Mapping class groupoids & motion groupoids

Talk 1

Fiona Torzewska

University of Leeds

Motivation

PHYSICS:

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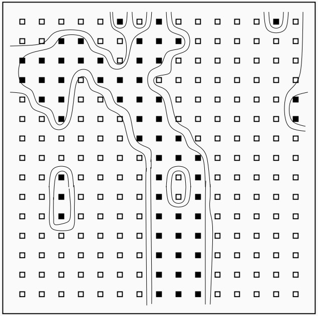

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Example in 2-Cob

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  AIM: Motion groupoids should describe movement of higher dimensional objects in higher dimensional ambient space. (And Functors from this category should model statistics of higher dimensional objects.)

ALSO: Lots of really nice maths, invariants of closed manifolds...

Plan of talks

1. Preliminaries
2. Braid group(oid)s as fundamental groupoids of configuration spaces
3. Motion groupoids
4. Mapping class groupoids
5. Relationship between the two
6. Motion groupoids and categories of (embedded) cobordisms

Braid groupoids - the idea

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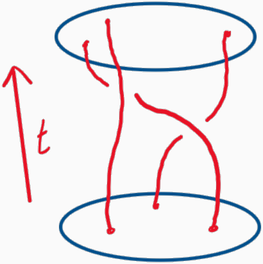

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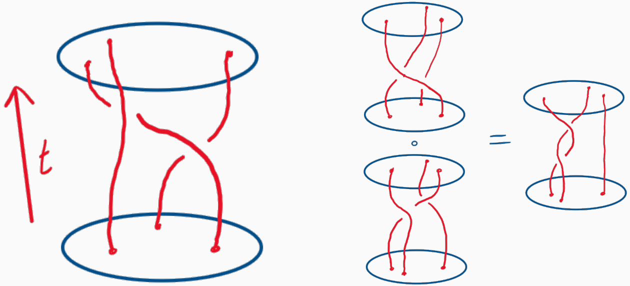

Preliminaries

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Let X and Y be Hausdorff topological spaces.

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Example

- If X is the one point space, then maps can be labelled by elements of Y and the compact open topology is the same as the topology on Y.
- Similarly if X is the n point space with the discrete topology, the compact open topology on maps X to Y is the same as the topology on $Y \times \ldots \times Y$, the product of Y with itself n times.

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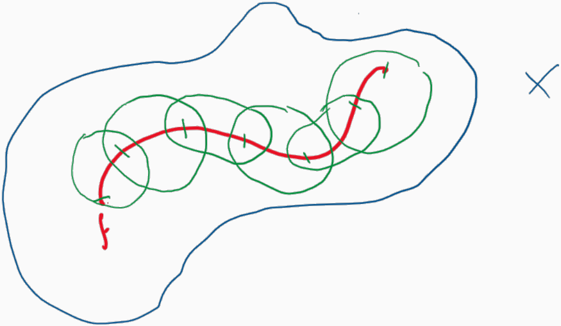

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(Coming from the adjunction of $\mathbf{TOP}(Y, -)$ and $- \times Y$.)

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We use $\gamma \colon x \to x'$ for a path $\gamma \colon I \to X$ with $\gamma(0) = x$ and $\gamma(1) = x'$.

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$$h\colon I \times I \to X$$

such that $h(t,0) = \gamma(t) \forall\, t \in I$, $h(t,1) = \gamma'(t) \forall\, t \in I$ and $h|_{I\times\{s\}}$ is a path x to x' $\forall s \in I$.

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- Using the lemma, this is the same as a continuous map

$$\tilde{h}\colon I \to PX$$

such that $\tilde{h}(0) = \gamma$, $\tilde{h}(1) = \gamma'$ and $\tilde{h}(t)$ is a path x to x' $\forall t \in I$ i.e. a path in the space of paths.

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We have that $\pi(X, *)$ is the fundamental group based at $* \in X$.

Braid groupoids

Historically the first motivation that led to the study of motion groups was braid groups as configuration spaces.

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Definition

Let the ordered configuration space of n points in the disk be the set

$$OC_n = \{(x_1, ..., x_n) | \, x_i \in int(D), x_i \neq x_j \, \forall \, i \neq j\}$$

with the subspace topology from $D^n = D \times ... \times D$.

