THE COORDINATE BETHE ANSATZ
LMS POSTGRADUATE SCHOOL INTRODUCTION TO INTEGRABILITY

BART VLAAR, HERIOT-WATT UNIVERSITY

Contents

1. Introduction

In this lecture we will discuss a particular method for solving one-dimensional quantum integrable systems called the *coordinate Bethe ansatz* (CBA) or sometimes simply *Bethe ansatz*. Its origins lie in the exact solution by Bethe of the Heisenberg spin chain, a one-dimensional model for (anti)ferromagnetism [Be31]. Many one-dimensional models have been solved with this method, for instance the quantum delta Bose gas [LL63], which we will focus on in this lecture, the Hubbard model [LW68] and the massive Thirring model [BT79]. This method has naturally developed into more modern and powerful methods, of which we should highlight Baxter's *Q-operator* method [Ba72, Ba73] and the *algebraic Bethe ansatz* [SF78, STF79, Sk82, KBI93, Fa95], also known as the *quantum inverse scattering method*.

The main output of this method is initially an explicit construction of an eigenfunction of a Hamiltonian. Imposing boundary conditions (we will focus on periodic boundary conditions) is guaranteed by the *Bethe (ansatz) equations*. This system of equations can also be derived from a variational principle of an action and when diagonalizing the transfer matrix in the algebraic Bethe ansatz formalism. Finally, the Bethe equations are useful in the thermodynamic limit. Good textbook references for the coordinate Bethe ansatz are [Ga14, KBI93]

1.1. **Group algebras and symmetrizers.** Before we explain the general idea of the CBA, let us introduce some basic terminology of representation theory. The physical motivation is that integrability is associated with symmetries, which we classically think of as a group acting on the space of states of a model. Let G be a group, written multiplicatively, with unit element e; the example to keep in mind is the symmetric group S_n. If X is a set, then a *(group) action* of G on X is a map $m : G \times X \to X$, written $m(g, v) = g \cdot v$ for all $g \in G$, $v \in V$, with the following properties:

$$e \cdot x = x, \qquad (gh) \cdot x = g \cdot (h \cdot x), \qquad \text{for all } g, h \in G, x \in X. \tag{1.1}$$

As a consequence of $gg^{-1} = g^{-1}g = e$ it follows that for all $g \in G$, the map $\pi(g) : X \to X$ defined by $x \mapsto g \cdot x$ is invertible.

On the other hand, in the quantum-mechanical treatment we work with complex-linear structures and hence it is useful to highlight $\mathbb{C}$-linear group actions. If V is a vector space (always over $\mathbb{C}$), then a *linear action* of G on V is an action that is compatible with the linear structure as follows:

$$g \cdot (v + w) = g \cdot v + g \cdot w, \qquad g \cdot (cv) = c(g \cdot v), \qquad \text{for all } g \in G, v, w \in V, c \in \mathbb{C}. \tag{1.2}$$

In other words, the map $\pi(g)$ is linear for all $g \in G$; combining this with invertibility of $\pi(g)$ we obtain a group homomorphism π from G to the group

$$\mathrm{GL}(V) = \{f : V \to V \mid f \text{ invertible and linear}\}. \tag{1.3}$$

Such a map π is called a *(group) representation* of G on V. Conversely, from a group representation one can construct a linear group action.

The *group algebra* $\mathbb{C}G$ is a unital associative algebra containing G which, given a linear action on a vector space V, allows us to consider more general (including non-invertible) linear maps on V. Consider the algebra

$$\mathrm{End}(V) = \{f : V \to V \mid f \text{ linear}\} \tag{1.4}$$

containing the group $\mathrm{GL}(V)$. Analogously we can define the group algebra as an algebra containing G; we do this as follows. As a vector space, $\mathbb{C}G$ is simply the free vector space with basis G. The multiplicative structure on $\mathbb{C}G$ is obtained by linearly extending the group multiplication of G; in particular, the unit element e of the group G becomes the unit element of the algebra $\mathbb{C}G$. In other words, an arbitrary element of $\mathbb{C}G$ is a linear combination $\sum_{g \in G} c_g g$ with $c_g \in \mathbb{C}$, all but finitely many equal to zero; the product of two such elements is well-defined and is again of this form:

$$\Big(\sum_{g \in G} c_g g\Big)\Big(\sum_{h \in G} d_h h\Big) := \sum_{g,h \in G} c_g d_h gh = \sum_{k \in G} \Big(\sum_{g \in G} c_g d_{g^{-1}k}\Big) k. \tag{1.5}$$

