continuous function $\kappa_\xi : [0, \infty) \to (-\infty, 0]$ such that for all $0 \leq s < t < \infty$,

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**Proof Technique.** We compute $\kappa_\xi$ explicitly.
Absolute Continuity of $\mathcal{H}_\xi$

**Theorem 7.1 (PW '16).** Let $(\alpha, \nu)$ be critical data such that $\langle \chi^2, \nu \rangle < \infty$ and let $u, l > 0$. For each $\xi \in K_{u,l}$, there exists a continuous function $\kappa_\xi : [0, \infty) \to (-\infty, 0]$ such that for all $0 \leq s < t < \infty$,

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and $\kappa_\xi(u) = 0$ if and only if $\zeta_\xi(u) \in I$.

**Proof Technique.** We compute $\kappa_\xi$ explicitly.

Henceforth, $u, l > 0$ and critical data $(\alpha, \nu)$ such that $\langle \chi^2, \nu \rangle < \infty$ are fixed.
For $x \in (0, \infty)$, set $k(x) = x - 1 - \ln(x)$ and set $k(0) = \infty$. 
Let $\xi \in \mathbf{K}_{u,l}$. For $t, x \in [0, \infty)$, set

\[
q^{\xi}(t) = \langle 1, \xi(t) \rangle, \\
q^{\xi}(t, x) = \langle 1_{(x, \infty)}, \xi(t) \rangle, \\
\mathcal{N}_e(x) = \langle 1_{(x, \infty)}, \nu_e \rangle.
\]
An Explicit Expression for $\kappa_\xi$

Let $\xi \in K_{u,l}$. For $t, x \in [0, \infty)$, set

$$q_\xi(t) = \langle 1, \zeta_\xi(t) \rangle,$$
$$\overline{q}_\xi(t, x) = \langle 1_{(x, \infty)}, \zeta_\xi(t) \rangle,$$
$$\overline{N}_e(x) = \langle 1_{(x, \infty)}, \nu_e \rangle.$$

Then, for $t > 0$,

$$\kappa_\xi(t) = \frac{-1}{\langle \chi, \xi \rangle} \mathbb{E}_{\nu_e} \left[ k \left( \frac{\overline{q}_\xi(t, X)}{q_\xi(t) \overline{N}_e(X)} \right) \right].$$
Corollary 7.1 (PW ‘16)

Let \( \xi \in K \). Suppose that

- \( \nu \) does not have atoms, and
- \( \xi \) is nonzero and has a continuous density.
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Let $\xi \in K$. Suppose that

- $\nu$ does not have atoms, and
- $\xi$ is nonzero and has a continuous density.

Then for all $(t, x) \in [0, \infty)^2$,

$$
\frac{\partial}{\partial t} q_\xi(t, x) = \alpha \langle 1_{(x, \infty)}, \nu \rangle - \frac{\partial}{\partial x} \frac{q_\xi(t, x)}{q_\xi(t)}.
$$
Corollary 7.1 *(PW ‘16)*

Let \( \xi \in \mathbf{K} \). Suppose that

- \( \nu \) does not have atoms, and
- \( \xi \) is nonzero and has a continuous density.

Then for all \((t, x) \in [0, \infty)^2\),

\[
\frac{\partial}{\partial t} \bar{q}^\xi(t, x) = \alpha \langle 1_{(x, \infty)}, \nu \rangle - \frac{\partial}{\partial x} \frac{\bar{q}^\xi(t, x)}{q^\xi(t)}.
\]

**Remark.** Used by Paganini et. al. ‘12 to study stability properties of subcritical Bandwidth sharing models.
Prf of Theorem 7.1: Absolute Continuity of $\mathcal{H}_\xi$

1. Verify that $\kappa_\xi$ is finite and continuous.

2. Restrict to absolutely continuous $\xi \in K_{u,l}$.
   a) Prove that a weak formulation of the PDE holds.
   b) Use integration-by-parts together with the weak formulation of the PDE and other identities to verify that $\kappa_\xi$ is the density of $\mathcal{H}_\xi$.

3. Use approximation arguments to extend to $\xi \in K_{u,l}$. 
Fix $u, l, T, \varepsilon > 0$. We show that there exist

1. $B > 0$ such that $\mathcal{H}_\xi(t) \leq B$ for all $t \geq 0$ and $\xi \in K_{u,l}$. 

Prf of Theorem 3.2: $\mathcal{H}_\xi(t) \downarrow 0$ uniformly on $K_{u,l}$. 
Prf of Theorem 3.2: $\mathcal{H}_\xi(t) \searrow 0$ uniformly on $K_{u,l}$.

Fix $u, l, T, \varepsilon > 0$. We show that there exist

1. $B > 0$ such that $\mathcal{H}_\xi(t) \leq B$ for all $t \geq 0$ and $\xi \in K_{u,l}$,

2. a compact set $M_{u,l,T}$ that does not contain the zero measure and such that for all $\xi \in K_{u,l}$, $\zeta_\xi(t) \in M_{u,l,T}$ for all $t \geq T$. 
Fix $u, l, T, \varepsilon > 0$. We show that there exist

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2. a compact set $M_{u,l,T}$ that does not contain the zero measure and such that for all $\xi \in K_{u,l}$, $\zeta_\xi(t) \in M_{u,l,T}$ for all $t \geq T$.

3. $\delta > 0$ such that if $t \geq T$ and $H_\xi(t) \geq \varepsilon$, then $\kappa_\xi(t) \leq -\delta$. 

