Non-Commutative Stone duality:

groups,

groupoids,

and pseudogroups

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A Couple of restrictions

1. We work with monoids rather than semigroups.

This means certain spaces will be compact rather than locally compact.

2. We focus on Boolean inverse monoids.

But the theory extends to distributive inverse semigroups and pseudogroups.

The talk is in two parts

Part 1: The theory of non-commutative Stone duality.

Part 2: Application of the theory in Part 1 to "k-monoids" = 1-vertex higher rank graphs.

Part 1

The theory of non-commutative Stone duality

Classical Stone duality

A.K.A. the 'Commutative case'.

M.H. Stone, Applications of the theory of Boolean rings to general topology, *T.A.M.S.* **41** (1937), 375-481.

A topological space is called <u>Boolean</u> if it is compact, Hausdorff and has a basis of clopen subsets.

<u>Proposition</u> If X is a Boolean space, then the set $\underline{B}(X)$ of clopen subsets of X is a Boolean algebra.

Let B be a Boolean algebra.

A subset $F \subseteq B$ is called a <u>filter</u> if (1) $a, b \in F \Rightarrow a \wedge b \in F$

(2) $a \in F$ and $a \leq b \Rightarrow b \in F$. A filter is <u>proper</u> if

$0 \notin F$. A maximal proper filter is called an

<u>ultrafilter</u>.

Denote the set of all ultrafilters on B by $\underline{X}(B)$.

Let $a \in B$. Denote by U_a the set of all ultrafilters that contain a.

Put $\tau = \{ U_a : a \in B \}$.

<u>Proposition</u> τ is the basis for a topology on $\underline{X}(B)$ that makes it a Boolean space.

We call $\underline{X}(B)$ the <u>Stone space</u> of B.

<u>Theorem</u> [Commutative Stone duality]

(1) Let B be a Boolean algebra.
Then $B \cong \underline{B}(\underline{X}(S))$.

(2) Let X be a Boolean space.
Then $X \cong \underline{X}(\underline{B}(X))$.

<u>Examples</u>

(1) Finite discrete spaces are Boolean spaces. Their associated Boolean algebras are the powerset Boolean algebras.

(2) Upto isomorphism, there is exactly one countable, atomless Boolean algebra called the <u>Tarski algebra</u>. Its Stone space is the <u>Cantor space</u>.

We now go non-commutative...

Boolean inverse monoids

An *inverse semigroup* is a semigroup in which for each element a there is a unique element a^{-1} s.t. $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Key facts

1. Idempotents commute and form a meet semilattice where $e \wedge f = ef$. $\underline{E}(S)$ is the set of idempotents.
2. Define $a \leq b$ by $a = be$ for some idempotent e. $\leq$ is the *natural partial order*.

$a \le b \Rightarrow a^{-1} \le b^{-1}$. If $c \le d$ then $ac \le bd$.

3. If $a, b \le c$ then $a^{-1}b$ and ab^{-1} are idempotents.

We say <u>a is compatible with b</u> if $a^{-1}b$ and ab^{-1} are idempotents, denoted $a \sim b$.

A subset whose elements are pairwise compatible is called a <u>compatible subset</u>.

An inverse monoid is called <u>distributive</u> if

(1) $a \sim b \Rightarrow \exists\, a \vee b$.

(2) $c(a \vee b) = ca \vee cb$ and $(a \vee b)c = ac \vee bc$

when $\exists\, a \vee b$.

A distributive inverse monoid is called <u>Boolean</u> if its set of idempotents forms a Boolean algebra.

[A <u>pseudogroup</u> has arbitrary compatible joins, and left and right distributivity by multiplication].

Commutative	Non-Commutative
Frame	Pseudogroup
Distributive lattice	Distributive inverse monoid
Boolean algebra	Boolean inverse monoid

A <u>groupoid</u> G is a (small) category in which each arrow is invertible. Denote the set of identities of G by G_0. There are maps $d, r : G \to G_0$ given by $d(g) = g^{-1}g$, $r(g) = gg^{-1}$. Put

$$G * G = \{(g,h) \in G \times G : d(g) = r(h)\}.$$

There is a map $m : G * G \to G$ given by $(g,h) \mapsto gh$. There is a map $i : G \to G$ given by $i(g) = g^{-1}$.

A <u>topological groupoid</u> is a groupoid G equipped with a topology such that d, r, m, i are all continuous.

A topological groupoid is <u>étale</u> if $d : G \to G_0$ is a local homeomorphism.

What does being étale really mean?

<u>Theorem</u> (Resende, 2006)

If G is an étale groupoid then $\Omega(G)$, the set of all open subsets of G, is a monoid under subset multiplication.

A <u>Boolean groupoid</u> is an étale groupoid whose identity space is Boolean.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{Boolean inverse monoids} \arrow[r, bend left, "?"] & \text{Boolean groupoids} \arrow[l, bend left, "?"] \\
\text{Boolean algebras} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, bend left, "\underline{X}"] & \text{Boolean spaces} \arrow[u, phantom, sloped, "\subseteq"] \arrow[l, bend left, "\underline{B}"]
\end{tikzcd}$$

Let G be a groupoid.

A subset $A \subseteq G$ is called a <u>partial bisection</u> if $A^{-1}A,\ AA^{-1} \subseteq G_0$.

<u>Proposition</u> If G is a Boolean groupoid then the set $\underline{KB}(G)$ of compact-open partial bisections is a Boolean inverse monoid under subset multiplication.

Let S be a Boolean inverse monoid.

