The groupoids of adaptable separated graphs and their type semigroups (II)

Pere Ara

Universitat Autònoma de Barcelona

Higher rank graphs: geometry, symmetry, dynamics
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P.A., J. BOSA, E. PARDO, A. SIMS,
The groupoids of adaptable separated graphs and their type semigroups
arXiv:1904.05197v2 [math.RA].

P.A., J. BOSA, E.PARDO,
The realization problem for finitely generated refinement monoids
arXiv:1907.03648 [math.RA].

Outline

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Definition (Steinberg)

Given an ample groupoid $\mathcal{G}$, and a field with involution $(K, *)$, the *Steinberg algebra* associated to $\mathcal{G}$ is defined to be the $*$-algebra over K

$$A_K(\mathcal{G}) = \text{span}\{1_B : B \text{ is an open compact bisection }\}$$

with the *convolution product*

$$(fg)(\gamma) = \sum_{\substack{(\gamma_1,\gamma_2)\in\mathcal{G}^{(2)} \\ \gamma_1\gamma_2=\gamma}} f(\gamma_1)g(\gamma_2).$$

and the involution $f^*(\gamma) = f(\gamma^{-1})^*$.

When $\mathcal{G}$ is Hausdorff, $A_K(\mathcal{G})$ is just the $*$-algebra of compactly supported, locally constant functions $f\colon \mathcal{G} \to K$.

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Let S be an inverse semigroup with 0, and denote by $\mathcal{E}$ its semilattice of idempotents.

If $F \subseteq \mathcal{E}$ is any subset, then a finite subset $\Sigma \subseteq F$ is a **finite cover** of F when for any $0 \neq f \in F$ there exists $e \in \Sigma$ such that $fe \neq 0$.

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Definition

[Exel, Steinberg] Let S be an inverse semigroup, and A be a $*$-algebra over a field with involution K. Then, we say that $\pi : S \to A$ is a *representation* if $\pi(st) = \pi(s)\pi(t)$ and $\pi(s^*) = \pi(s)^*$ for all $s, t \in S$. A representation π is said to be a *tight representation* if for every idempotent $e \in \mathcal{E}$ and every finite cover Z of $\mathcal{F}_e := \{f \in \mathcal{E} \mid f \leq e\}$, we have

$$\pi(e) = \bigvee_{z \in Z} \pi(z)$$

in the (generalized) Boolean algebra of idempotents of the commutative $*$-subalgebra $A_{\mathcal{E}}$ of A generated by $\pi(\mathcal{E})$.

We say that a $*$-algebra A, together with a tight representation $\iota\colon S \to A$, is *universal for tight representations* if given any $*$-algebra B and any tight representation $\phi\colon S \to B$, there is a unique $*$-homomorphism $\tilde{\phi}\colon A \to B$ such that $\tilde{\phi} \circ \iota = \phi$. By the usual argument, such a universal tight $*$-algebra is unique up to $*$-isomorphism.

Theorem (Steinberg 2016)

Let S be a Hausdorff inverse semigroup with zero and let K be a field with involution. Then $A_K(\mathcal{G}_{tight}(S))$ is the universal $*$-algebra for tight representations of S.

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Recall that we have introduced in the first talk a list of generators and relations associated to an adaptable separated graph.

Definition

Let (E, C) be an adaptable separated graph and K be a field. The K-algebra $\mathcal{S}_K(E, C)$ is the $*$-algebra over K with generators

$$E^0 \cup E^1 \cup \{(t_i^v)^{\pm} : i \in \mathbb{N}, v \in E^0\}$$

and defining relations given in the first talk (Block 1 and Block 2) (including this time *all* the relations)

Let $\iota : S(E,C) \to \mathcal{S}_K(E,C)$ be the natural representation of $S(E,C)$ into $\mathcal{S}_K(E,C)$.

Theorem

The map $\iota : S(E,C) \to \mathcal{S}_K(E,C)$ is universal for tight representations of $S(E,C)$ on $*$-algebras over K.

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Theorem

Let (E,C) be an adaptable separated graph, let $S(E,C)$ be the inverse semigroup associated to (E,C), let K be a field with involution and let $\mathcal{S}_K(E,C)$ be the $*$-algebra over K associated to (E,C). Let $A_K(\mathcal{G}_{tight}(S(E,C)))$ be the Steinberg algebra of the tight groupoid $\mathcal{G}_{tight}(S(E,C))$. There is a $*$-isomorphism

$$\mathcal{S}_K(E,C) \cong A_K(\mathcal{G}_{tight}(S(E,C)))$$

sending $\iota(s) \in \mathcal{S}_K(E,C)$ to $1_{\Theta(s,D_{s^*s})}$ for each $s \in S(E,C)$.