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There is a properly discontinuous action of the symmetric group which permutes indices. For $s \in S_n$

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The unordered configuration space of n points in the disk is the quotient space

$$UC_n = OC_n / S_n.$$

Definition
The n^{th} braid groupoid denoted $\mathcal{B}_n$ is

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Let $\mathsf{x} = (x_1, \ldots, x_n) \in UC_n$. The n^{th} braid group at x denoted B_n^{x} is

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with fibre $int(D) \setminus *$. And hence a short exact sequence

$$\pi_2(int(D)) \to \pi_1(int(D) \setminus *) \to \pi_1(OC_2) \to \pi_1(int(D))$$

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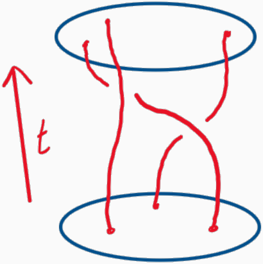

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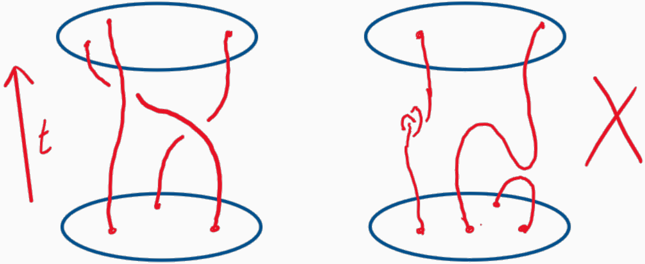

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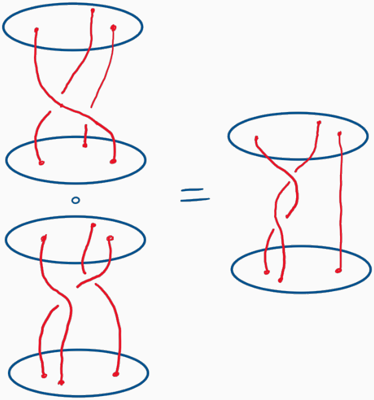
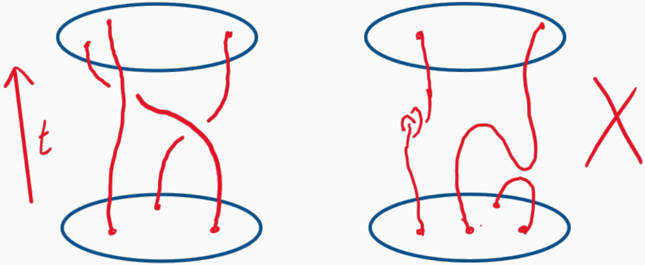
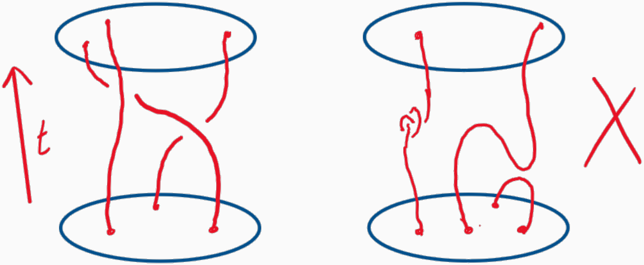


Braid groupoids - A question





Question

Can we construct a category which does contain things like the picture on the right? Is there a well defined/ faithful functor from $\mathcal{B}_n$?

Generalising braids to embeddings

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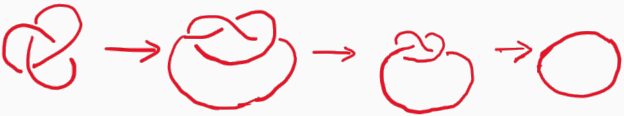
BUT: The topology of this quotient is (usually) unmanageable.
For N the n point space with discrete topology,

$$E(N, M) = OC_n$$

And we cannot extend to neighbourhood of N.