A linear action of the group G on a vector space V, which corresponds to a group representation of G on V, now produces an algebra representation of $\mathbb{C}G$ on V (and indeed produces a $\mathbb{C}G$-module structure on V). That is, we can simply linearly extend the group homomorphism $\pi : G \to \mathrm{GL}(V)$ to an algebra homomorphism: $\overline{\pi} : \mathbb{C}G \to \mathrm{End}(V)$. Conversely, any algebra homomorphism $\rho : \mathbb{C}G \to \mathrm{End}(V)$ produces a group homomorphism from G to $\mathrm{GL}(V)$ simply by restricting to $G \subset \mathbb{C}G$. By mild abuse of notation, we will use the notation $\cdot$ also for the action of an element of $\mathbb{C}G$ on V.

Now let G be finite. The *G-symmetrizer* is the special element

$$\Sigma_G = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}G. \tag{1.6}$$

When $G = S_n$ we will simply write Σ_n instead of Σ_{S_n}.

Exercise 1.1. Show that

$$g\Sigma_G = \Sigma_G = \Sigma_G g \qquad \text{for all } g \in G \tag{1.7}$$

and deduce that Σ_G is an idempotent element of $\mathbb{C}G$, i.e. $\Sigma_G^2 = \Sigma_G$. ⌀

Given a group action on V, consider the subspace of G-fixed points:

$$V^G = \{v \in V \mid \forall g \in G : \, g \cdot v = v\}. \tag{1.8}$$

The action of Σ_G is a projection from V onto V^G. As with all projections, the image of Σ_G is precisely the set of fixed points of Σ_G:

$$V^G = \{v \in V \mid \Sigma_G \cdot v = v\}. \tag{1.9}$$

1.2. **Outline of the coordinate Bethe ansatz for quantum many-body systems.** The basic idea underpinning the CBA can now be stated as follows. Suppose that X carries an action of a group G; then G acts linearly on the vector space of all functions $X \to \mathbb{C}$ via

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \qquad \text{for all } g \in G,\ f : X \to \mathbb{C},\ x \in X. \tag{1.10}$$

Exercise 1.2. Prove that (1.10) defines a linear action on $\{f : X \to \mathbb{C}\}$. ⌀

Let $\mathcal{F}$ be a G-stable subspace of $\{f : X \to \mathbb{C}\}$, i.e. $g \cdot f \in \mathcal{F}$ for all $f \in \mathcal{F}$. Now consider a linear operator $\mathcal{H}_\gamma$ on $\mathcal{F}$ depending on a parameter $\gamma \in \mathbb{C}$ which (for all γ) commutes with the action of G:

$$g \cdot (\mathcal{H}_\gamma(f)) = \mathcal{H}_\gamma(g \cdot f), \qquad \text{for all } f \in \mathcal{F}, g \in G. \tag{1.11}$$

We are interested in finding G-fixed eigenfunctions of $\mathcal{H}_\gamma$; moreover in obtaining a basis of $\mathcal{F}$ consisting of eigenvectors of $\mathcal{H}_\gamma$ and to describe the space of operators on $\mathcal{F}$ which are simultaneously diagonalizable with $\mathcal{H}_\gamma$.

Remark 1.1. In the application to quantum physics, $\mathcal{F}$ is a Hilbert space describing the possible states of the model. Its elements f are called wavefunctions and should be interpreted as (relative) probability amplitudes; that is, $|f|^2 = f\overline{f}$ is a relative probability distribution. One obtains an honest probability amplitude by dividing f by its Hilbert space norm $\|f\|$. Furthermore, we view $\mathcal{H}_\gamma \in \mathrm{End}(\mathcal{F})$ as a Hamiltonian operator and γ as a coupling constant, with $\gamma = 0$ corresponding to the case of free (non-interacting) particles. ⌀

Suppose now that in the special case $\gamma = 0$ the operator $\mathcal{H}_0$ has an eigenfunction $f_0 \in \mathcal{F}$. Hence it has a G-fixed eigenfunction

$$F_0 = \Sigma_G \cdot f_0 = |G|^{-1} \sum_{g \in G} g \cdot f_0 \in \mathcal{F}^G. \tag{1.12}$$

