Graph algebras

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Higher rank graphs: geometry, symmetry, dynamics
ICMS

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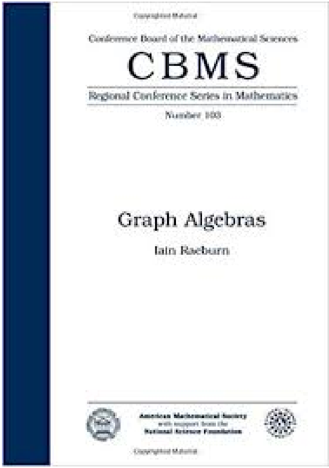
Everything in this talk can be found in:



I. Raeburn, *Graph Algebras*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.

Plan

1. C^*-algebras
2. Directed graphs (setting up notation)
3. Cuntz-Krieger E-families and graph algebras
4. Simplicity of graph algebras
5. Two uniqueness theorems

C^*-algebras

Definition

A Banach $*$-algebra A is called a *C^*-algebra* if

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in A.$$

Theorem (Gelfand-Naimark)

Every C^*-algebra A is isomorphic to a closed $*$-subalgebra of the bounded operators on a Hilbert space

Theorem (Gelfand-Naimark)

Suppose A is a commutative C^*-algebra. Then $A \cong C_0(X)$ for a locally compact and Hausdorff topological space X.

The Gelfand-Naimark Theorem implies that we should think of C^*-algebras as noncommutative topological spaces.

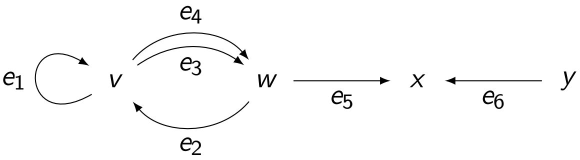
Definition

Suppose A is a C^*-algebra. We say

1. An element a is self-adjoint if $a = a^*$.
2. An element p is a projection if $p = p^2 = p^*$; that is, p is a self-adjoint idempotent.
3. If A is unital, an element u is a unitary if $u^*u = 1 = uu^*$.
4. An element s is a partial isometry if s^*s is a projection (equivalently $s = ss^*s$ or $s^* = s^*ss^*$).

Directed graphs

- Let $E = (E^0, E^1, r, s)$ be a finite directed graph with vertex set E^0, edge set E^1, and range and source maps from E^1 to E^0.

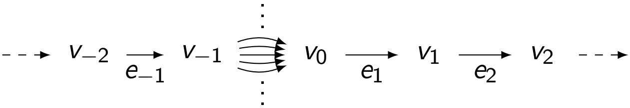


- A vertex that does not receive any edges is called a *source*.
- A vertex that does not emit any edges is called a *sink*.

Directed graphs

- A graph can have an infinite number of vertices:

- A graph can have an infinite number of edges between vertices:



- We typically rule out graphs that have a vertex with an infinite number of incoming edges. A graph is called *row-finite* if there are no vertices with this property; that is, we require that $r^{-1}(v) < \infty$ for all $v \in E^0$.

Directed graphs

- A *path* of length k in a graph is a sequence $\mu = \mu_1 \mu_2 \cdots \mu_k$ with $\mu_i \in E^1$ and $s(\mu_i) = r(\mu_{i+1})$. A path of length zero is defined to be a vertex. We let E^k denote the paths of length k.

- The range and source maps extend to paths $\mu = \mu_1 \dots \mu_k$ in E^k by $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_k)$.

- The *path space* of a graph E is the set

$$E^* = \bigcup_{k \geq 0} E^k$$

Graph algebras

Suppose E is a row-finite directed graph and $\mathcal{H}$ is a Hilbert space. A *Cuntz-Krieger E-family* $\{S, P\}$ on $\mathcal{H}$ consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$ such that

(CK1) $$P_{s(e)} = S_e^* S_e \quad \text{for all } e \in E^1 \text{ and}$$

(CK2) $$P_v = \sum_{r(e)=v} S_e S_e^* \quad \text{whenever } v \text{ is not a source.}$$

- (CK1) implies that the source projection of S_e is $P_{s(e)}$.
- (CK2) implies that the range projection of S_e is a subprojection of $P_{r(e)}$.
- We now see why we want to restrict to row-finite graphs, if E is not row-finite (CK2) doesn't make sense.
- We can assume that $\mathcal{H} = \bigoplus_{v \in E^0} P_v \mathcal{H}$; then we say $\mathcal{H}$ is non-degenerate.