$M = S^3 \quad N = S^1$



Self homeomorphisms

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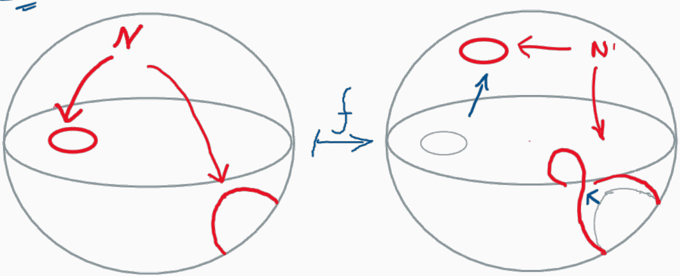
Ex $M = B^3$



Lemma
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This means the function mapping elements to their inverses and the group operation are continuous.

Propostion
Let M be a compact, connected, orientable manifold. There is a groupoid denoted $Homeo_M$ such that

- objects are subsets of M,
- and morphisms between subspaces are elements of $Homeo_M(N, N')$.

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Proof

Identity and inverse are obvious homeomorphisms. Composition of homeomorphisms is associative.

Closure: $g \circ f(N) = g(N') = N''$ and $g \circ f(\partial M) = g(\partial M) = \partial M$

Motion Groupoids

Definition
A motion from N to N' denoted $f\colon N \to N'$ is a path $f\colon I \to Homeo_M(\emptyset, \emptyset)$ with $f_0 = id_M$ and $f_1 \in Homeo_M(N, N')$.

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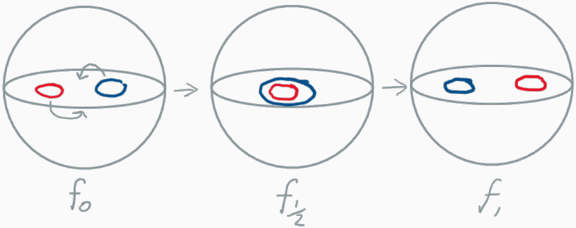
Using lemma $\mathbf{Top}(X, \mathbf{TOP}(Y, Z)) \to \mathbf{Top}(X \times Y, Z)$

$$\{I \to Homeo_M(\emptyset, \emptyset)\} \leftrightarrow \{M \times I \to M\} \leftrightarrow \{M \times I \to M \times I\}$$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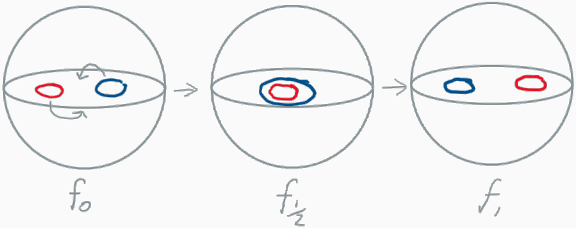
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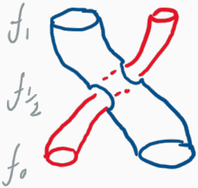
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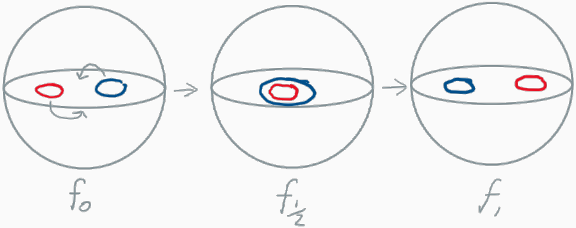
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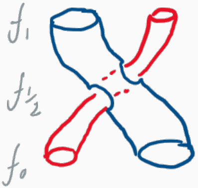
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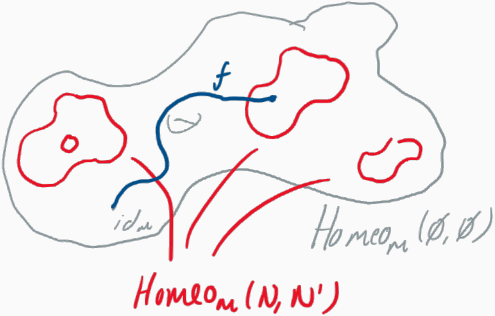
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$Homeo_M(N, N')$