Then use as an *ansatz* for a G-fixed eigenfunction of $\mathcal{H}_\gamma$ for arbitrary $\gamma \in \mathbb{C}$ an expression of the form

$$F_\gamma(\boldsymbol{q}) := |G|^{-1} \sum_{g \in G} a_\gamma(g; \boldsymbol{q})(g \cdot f_0)(\boldsymbol{q}) \tag{1.13}$$

where $a_\gamma(g; \boldsymbol{q}) \in \mathbb{C}$ are coefficients to be determined. It is natural to require that

$$a_0(g; \boldsymbol{q}) = 1 \qquad \text{for all } g \in G, \boldsymbol{q} \in X, \tag{1.14}$$

so that F_0 is recovered as a special case of F_γ. Moreover, the coefficients $a_\gamma(g)$ are constrained by the eigenvalue problem $\mathcal{H}_\gamma F_\gamma = E_\gamma F_\gamma$; the details of the derivation of $a_\gamma(g)$ of course depend on the nature of $\mathcal{F}$ and the distinguished element $\mathcal{H}_\gamma$.

Exercise 1.3. Assume that $a_\gamma(hg, h \cdot \boldsymbol{q}) = a_\gamma(g, \boldsymbol{q})$ for all $g, h \in G$, $\boldsymbol{q} \in X$. Show that the function F_γ defined by (1.13) is indeed G-fixed. ⌀

The main drawback of (1.13) is that we do not want $a_\gamma(g; \boldsymbol{q})$ to have a complicated $\boldsymbol{q}$-dependence (for instance, this makes it cumbersome to differentiate F_γ and verify whether it is an eigenfunction of $\mathcal{H}_\gamma$). Instead of requiring (1.13), it is more convenient to specify F_γ only in a subset X_+ of X which is fundamental for the G-action, in the following sense: the requirement that F_γ is G-fixed allows us to recover F_γ from $F_\gamma|_{X^+}$. This allows us to choose $a_\gamma(g; \cdot)$ to be piecewise constant.

To be more precise, consider a subset $X_+ \subseteq X$ with the following two properties.

(1) For all $g, h \in G$, $(g \cdot X_+) \cap (h \cdot X_+) = \emptyset$ if $g \neq h$;
(2) Any $f_1, f_2 \in \mathcal{F}$ which coincide on the set of *regular points*

$$X_{\mathrm{reg}} := \bigcup_{g \in G} g \cdot X_+ \tag{1.15}$$

are the same function on all of X (in other words, knowledge of $f(x)$ for all $x \in X_{\mathrm{reg}}$ determines $f(x)$ uniquely for all $x \in X$).

Example 1.2. It is good to have an example in mind now. Take $X = \mathbb{R}^n$ and $G = S_n$, whose action on X is simply given by permuting coordinates; furthermore take for $\mathcal{F}$ the space of *continuous* complex-valued functions on $\mathbb{R}^n$. Now let X_+ be the *fundamental chamber*

$$X_+ = \{(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n \,|\, q_1 > q_2 > \ldots > q_n\}. \tag{1.16}$$

Clearly, the subsets $g \cdot X_+$ are pairwise disjoint, and the regular points are those with all coordinates distinct:

$$X_{\mathrm{reg}} = \{(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n \,|\, q_j \neq q_k \text{ if } j \neq k\}. \tag{1.17}$$

Now note that continuous functions from $\mathbb{R}^n$ to $\mathbb{C}$ are uniquely determined by their values on dense subsets of $\mathbb{R}^n$. The set $X_{\rm reg}$ is indeed dense, since an arbitrary point in $X = \mathbb{R}^n$ can be approximated by a sequence of regular points. ⌀

We now arrive at the most common version of the CBA for G-fixed eigenfunctions of $\mathcal{H}_\gamma$:

$$F_\gamma(\boldsymbol{q}) = |G|^{-1} \sum_{g \in G} a_\gamma(g) f_0(g^{-1} \cdot \boldsymbol{q}) \qquad \text{for all } \boldsymbol{q} \in X_+. \tag{1.18}$$

Furthermore, we will require $a_0(g) = 1$ for all $g \in G$. The function F_γ thus uniquely defined is called the *Bethe wavefunction*.