Graph algebras

Theorem (Cuntz-Krieger 1980, Kumjian-Pask-Raeburn 1998)

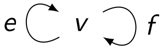
Suppose E is a row-finite directed graph. There is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s, p\}$ such that for every Cuntz-Krieger E-family $\{T, Q\}$ in a C^*-algebra B, there is a $*$-homomorphism $\pi : C^*(E) \to B$ satisfying $\pi(s_e) = T_e$ for every $e \in E^1$ and $\pi(p_v) = Q_v$ for every $v \in E^0$.

Definition

The C^*-algebra $C^*(E)$ is called the C^*-algebra of the graph E, or just the graph algebra of E.

- If both E^0 and E^1 are finite, then $C^*(E)$ is a Cuntz-Krieger algebra. Graph algebras are a natural extension of these.
- We will always use lower case letters to denote the universal algebras.

The Cuntz algebra $\mathcal{O}_2$



The Cuntz-Krieger relations yield

$$\begin{aligned} &\text{(CK1)} \qquad && s_e^* s_e = p_v = s_f^* s_f \quad \text{and} \\ &\text{(CK2)} \qquad && p_v = s_e s_e^* + s_f s_f^*. \end{aligned}$$

To ensure we can realise these as operators, consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ and define

$$S_e(x_0, x_1, \dots) = (x_0, 0, x_1, 0, \dots) \ \& \ S_e^*(x_0, x_1, \dots) = (x_0, x_2, \dots)$$
$$S_f(x_0, x_1, \dots) = (0, x_0, 0, x_1, \dots) \ \& \ S_f^*(x_0, x_1, \dots) = (x_1, x_3, \dots)$$

Then

$$S_e^* S_e(x_0, x_1, \dots) = (x_0, x_1, \dots) \ \& \ S_f^* S_f(x_0, x_1, \dots) = (x_0, x_1, \dots)$$
$$S_e S_e^*(x_0, x_1, \dots) = (x_0, 0, x_2, 0, \dots)$$
$$S_f S_f^*(x_0, x_1, \dots) = (0, x_1, 0, x_3, \dots).$$

Paths in graph algebras

Proposition

Suppose E is a row-finite directed graph and $C^*(E)$ is the graph algebra of E. Then

- $\{s_e s_e^* : e \in E^1\}$ *is a mutually orthogonal set of projections;*
- $s_e^* s_f \neq 0 \implies e = f$;
- $s_e s_f \neq 0 \implies s(e) = r(f)$; *and*
- $s_e s_f^* \neq 0 \implies s(e) = s(f)$.

- These relations imply that the graph algebra keeps track of paths in the graph; that is, if $\mu \in \prod_{i=1}^k E^1$ and we define $s_\mu = s_{\mu_1} \cdots s_{\mu_k}$, then $s_\mu = 0$ unless μ is a path in E^k.
- Noting that $s_\nu^* = s_{\nu_k}^* \cdots s_{\nu_1}^*$ and using the relations above, it is a great exercise to prove

$$C^*(E) = \overline{span}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

Simplicity of graph algebras

- A path $\mu = \mu_1 \cdots \mu_k$ in a graph is called a *cycle* if $r(\mu_1) = s(\mu_k)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. An edge e is called an *entry* to the cycle μ if there exists i such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$.
- For vertices $v, w \in E^0$, we say that $w \leq v$ if there is a path $\mu \in E^*$ such that $s(\mu) = v$ and $r(\mu) = w$. Let E^∞ denote the set of infinite paths and let $E^{\leq\infty}$ denote the set of infinite paths and the paths that begin at sources. We say E is *cofinal* if for every $\mu \in E^{\leq\infty}$ and $v \in E^0$ there exists a vertex w on μ such that $v \leq w$.