Outline

Let S be an inverse semigroup (always with 0). We denote by $\mathcal{E}(S)$ the semilattice of idempotents of S. We say that $x, y \in S$ are orthogonal, written $x \perp y$ if $x^*y = yx^* = 0$

Recall that a *Boolean inverse semigroup* is an inverse semigroup S such that $\mathcal{E}(S)$ is a generalized Boolean lattice, and such that every pair $x, y \in S$ satisfying $x \perp y$ has a supremum, denoted $x \oplus y \in S$

Definition (Wehrung 2017)

Let S be a Boolean inverse semigroup. The *type semigroup* (or type monoid) of S is the commutative monoid $\mathrm{Typ}(S)$ freely generated by elements $\mathrm{typ}(x)$, where $x \in \mathcal{E}(S)$, subject to the relations

1. $\mathrm{typ}(0) = 0$,
2. $\mathrm{typ}(x) = \mathrm{typ}(y)$ whenever $x, y \in \mathcal{E}(S)$ and there is $s \in S$ such that $ss^* = x$ and $s^*s = y$.
3. $\mathrm{typ}(x \oplus y) = \mathrm{typ}(x) + \mathrm{typ}(y)$ whenever x, y are orthogonal elements in $\mathcal{E}(S)$.

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Let $\mathcal{G}$ be a (not-necessarily-Hausdorff) étale groupoid, with a Hausdorff locally compact unit space $X := \mathcal{G}^{(0)}$. Then the collection $S(\mathcal{G})$ of all compact open bisections of $\mathcal{G}$ forms a Boolean inverse semigroup.

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If $\mathcal{G}$ is second countable, then the type semigroup $\mathrm{Typ}(\mathcal{G})$ is a countable conical refinement monoid.

Theorem

Let (E, C) be an adaptable separated graph, $S(E, C)$ be the inverse semigroup associated to (E, C), and let $\mathcal{G}_{tight}(S(E, C))$ be the groupoid of germs associated to the canonical action of $S(E, C)$ on the space of ultrafilters $\hat{\mathcal{E}}_\infty$. Then, there is a monoid isomorphism

$$\psi \colon M = M(E, C) \to \mathrm{Typ}(\mathcal{G}_{tight}(S(E, C)))$$

such that $\psi(a_v) = [\mathcal{Z}(v)]$ for every $v \in E^0$.

Corollary

Let M be a finitely generated conical refinement monoid. Then there is an adaptable separated graph (E, C) such that

$$M \cong \mathrm{Typ}(\mathcal{G}_{tight}(S(E, C))).$$

In particular, all finitely generated conical refinement monoids arise as type semigroups of ample Hausdorff groupoids.

Outline

Definition

A ring R is *von Neumann regular* if $\forall x \in R\ \exists y \in R$ such that $x = xyx$.

Von Neumann regular rings were invented by John von Neumann in 1936 to coordinatize certain lattices L (meaning that $L \cong L(R_R)$).

Murray and von Neumann also considered a particular example of regular rings in Analysis: If $\mathcal{N} \subseteq B(H)$ is a finite von Neumann algebra, then the $*$-algebra $\mathcal{U}$ of all the unbounded densely defined operators affiliated to $\mathcal{N}$ is a $*$-regular complex algebra.

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• The $*$-algebra $\mathcal{U}$ coordinatizes the lattice L of all projections of $\mathcal{N}$.

• If Γ is a discrete group, the algebra $\mathcal{U}(\Gamma)$ is the $*$-regular ring of $\mathcal{N}(\Gamma)$, the von Neumann algebra of Γ. The $*$-regular ring $\mathcal{U}(\Gamma)$ and its various subrings play an important role in the study of various conjectures, such as the Atiyah Conjecture on l^2-Betti numbers.

• It is well-known that $\mathcal{U}$ is the *classical ring of quotients* of $\mathcal{N}$, so the extension $\mathcal{N} \subset \mathcal{U}$ is given by an *Ore localization*

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For a ring R, let $\mathcal{V}(R)$ be the monoid of isomorphism classes of finitely generated projective right R-modules, with the operation $[A] + [B] = [A \oplus B]$.