We can define composition of motions $f\colon N \to N'$ and $g\colon N' \to N''$

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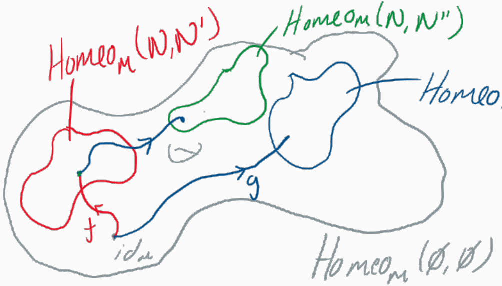

$g_1 \circ f_1 \in Homeo_M(N,N'')$

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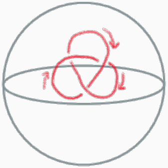

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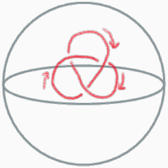

As a picture in $M \times I$ of a stationary motion looks like N remains fixed in M at all t.

Define motion corresponding to each $f\colon N \to N'$

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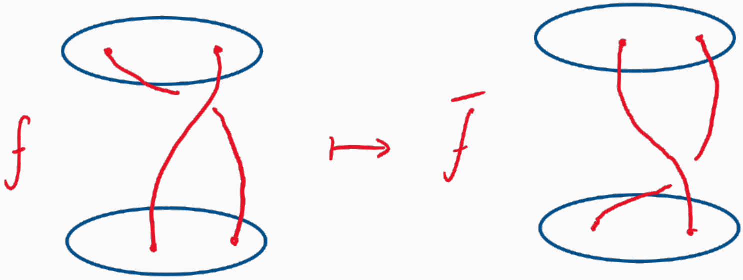

Two motions $f, g \colon N \to N'$ are equivalent if $\bar{g} * f$ is homotopic, keeping endpoints fixed, to a stationary motion.

Motion groupoids - Equivalence

Two motions $f, g\colon N \to N'$ are equivalent if $\bar{g} * f$ is homotopic, keeping endpoints fixed, to a stationary motion.
This means a continuous map

$$H\colon (M \times I) \times I \to M \times I$$

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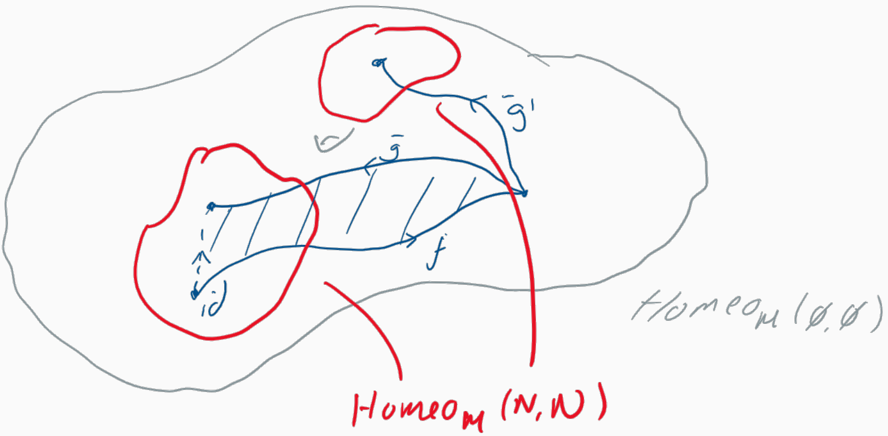
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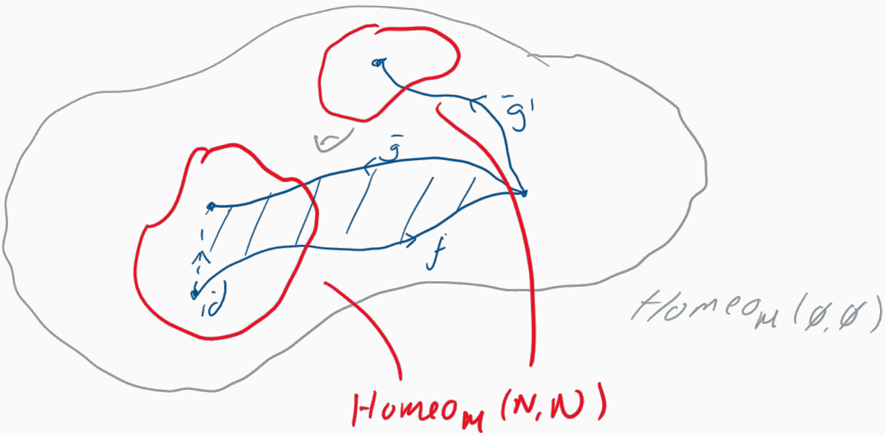

(Notice this implies stationary motions are equivalent to the trivial motion $f_t = i_M$ for all t.)