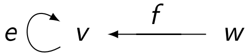
Theorem (Cuntz 1981, Kumjian-Pask-Raeburn-Renault 1997, an Huef-Raeburn 1997, Bates-Pask-Raeburn-Szymański 2000, Kajiwara-Pinzari-Watatani 2001)

Suppose E is a row-finite directed graph. Then $C^*(E)$ is simple if and only if every cycle has an entry and E is cofinal.

Simplicity of graph algebras

- Using this characterisation, we can see straight away that the Cuntz-algebras $\mathcal{O}_n$ for $n \geq 2$ are simple and that the graph algebra of any strongly connected graph E (there is a path between any two vertices) such that at least one vertex v has $|r^{-1}(v)| \geq 2$ is simple.

Consider the following graph



- This graph is not cofinal and hence is not simple. One can check (using Coburn's Theorem) that the graph algebra is the Toeplitz algebra, and we know that the Toeplitz algebra contains the compact operators as a non-trivial ideal and the quotient by the compacts is $C(\mathbb{T})$.

Purely infinite or approximately finite

- A C^*-algebra A is said to be *approximately finite dimensional* if there is a finite dimensional sequence of C^*-subalgebras $\{A_n\}$ of A such that $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_{n=1}^{\infty} A_n}$.

- A projection p in a C^*-algebra A is said to be *infinite* if it is equivalent to a proper subprojection of itself; that is, there exists a partial isometry $s \in A$ such that $p = s^*s$ and $ss^* < p$. If there exist projections q_1 and q_2 in A such that $p \sim q_1 \sim q_2$ and $q_1 + q_2 \leq p$ then p is said to be *properly infinite*.

- A C^*-algebra is said to be *purely infinite* if it has real rank zero and every non-zero projection is properly infinite.

  Note: A C^*-algebra has real rank zero if the invertible self-adjoint elements are dense in the self-adjoint elements

Purely infinite or approximately finite

Theorem (Kumjian-Pask-Raeburn 1998)

Suppose E is a row-finite directed graph E, then $C^*(E)$ is AF if and only if E has no cycles.

Theorem (Kumjian-Pask-Raeburn 1998, Bates-Pask-Raeburn-Szymański 2000)

Suppose E is a row-finite directed graph E and $C^*(E)$ is simple. Then $C^*(E)$ is purely infinite if and only if for every $v \in E^0$ there is a cycle μ with $r(\mu) \geq v$.

Corollary

Suppose E is a row-finite directed graph E and $C^*(E)$ is simple. Then $C^*(E)$ is either AF or purely infinite.

The Gauge-Invariant Uniqueness Theorem

- A gauge action on a C^*-algebra B is a homomorphism $\alpha : \mathbb{T} \to \mathrm{Aut}(B)$ such that $z \mapsto \alpha_z(a)$ is continuous for each $a \in B$.
- For a row-finite directed graph, The graph algebra $C^*(E)$ always has a canonical gauge action $\alpha : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ such that $\alpha_z(s_e) = zs_e$ and $\alpha_z(p_v) = p_v$.

Theorem (an Huef-Raeburn 1997)

Suppose E is a row-finite directed graph. If $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B such that $P_v \neq 0$ for all $v \in E^0$ and there is a gauge action $\alpha : \mathbb{T} \to Aut(B)$ such that $\alpha_z(S_e) = zS_e$ and $\alpha_z(P_v) = P_v$, then the $*$-homomorphism $\pi_{S,P} : C^*(E) \to B$ is an isomorphism of $C^*(E)$ onto $C^*(S, P)$.

The Cuntz-Krieger Uniqueness Theorem

Theorem (Cuntz-Krieger 1980, Pask-Raeburn 1998, Kumjian-Pask-Raeburn 1998)

Suppose E is a row-finite directed graph and every cycle has an entry. If $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B such that $P_v \neq 0$ for all $v \in E^0$, then the $*$-homomorphism $\pi_{S,P} : C^*(E) \to B$ is an isomorphism of $C^*(E)$ onto $C^*(S, P)$.

- Both of these results imply that we don't often need to check the universal properties of graph algebras, but only need to find a Cuntz-Krieger E-family $\{S, P\}$ that satisfies the hypotheses of either uniqueness theorem.
- These theorems were instrumental in understanding of the ideal structure of graph algebras, and have many other applications.