Goodearl's question 1995:

> *Which monoids arise as $\mathcal{V}(R)$ for (von Neumann) regular rings R?*

For regular R, $\mathcal{V}(R)$ must be a conical refinement monoid. Wehrung 1998 produced an example of such a monoid of size $\aleph_2$ which cannot be realized by any regular ring.

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Definition

Let R be a unital ring and Σ a family of square matrices over R. The *universal (or Cohn) localization* of R with respect to Σ is a ring $R\Sigma^{-1}$ with a ring homomorphism $\iota\colon R \to R\Sigma^{-1}$ such that:

1. All matrices $\iota(A)$, for $A \in \Sigma$ are invertible over $R\Sigma^{-1}$.
2. If $f\colon R \to S$ is such that $f(A)$ are invertible for all $A \in \Sigma$ then there is a unique $\tilde{f}\colon R\Sigma^{-1} \to S$ such that $f = \tilde{f} \circ \iota$.

Example: Any Ore localization is a universal localization. In particular the extension $\mathcal{N} \subset \mathcal{U}$ is a universal localization.

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The case of finite graphs

Definition (A-Brustenga)

Let E be a finite directed graph with $|E^0| = d$. Let $P(E)$ be the path algebra of E and $\epsilon\colon P(E) \to K^d$ the augmentation map. Let Σ be the set of all square matrices A over $P(E)$ such that $\epsilon(A)$ is invertible over K^d. The *regular algebra* of E is the K-algebra $Q_K(E) = L_K(E)\Sigma^{-1}$, where $L_K(E)$ is the Leavitt path algebra of E.

Theorem (A-Brustenga)

$Q_K(E)$ is von Neumann regular and the natural map

$$M(E) \to \mathcal{V}(Q_K(E))$$

is a monoid isomorphism.

We want to generalize this to the setting of adaptable separated graphs. Namely we want to build a suitable universal localization

$$Q_K(E,C) = \mathcal{S}_K(E,C)\Sigma^{-1}.$$

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First step: inverting polynomials in t_i^v

For $v \in E^0$, let $\Sigma_1^v \subseteq v\mathcal{S}_K(E,C)v$ be the set

$$p(t_i^v) = 1 + \lambda_1 t_i^v + \cdots + \lambda_n (t_i^v)^n \in v\mathcal{S}_K(E,C)v,$$

($n \geq 1$, $\lambda_n \neq 0$). We consider the universal localization

$$\mathcal{S}_K^1(E,C) := \mathcal{S}_K(E,C)\Big(\bigcup_{v\in E^0} \Sigma_1^v\Big)^{-1}.$$

Let $L = K(t_1, t_2, \ldots,)$ be an infinite purely transcendental extension of K. For each $v \in E^0$ there is a natural unital embedding $L \to v\mathcal{S}_K^1(E,C)v$ sending t_i to t_i^v. For $p(t_i) \in L$, we will denote by $p(t_i^v)$ its image under this embedding.

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We now define sets $\Sigma(p)$ for $p \in I$. We will differentiate between the free and regular cases.

- Take $p \in I_{\mathrm{free}}$. We have a well-defined evaluation map

$$L[x_1, \dots, x_{k(p)}] \to L[\alpha(p,1), \dots, \alpha(p,k(p))], \quad f(x_i) \mapsto f(\alpha(p,i)).$$

Let $\Sigma(p)$ be the set of all elements of $v^p S_K^1(E,C) v^p$ given by

$$\Sigma(p) = \{ f(\alpha(p,i)) : f \in L[x_i] \text{ and } x_j \text{ does not divide } f \ \forall j \}.$$

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• Take $p \in I_{\text{reg}}$ and E_p finite.
Consider the path L-algebra $P_L(E_p) \hookrightarrow v_p \mathcal{S}_K^1(E,C) v_p$, where $v_p = \sum_{v \in E_p^0} v$, and the canonical augmentation map

$$\epsilon^p \colon P_L(E_p) \to \oplus_{v \in E_p^0} vL.$$

Then $\Sigma(p)$ is the set of all square matrices A over $P_L(E_p)$ such that $\epsilon^p(A)$ is invertible as a matrix over $\oplus_{v \in E_p^0} vL$.

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Theorem

Let (E, C) be an adaptable separated graph and let K be a field. Then there exists a von Neumann regular K-algebra $Q_K(E, C)$ and a natural monoid isomorphism

$$M(E, C) \longrightarrow \mathcal{V}(Q_K(E, C)).$$

Theorem

Let M be a finitely generated conical refinement monoid and let K be a field. Then there exists a von Neumann regular (unital) K-algebra R such that $M \cong \mathcal{V}(R)$.