1.3. **Bosons vs. fermions.** Let $\mathcal{F}$ be the vector space of wavefunctions associated to a particular model of n indistinguishable particles. We interpret its elements as probability amplitudes depending on variables associated to the particles, e.g. the n positions $q_1, \ldots, q_n$. Such models come into two types: either different particles are allowed to be in the same 1-particle state, or not. If particles are allowed to be in the same state, then by indistinguishability of the particles the two tuples $(q_1, \ldots, q_j, \ldots, q_k, \ldots, q_n)$ and $(q_1, \ldots, q_k, \ldots, q_j, \ldots, q_n)$ represent the same physical state, so we need to impose

$$f(q_1, \ldots, q_j, \ldots, q_k, \ldots, q_n) = f(q_1, \ldots, q_k, \ldots, q_j, \ldots, q_n) \qquad \text{for all } 1 \leq j < k \leq n. \tag{1.19}$$

In other words, we need to restrict to the S_n-fixed subspace $\mathcal{F}^{S_n} = \Sigma_n \cdot \mathcal{F}$ of *symmetric wavefunctions*. Such particles are called *bosons*.

In the second case, we are required to restrict to *antisymmetric wavefunctions*, i.e. elements $f \in \mathcal{F}$ satisfying

$$f(q_1, \ldots, q_j, \ldots, q_k, \ldots, q_n) = -f(q_1, \ldots, q_k, \ldots, q_j, \ldots, q_n) \qquad \text{for all } 1 \leq j < k \leq n. \tag{1.20}$$

Hence $f(q_1, \ldots, q_n) = 0$ if $q_j = q_k$ for some $1 \leq j < k \leq n$; considering our interpretation of elements of $\mathcal{F}$, such a state $(q_1, \ldots, q_k, \ldots, q_k, \ldots, q_n)$ is forbidden: two particles cannot be in the same 1-particle state. Such particles are called *fermions*.

Remark 1.3. By the spin-statistics theorem, three-dimensional particles with half-integer spin ("intrinsic angular momentum") are fermions and those with integer spin are bosons. However we will see that for systems in one spatial dimension can show fermionic behaviour, even if the constituent particles are bosons. ⌀

In the above formulation of the CBA, one can take $G = S_n$ for bosons. To cover the fermionic case, one can also take $G = S_n$ but then one should require that the coefficients $a_\gamma(g)$ satisfy $a_0(g) = \mathrm{sgn}(g)$ (instead of $a_0(g) = 1$) for all $g \in S_n$. Recall that the *sign* of a permutation is $\mathrm{sgn}(g) = (-1)^{\ell(g)}$ where $\ell(g)$ is the number of *inversions* of g, i.e. the size of the set

$$\{(j,k) \in \{1, \ldots, n\}^2 \,|\, j < k \text{ and } g(j) > g(k)\}. \tag{1.21}$$

Equivalently, g can be written as a product of simple transpositions in many ways, and $\ell(g)$ is the length of g, i.e. the minimum number of simple transpositions appearing in such decompositions for g. The sign map $\mathrm{sgn} : S_n \to \{-1, 1\}$ is a *group homomorphism*, i.e. $\mathrm{sgn}(gh) = \mathrm{sgn}(g)\mathrm{sgn}(h)$ for all $g, h \in S_n$.

Exercise 1.4. Find the values $\ell(g)$ and $\mathrm{sgn}(g)$ for all $g \in S_3$. ⌀

2. The one-dimensional quantum delta Bose gas on a circle

The model we will be using to illustrate the coordinate Bethe ansatz is the one-dimensional quantum delta Bose gas on a circle. This model is interesting for many reasons:

- It was the first quantum many-body model with nontrivial interactions which was solved using the CBA, namely by Lieb and Liniger in [LL63]. More precisely, it was the first quantum many-body model to be solved by the CBA whose spectrum depends nontrivially on a parameter, namely the coupling constant γ. In particular, for small values of γ, it can be used to verify results obtained by perturbation theory. It is useful here to contrast with an earlier study in [Gi60] of a one-dimensional gas of impenetrable bosons, whose spectrum does not depend on γ.
- It is a model which has been manufactured in condensed matter (fairly recently), namely by optically and magnetically confining ultracold atoms to one dimension, see for instance [Am08, Es06, KWW04, KWW06, Pa04].
- It has a rich mathematical structure, in particular from the points of view of representation theory and functional analysis.

It is also known as the *Lieb-Liniger model* and the *quantum nonlinear Schrödinger model*. It has been investigated by many authors; in the context of the CBA, in addition to [LL63], here we should at least mention C.N. Yang's extension [Y67] to the case where there is no restriction on the behaviour of the particles under S_n and the study [YY69] into the thermodynamics of the model. A useful introductory review from a physical perspective is [JCG15].