Denote the set of all ultrafilters on S by $\underline{G}(S)$.

Put $\tau = \{ U_s : s \in S \}$.

<u>Proposition</u> $\underline{G}(S)$ is a groupoid for an appropriate definition of multiplication. τ is the basis for a topology on $\underline{G}(S)$ that makes it a Boolean groupoid. We call $\underline{G}(S)$ the <u>Stone groupoid</u> of S.

__Theorem__ [Non-Commutative Stone duality]
(Lenz & Lawson, Resende).

(1) Let S be a Boolean inverse monoid.
Then $S \cong \underline{KB}(\underline{G}(S))$.

(2) Let G be a Boolean groupoid.
Then $G \cong \underline{G}(\underline{KB}(G))$.

Boolean inverse monoid	Boolean groupoid
$\wedge$-monoid	Hausdorff
fundamental	effective
Countable	second-countable
Tarski algebra of idempotents	Identity space is the Cantor space
0-simplifying	Minimal
0-simple	purely infinite + minimal
group of units	topological full group

A <u>Tarski inverse monoid</u> is a countable Boolean inverse $\wedge$-monoid whose set of idempotents forms a Tarski algebra.

<u>Congruence-free</u> = 0-simple + fundamental

The following uses a result of Matui.

<u>Theorem</u> The group of units of a congruence-free Tarski inverse monoid is:

> a full, countable subgroup of the group of self-homeomorphisms of the Cantor space which act minimally and in which each element has clopen support. In addition, the commutator subgroup is simple.

Part 2

Constructing a Thompson-esque group from a 1-vertex higher rank graph (called here a k-monoid)

This is joint work with Alina Vdovina.

Assumed "nice"

A <u>k-monoid</u> is a countable monoid S equipped with a homomorphism $d : S \to \mathbb{N}^k$ satisfying the <u>unique factorization property (UFP)</u>: if $d(s) = \underline{m} + \underline{n}$ then there exist unique elements $x, y \in S$ s.t

(1) $s = xy$ (2) $d(x) = \underline{m}$ and $d(y) = \underline{n}$.

<u>Examples</u> 1-monoids $\equiv$ Countable free monoids

Finite direct products of free monoids.

Let S be a monoid.

A subset $R \subseteq S$ is a <u>right ideal</u> if $RS \subseteq R$.

R is <u>finitely generated</u> if $R = XS$ where $X \subseteq S$ is a finite set.

A function $\theta : R_1 \to R_2$ between two right ideals is a <u>morphism</u> if $\theta(as) = \theta(a)s$ $(\forall s \in S)(\forall a \in R_1)$

Let S be a k-monoid.

Define $\underline{R}(S)$ to consist of all bijective morphisms between finitely generated right ideals.

<u>Proposition</u> $\underline{R}(S)$ is a distributive inverse $\wedge$-monoid.

A right ideal is <u>essential</u> if it has a non-empty intersection with every (non-empty) right ideal.

$\underline{R}(S)^e$ is defined to be all the bijective morphisms between finitely generated essential right ideals.

Let T be any inverse semigroup.

Define a relation σ on T by

$$a \,\sigma\, b \iff (\exists c)(c \le a, b).$$

Then σ is a congruence on T, T/σ is a group and if ρ is any congruence on T such that T/ρ is a group then $\sigma \subseteq \rho$.

$$\boxed{\text{Define } \quad \mathcal{G}(S) = \frac{\underline{R}(S)^e}{\sigma}}$$

Let T be an inverse $\wedge$-monoid.

Define $\equiv$ on T as follows:

$$a \equiv b \iff (\forall x < a)(x \neq 0)\,(x \wedge b \neq 0)$$

$$\&$$

$$(\forall x < b)\,(x \neq 0)\,(x \wedge a \neq 0)$$

It can be proved that $\equiv$ is a congruence.

<u>Theorem</u> Let S be a K-monoid.

$\underline{C}(S) = \underline{R}(S)/\equiv$ is a Boolean inverse monoid.

Its Stone groupoid is the usual groupoid associated to S.

Under mild conditions on S, the monoid $\underline{C}(S)$ is a congruence-free Tarski monoid.

Its group of units is isomorphic to $\mathcal{G}(S)$.

It follows that the group $\mathcal{G}(S)$ is Thompson-esque.

We now give a more direct description of this group.

Details will be omitted.

Elements of S, x and y, are called <u>incomparable</u> if $xS \cap yS = \emptyset$. Otherwise, they are <u>comparable</u>.

Maximal generalized prefix code $\equiv$ maximal finite set of incomparable elements

S^∞ = set of sectors in S

[Generalization of right-infinite strings.]

Usually, S^∞ is the Cantor space.

If X is a maximal generalized prefix code, then $S^\infty = X S^\infty$ and $\{ x S^\infty : x \in X \}$ is a finite partition of S^∞.

Let $x, y \in S$.

Define $yx^{-1} : xS^{\infty} \longrightarrow yS^{\infty}$ by

$xw \longmapsto yw$ where $w \in S^{\infty}$.

This is a partial homeomorphism of S^{∞}.

Let $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_m)$ be maximal generalized prefix codes.

Define $f_{(X,Y)} : S^\infty \to S^\infty$ by

$$f_{(X,Y)} = \bigcup_{i=1}^{m} y_i x_i^{-1}$$

<u>Theorem</u> $\mathcal{G}(S)$ is the group consisting of all such self-homeomorphisms of S^∞ of the form $f_{(X,Y)}$.