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The proof consists in decomposing our original adaptable separated graph (E, C) into a family of non-separated graphs, where we can apply the results from [A-Brustenga], and then reconstruct (E, C), the monoid $M(E, C)$ and the K-algebra $Q_K(E, C)$ in terms of the ones corresponding to the above-mentioned family of non-separated graphs.

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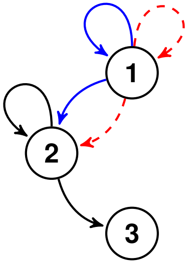
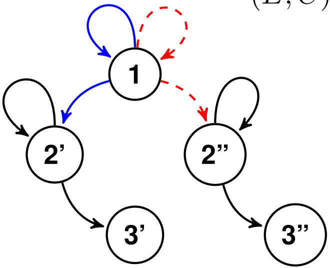
Step 1: In this step, for each adaptable separated graph (E, C), we find another $(\tilde{E}, \tilde{C})$ satisfying “condition (F)”
Example:

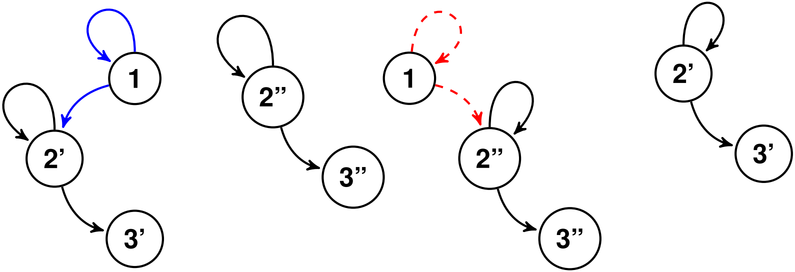
(E, C)



$(\tilde{E}, \tilde{C})$



Step 2: Consider $(\tilde{E}, \tilde{C})$ with condition (F). We reconstruct $(\tilde{E}, \tilde{C})$ via successive pullbacks of “building blocks”. These are the connected components of the non-separated graphs obtained by choosing a single set $X \in \tilde{C}_v$ at each $v \in \tilde{E}^0$:



Step 3: In this final step, we revert the cover map $\phi : (\tilde{E}, \tilde{C}) \to (E, C)$ described in Step 1 in order to move back from the auxiliary separated graph $(\tilde{E}, \tilde{C})$ to our original separated graph (E, C). To this end, we use the **crowned push-out** construction. We consider diagrams of the form:

$$\begin{array}{ccc} I & \xrightarrow{=} & I \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\iota_1} \\ I' & \xrightarrow{\iota_2} & P \end{array}$$

where I and I' are isomorphic (via φ) order-ideals in P with $I \cap I' = \{0\}$.
Then, we define the crowned pushout of (P, I, I', φ) as the coequalizer of the maps ι_1 and $\iota_2 \circ \varphi$.

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Then, we define the crowned pushout of (P, I, I', φ) as the coequalizer of the maps ι_1 and $\iota_2 \circ \varphi$.

We build a finite chain of adaptable separated graphs and cover maps

$$(\tilde{E},\tilde{C})=(\tilde{E}_n,\tilde{C}_n)\xrightarrow{\phi_n}(\tilde{E}_{n-1},\tilde{C}_{n-1})\xrightarrow{\phi_{n-1}}\dots\xrightarrow{\phi_1}(\tilde{E}_0,\tilde{C}_0)=(E,C),$$

satisfying that each $M(\tilde{E}_{k-1},\tilde{C}_{k-1})$ is the crowned push-out of a quadruple determined by $(\tilde{E}_k,\tilde{C}_k)$ and ϕ_k.

The realization theorem is shown inductively along this chain.

We build a finite chain of adaptable separated graphs and cover maps

$$(\tilde{E},\tilde{C}) = (\tilde{E}_n,\tilde{C}_n) \xrightarrow{\phi_n} (\tilde{E}_{n-1},\tilde{C}_{n-1}) \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_1} (\tilde{E}_0,\tilde{C}_0) = (E,C),$$

satisfying that each $M(\tilde{E}_{k-1},\tilde{C}_{k-1})$ is the crowned push-out of a quadruple determined by $(\tilde{E}_k,\tilde{C}_k)$ and ϕ_k.

The realization theorem is shown inductively along this chain.

THANK YOU VERY MUCH FOR YOUR ATTENTION!!!