Let us now introduce the Hamiltonian $\mathcal{H}_\gamma$ of the model, initially as a formal object. We consider n indistinguishable particles constrained to a circle of circumference $L > 0$ and denote by $q_j \in \mathbb{R}$ the particle locations. Consider the Laplacian

$$\Delta = \sum_{j=1}^{n} \partial_j^2, \qquad \partial_j := \frac{\partial}{\partial q_j}. \tag{2.1}$$

Denoting by i the imaginary unit and choosing units in which $\hbar = 1$ and the particle mass equals $1/2$, we view $-\mathrm{i}\partial_j$ as the quantum-mechanical analogue of the classical momentum p_j and hence the operator

$$-\Delta = \sum_{j=1}^{n} (-\mathrm{i}\partial_j)^2 \tag{2.2}$$

as the quantum-mechanical operator analogue of the total kinetic energy of the particles. For $\gamma \geq 0$ consider the Hamiltonian

$$\mathcal{H}_\gamma = -\Delta + 2\gamma \sum_{1 \leq j < k \leq n} \delta(q_j - q_k) \tag{2.3}$$

where δ is the Dirac delta, i.e. a formal object with the following properties

$$\int_I f(x)\delta(x)\mathrm{d}x = \begin{cases} f(0) & \text{if } 0 \in I, \\ 0 & \text{if } 0 \notin I, \end{cases} \qquad \delta(x) = 0 \text{ if } x \neq 0,$$

where $I \subseteq \mathbb{R}$ is any interval with nonempty interior; in particular $\delta(-x) = \delta(x)$. Note that it is formally similar to the Calogero-Moser Hamiltonian from lecture 2, which has a long-range (but still pairwise) interaction. We will give a more rigorous interpretation of $\mathcal{H}_\gamma$ in due course, but note already that we should expect $\mathcal{H}_\gamma \to -\Delta$ as $\gamma \to 0$. Intuitively, the potential term has the interpretation that the pointlike particles have a pairwise repulsive contact interaction.

The fact that we are considering particles on a circle of circumference L means that we are interested in eigenfunctions F of $\mathcal{H}_\gamma$ satisfying the additional conditions

$$F(q_1, \ldots, q_j + L, \ldots, q_n) = F(q_1, \ldots, q_j, \ldots, q_n) \qquad \text{for all } 1 \leq j \leq n. \tag{2.4}$$

Furthermore, we want to consider a bosonic model, which means that we are interested in eigenfunctions F of $\mathcal{H}_\gamma$ which are fixed by the S_n-action. Note indeed that the Hamiltonian $\mathcal{H}_\gamma$ commutes with this action.

2.1. **Interpretation of the Hamiltonian.** Before we employ the CBA, we need to specify the domain of $\mathcal{H}_\gamma$ and explain how to interpret $\delta(q_j - q_k)$. For more detail specifically for the quantum delta Bose gas, we refer to the papers [Gu82, Gu88, Do93] and the theses [Em06, Vl11]. Good general textbook references for functional analysis are [La97] and, especially for the application to mathematical physics, [RS72].

Let us first look at a simpler example, and study a linear differential operator of the form

$$O = -\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V(q) \tag{2.5}$$

acting on complex-valued functions of one variable q; where $V(q)$ is some q-dependent potential. The eigenvalue problem of O is equivalent to

$$f''(q) = (V(q) - E)f(q), \tag{2.6}$$

where $f : (a, b) \to \mathbb{C}$ is defined on some open interval $(a, b) \subseteq \mathbb{R}$ containing 0. Let us assume that f is at least continuous. Let $\epsilon > 0$ be such that $(-\epsilon, \epsilon) \subseteq I$. We integrate (2.6) to obtain

$$\int_{-\epsilon}^{\epsilon} f''(q)\mathrm{d}q = \int_{-\epsilon}^{\epsilon} (V(q) - E)f(q)\mathrm{d}q. \tag{2.7}$$

By (the second part of) the fundamental theorem of calculus, if f'' is Riemann-integrable on $[-\epsilon,\epsilon]$ (for instance, if f'' is continuous) this implies

$$f'(\epsilon) - f'(-\epsilon) = \int_{-\epsilon}^{\epsilon} V(q)f(q)\mathrm{d}q - E\int_{-\epsilon}^{\epsilon} f(q)\mathrm{d}q. \tag{2.8}$$