Prf of Theorem 3.2: $H_\xi(t) \downarrow 0$ uniformly on $K_{u,l}$. 
Prf of Theorem 3.2: $\mathcal{H}_\xi(t) \downarrow 0$ uniformly on $K_{u,l}$.

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1. $B > 0$ such that $\mathcal{H}_\xi(t) \leq B$ for all $t \geq 0$ and $\xi \in K_{u,l}$,

2. a compact set $M_{u,l,T}$ that does not contain the zero measure and such that for all $\xi \in K_{u,l}$, $\zeta^\xi(t) \in M_{u,l,T}$ for all $t \geq T$.

3. $\delta > 0$ such that if $t \geq T$ and $\mathcal{H}_\xi(t) \geq \varepsilon$, then $\kappa^\xi(t) \leq -\delta$.

It follows by monotonicity of $\mathcal{H}_\xi$ that $\mathcal{H}_\xi(t) < \varepsilon$ for all $t \geq T + B/\delta$. 
Recall that for $\eta \in \mathbf{M}$ such that $0 < \langle \chi, \eta \rangle < \infty$, 

$$H(\eta) = \mathcal{E}(\eta_e, (\nu_e)_e).$$
Prf of Main Result: Properties of $H$

Recall that for $\eta \in \mathcal{M}$ such that $0 < \langle \chi, \eta \rangle < \infty$,

$$H(\eta) = \mathcal{E}(\eta_e, (\nu_e)_e).$$

Let $J = \{ \eta \in \mathcal{M} : \eta = a\delta_0 + c\nu_e \text{ for some } a, c \in [0, \infty) \}$. 
Recall that for $\eta \in M$ such that $0 < \langle \chi, \eta \rangle < \infty$, 

$$H(\eta) = \mathcal{E}(\eta_e, (\nu_e)_e).$$

Let $J = \{\eta \in M : \eta = a\delta_0 + c\nu_e \text{ for some } a, c \in [0, \infty)\}$. 

Note $I = \{\eta \in J : \langle 1_{\{0\}}, \eta \rangle = 0\}$, and $I \subset J$. 
Prf of Main Result: Properties of $H$

Recall that for $\eta \in \mathbb{M}$ such that $0 < \langle \chi, \eta \rangle < \infty$,

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Note $I = \{ \eta \in J : \langle 1_{\{0\}}, \eta \rangle = 0 \}$, and $I \subset J$.

Proposition.

1. For $\eta \in \mathbb{M}$ such that $0 < \langle \chi, \eta \rangle < \infty$, $H(\eta) = 0$ if and only if $\eta \in J$.

2. $H$ is continuous on $\mathbb{M}_{u,l}$. 
Corollary 6.1 \textit{(PW '16)}.

\[
\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\zeta^\xi(t), J) = 0.
\]
Corollary 6.1 \((PW \ '16)\).

\[
\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\zeta(\xi(t)), J) = 0.
\]

Pf Sketch. \(\exists u^*, l^* > 0 \text{ s.t. } \zeta(\xi(t)) \in K_{u^*, l^*} \text{ for all } t \geq 0.\)
Corollary 6.1 (PW ‘16).

\[
\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\xi(t), J) = 0.
\]

Pf Sketch. \( \exists u^*, l^* > 0 \) s.t. \( \xi(t) \in K_{u^*, l^*} \) for all \( t \geq 0 \).

By continuity of \( H \) and compactness of \( M_{u^*, l^*} \), given \( \varepsilon > 0 \), there exists \( \gamma > 0 \) such that

\[
\{ \eta \in M_{u^*, l^*} : d(\eta, J) \geq \varepsilon \} \subseteq \{ \eta \in M_{u^*, l^*} : H(\eta) \geq \gamma \}.
\]
Corollary 6.1 \((PW \text{ ‘}16)\).

\[
\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\zeta^\xi(t), J) = 0.
\]

**Pf Sketch.** \(\exists u^*, l^* > 0\ \text{s.t.} \ \zeta^\xi(t) \in K_{u^*,l^*} \text{ for all } t \geq 0.\)

By continuity of \(H\) and compactness of \(M_{u^*,l^*}\), given \(\varepsilon > 0\), there exists \(\gamma > 0\) such that

\[
\{ \eta \in M_{u^*,l^*} : d(\eta, J) \geq \varepsilon \} \subseteq \{ \eta \in M_{u^*,l^*} : H(\eta) \geq \gamma \}.
\]

By Theorem 3.2, \(H(\zeta^\xi(t))\) is uniformly close to zero.
Prf of Main Result

Theorem 3.1 \((PW \ '16)\)

\[
\lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\zeta^\xi(t), I) = 0.
\]

Pf Sketch.
**Prf of Main Result**

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But \(\langle 1_{\{0\}}, \zeta^\xi(t) \rangle = 0\) for all \(t \geq 0\) and \(\xi \in K\).
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Theorem 3.1 (PW ’16)

\[ \lim_{t \to \infty} \sup_{\xi \in K_{u,l}} d(\zeta^\xi(t), I) = 0. \]

Pf Sketch.

*By Corollary 6.1, the above holds with \( I \) replaced by \( J \).*

*But \( \langle 1_{\{0\}}, \zeta^\xi(t) \rangle = 0 \) for all \( t \geq 0 \) and \( \xi \in K \).*

*Using this and other properties of \( \zeta^\xi \) for \( \xi \in K_{u,l} \), it can be shown that \( J \) in Corollary 6.1 can be replaced by \( I \).*
Work in Progress

Extension to multiclass processor sharing queues (w/ J. Mulvany & R. Williams).
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Thank you for your attention.