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Let M be a compact, connected, orientable manifold. There is a groupoid denoted Mot_M such that

- objects are subsets of M,
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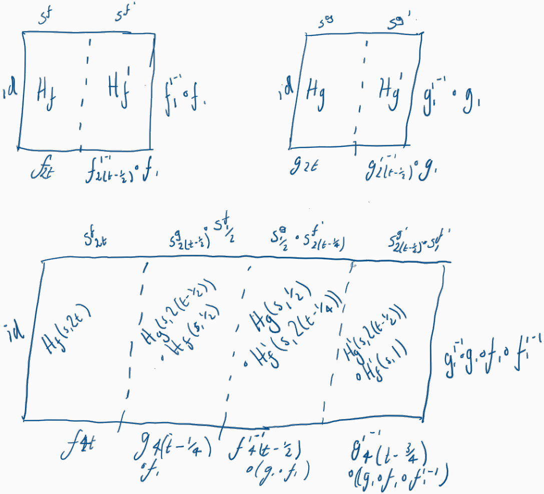

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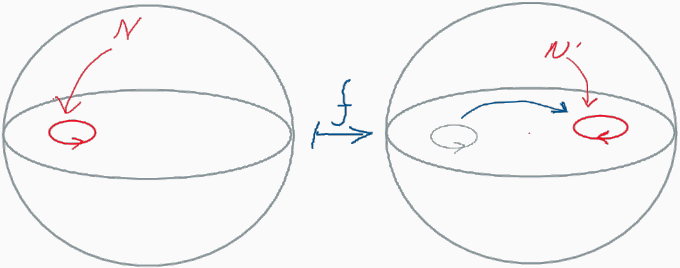
Ex: $M = B^3$, N, N' one component trivial links



Oriented groupoids of self homeomorphisms

Propostion
Let M be a compact, connected, orientable manifold. There is a groupoid denoted resp. $Homeo_M^{or}$ such that

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Lemma

There is a faithful functor $Homeo_M^{or} \to Homeo_M$ given by forgetting the orientation on manifolds N and N'.

Oriented motion groupoid

Theorem

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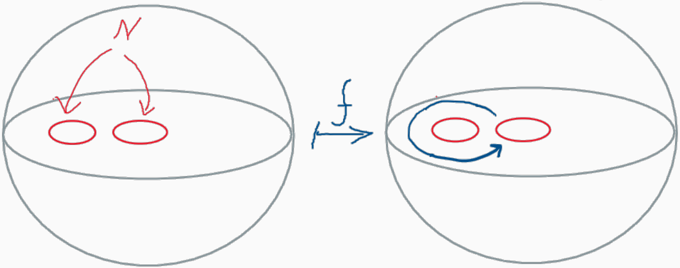

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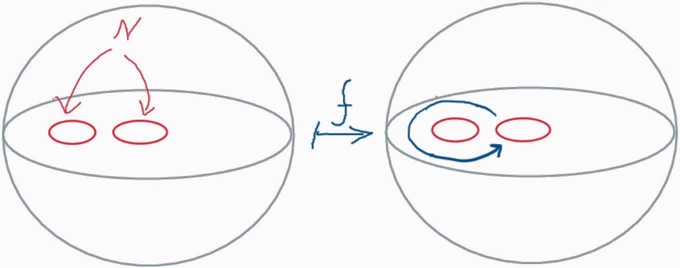

Define in the same way P$Homeo_M^{or}(N, N) \subset Homeo_M^{or}(N, N)$.

References

Motion groups: Goldsmith, Deborah L. ”The theory of motion groups.” The Michigan Mathematical Journal 28.1 (1981): 3-17.

Braid groups as configuration spaces: Fox, Ralph, and Lee Neuwirth. ”The braid groups.” Mathematica Scandinavica 10 (1962): 119-126.