We now study the limit $\epsilon \to 0$. Since ϵ was assumed positive, the limit is from above, denoted $\epsilon \to 0^+$:

$$\lim_{\epsilon\to 0^+} f'(\epsilon) - \lim_{\epsilon\to 0^+} f'(-\epsilon) = \lim_{\epsilon\to 0^+} \int_{-\epsilon}^{\epsilon} V(q)f(q)\mathrm{d}q - E\lim_{\epsilon\to 0^+}\int_{-\epsilon}^{\epsilon} f(q)\mathrm{d}q. \tag{2.9}$$

By continuity we have

$$\lim_{\epsilon\to 0^+} f'(\epsilon) - \lim_{\epsilon\to 0^-} f'(\epsilon) = \lim_{\epsilon\to 0^+} \int_{-\epsilon}^{\epsilon} V(q)f(q)\mathrm{d}q, \tag{2.10}$$

where $\epsilon \to 0^-$ indicates that the limit is taken from below. Now assume that $V(q) = \kappa\delta(q)$ for some constant $\kappa \in \mathbb{C}$. Since δ appears inside an integral, we know what to do. It gives the following derivative jump condition:

$$\lim_{\epsilon\to 0^+} f'(\epsilon) - \lim_{\epsilon\to 0^-} f'(\epsilon) = \kappa f(0). \tag{2.11}$$

There is a problem here: since we had to assume that f is twice-differentiable on $(-\epsilon,\epsilon)$ in order to apply the fundamental theorem of calculus, f' is continuous on $(-\epsilon,\epsilon)$, so the left-hand side is 0. This forces either $\kappa = 0$ (thereby reducing the problem to the eigenvalue problem of the Laplacian on $(-\epsilon,\epsilon)$) or $f(0) = 0$, in which case we need to restrict the study of (2.6) to the subspace of twice-differentiable functions on $(-\epsilon,\epsilon)$ which vanish at 0. The latter amounts to a study of the eigenvalue problem of the Laplacian for twice-differentiable functions f_+ on $(0,\epsilon)$ with a continuous extension to $[0,\epsilon)$ such that $f_+(0) = 0$, and a similar problem for $(-\epsilon, 0)$. In both cases κ does not play a role which is undesirable.

In other words, we need to make sense of (2.6) and derive (2.11) in the case $f \in C(X)$. We now outline a rigorous explanation why this replacement of a delta-function potential by derivative jump conditions is still justified. We introduce some standard notation for function spaces. Let X be an open subset of $\mathbb{R}^n$ and let $m \in \mathbb{Z}_{>0}$.

$$\begin{aligned} C(X) &= \{f : X \to \mathbb{C} \,|\, f \text{ is continuous}\}, \\ C^m(X) &= \{f : X \to \mathbb{C} \,|\, f \text{ is } m \text{ times differentiable with continuous } m\text{-th partial derivatives}\}, \\ C^\infty(X) &= \{f : X \to \mathbb{C} \,|\, f \text{ is smooth}\}. \end{aligned}$$

Recall that a function is called *smooth* if it is m times differentiable with continuous m-th partial derivatives, for all $m \in \mathbb{Z}_{>0}$.

The key idea is to understand certain eigenvalue problems, for instance (2.6), in a distributional sense. *Distributions* on a given open subset $X \subseteq \mathbb{R}^n$ are a generalization of (real- or) complex-valued functions with domain X. They are defined by how they affect certain test functions when integrated against them. Hence, in order to define a distribution f on an open set $X \subseteq \mathbb{R}^n$, it suffices to define $\int_X f(\boldsymbol{q})\phi(\boldsymbol{q})\mathrm{d}\boldsymbol{q}$ for all ϕ in the space of test functions, namely *compactly supported smooth functions*,

$$C_c^\infty(X) = \{\phi \in C^\infty(X) \,|\, \exists \text{ a compact subset } K \subset X \text{ such that } \phi|_{X\setminus K} = 0\} \tag{2.12}$$

also known as "bump functions". Two distributions f_1, f_2 on X are equal if

$$\int_X f_1(\boldsymbol{q})\phi(\boldsymbol{q})\mathrm{d}\boldsymbol{q} = \int_X f_2(\boldsymbol{q})\phi(\boldsymbol{q})\mathrm{d}\boldsymbol{q} \qquad \text{for all } \phi \in C_c^\infty(X). \tag{2.13}$$

This is sensible, since if f_1, f_2 are both in $C(X)$ then (2.13) holds if and only if $f_1 = f_2$. Distributions on X have a natural linear structure and this vector space is denoted $C_c^\infty(X)'$.

We can now define the Dirac delta distribution δ on any open subset $X \subset \mathbb{R}^n$ containing $\mathbf{0}$ as the unique element of $C_c^\infty(X)'$ satisfying

$$\int_X \delta(\boldsymbol{q})\phi(\boldsymbol{q})\mathrm{d}\boldsymbol{q} = \phi(\mathbf{0}) \qquad \text{for all } \phi \in C_c^\infty(X). \tag{2.14}$$

Let $I \subseteq \mathbb{R}$ be an open interval and consider the vector space of test functions $C_c^\infty(I)$. For $\phi \in C_c^\infty(I)$ consider

$$\mathrm{supp}(\phi) = \text{closure of } \{q \in I \,|\, \phi(q) \neq 0\}. \tag{2.15}$$

There is a notion of sequential continuity in $C_c^\infty(I)$. Namely, if (ϕ_n) is a sequence in $C_c^\infty(I)$ then we say that $\phi_n \to \phi \in C_c^\infty(I)$ if there is a compact subset $K \subset I$ such that for all $n \in \mathbb{Z}_{\geq 0}$ we have (1) $\mathrm{supp}(\phi_n) \subseteq K$ and (2) for each $m \in \mathbb{Z}_{\geq 0}$ we have $\phi_n^{(m)} \to \phi^{(m)}$ uniformly on K. Then the distributions on I are defined as the continuous linear functionals on $C_c^\infty(I)$ (linear functionals on a complex vector space V are linear maps from V to $\mathbb{C}$). For instance, $\delta \in C_c^\infty(\mathbb{R})'$ is the continuous linear functional defined by $\delta(\phi) = \phi(0)$.

Before we return to the quantum delta Bose gas, we briefly discuss some properties of distributions defined on subsets of $\mathbb{R}$. Given a distribution f on (a,b), we define its k-th derivative $f^{(k)}$ by the rule

$$\int_a^b f^{(k)}(q)\phi(q)\mathrm{d}q = (-1)^k \int_a^b f(q)\phi^{(k)}(q)\mathrm{d}q \qquad \text{for all } \phi \in C_c^\infty((a,b)). \tag{2.16}$$

Note that for $f \in C^k((a,b))$ this is compatible with calculus, as follows from integrating by parts k times and using that ϕ and its derivatives have compact support.

Exercise 2.1. Let $\theta : \mathbb{R} \to \mathbb{R}$ be the step function defined by $\theta(q) = 1$ if $q \geq 0$ and $\theta(q) = 0$ if $q < 0$. Show that the distributional derivative of the absolute value function on $\mathbb{R}$ is $2\theta - 1$. Furthermore, show that $\theta' = \delta$. Finally, show that $\delta(q) = \delta(-q)$. ⌀

Note that the Laplacian Δ naturally acts on $C^\infty(X)$. By generalizing the above definition for the derivatives of a distribution on $\mathbb{R}$ to partial derivatives of a distribution on $\mathbb{R}^n$, the Laplacian Δ can be extended to a map from $C(X)$ to the space of distributions $C_c^\infty(X)'$ on X via the rule:

$$\int_X (\Delta f)(\boldsymbol{q})\phi(\boldsymbol{q})\mathrm{d}\boldsymbol{q} = \int_X f(\boldsymbol{q})(\Delta\phi)(\boldsymbol{q})\mathrm{d}\boldsymbol{q}, \qquad \text{for all } f \in C(X), \phi \in C_c^\infty(X). \tag{2.17}$$

We obtain that the quantum Hamiltonian $\mathcal{H}_\gamma$ defined by (2.3) is the linear map from $C(\mathbb{R}^n)$ to $C_c^\infty(\mathbb{R}^n)'$ defined by

$$(\mathcal{H}_\gamma f)(\phi) = -\int_{\mathbb{R}^n} f(\boldsymbol{q})(\Delta\phi)(\boldsymbol{q})\mathrm{d}\boldsymbol{q} + 2\gamma \sum_{j<k} \int_{L_{jk}} f(\boldsymbol{q})\phi(\boldsymbol{q})\mathrm{d}\boldsymbol{q} \tag{2.18}$$

where we have introduced the hyperplanes

$$L_{jk} := \{\boldsymbol{q} \in \mathbb{R}^n \,|\, q_j = q_k\}. \tag{2.19}$$

Now consider for $m \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ the space

$$CB^m(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) \,|\, \forall g \in S_n \; f|_{g\cdot\mathbb{R}^n_+} \text{ has a } C^m \text{ extension to some open set containing } g \cdot \overline{\mathbb{R}^n_+}\}. \tag{2.20}$$

We call $\boldsymbol{q} \in \mathbb{R}^n$ *subregular* if it lies on precisely one hyperplane L_{jk}. For $j \in \{1,\ldots,n\}$ we also denote by $\boldsymbol{e}_j$ the j-th unit vector $(0,\ldots,0,1,0,\ldots,0)$. For a choice of sign and a subregular point $\boldsymbol{q} \in L_{jk}$ we write

$$g(\boldsymbol{q}^\pm) = \lim_{\epsilon \to 0^\pm} g(\boldsymbol{q} + \epsilon(\boldsymbol{e}_j - \boldsymbol{e}_k)) \tag{2.21}$$

so that $((\partial_j - \partial_k)f)(\boldsymbol{q}^\pm)$ are the two directional derivatives normal to the hyperplane L_{jk}. We can now state a result due to Gutkin which puts the CBA for the quantum delta Bose gas on a rigorous footing. The key ingredient of the proof is the so-called propagation operator, an operator which intertwines $\mathcal{H}_0$ with $\mathcal{H}_\gamma$; due to its technicality we do not reproduce the proof here.

Proposition 2.1 ([Gu82, Thm. 2.7])**.** *For $f \in CB^1(\mathbb{R}^n)$ and $E \in \mathbb{C}$ the following statements are equivalent:*

(1) $\mathcal{H}_\gamma f = Ef$ as distributions on $\mathbb{R}^n$, with $\mathcal{H}_\gamma$ defined by (2.18);

(2) $-\Delta f = Ef$ as distributions on $\mathbb{R}^n_{\mathrm{reg}}$ and *for all $j < k$ such that $\boldsymbol{q}$ is a subregular element of L_{jk} we have the* derivative jump conditions

$$((\partial_j - \partial_k - \gamma)f)(\boldsymbol{q}^+) = ((\partial_j - \partial_k + \gamma)f)(\boldsymbol{q}^-). \tag{2.22}$$

Remark 2.2.

(1) Any $f \in CB^1(\mathbb{R}^n)$ which satisfies condition (1) or (2) in Proposition 2.1 restricts to a smooth function on $\mathbb{R}^n_{\mathrm{reg}}$.

(2) Note that the limits in the left-hand side of (2.22) exist since f is assumed to lie in the set $CB^1(\mathbb{R}^n)$.

(3) Equation (2.22) amounts to the statement that the $\epsilon \to 0^+$ limit of

$$((\partial_j - \partial_k - \gamma)f)(\boldsymbol{q} + \epsilon(\boldsymbol{e}_j - \boldsymbol{e}_k)) + ((\partial_k - \partial_j - \gamma)f)(\boldsymbol{q} + \epsilon(\boldsymbol{e}_k - \boldsymbol{e}_j)) \tag{2.23}$$

vanishes. The two terms are mapped to each other by the action of the transposition σ_{jk} swapping j and k. Hence, if f is fixed by σ_{jk}, condition (2.22) becomes

$$((\partial_j - \partial_k - \gamma)f)(\boldsymbol{q}^+) = 0. \tag{2.24}$$

(4) One can define a Hamiltonian $\widehat{\mathcal{H}}_\gamma$ and accordingly formulate Proposition 2.1 in terms of an infinite group, namely the affine symmetric group $\widehat{S}_n$ which is obtained by adjoining to the symmetric group all translations by L in each coordinate. This setup is appropriate for the system of bosons on a circle. We will work with the finite group S_n, thus treating initially the system of bosons on a line, and then impose the L-periodicity conditions.

The result [Gu82, Thm. 2.7] is in fact valid for an arbitrary finite or affine reflection group. The main application is the system of bosons on a line segment with reflecting boundary conditions. For earlier papers discussing such systems, see [Ga71, GS79]. $\varnothing$

2.2. **The coordinate Bethe ansatz in action.** The Lieb-Liniger approach is now as follows. We first solve the following problem for $F \in CB^1(\mathbb{R}^n)$:

$$(\Delta + E)(F) = 0, \tag{2.25}$$

$$\lim_{\epsilon\to 0^+} (\partial_j - \partial_k - \gamma)F|_{q_j=q_k+\epsilon} = \lim_{\epsilon\to 0^-} (\partial_j - \partial_k + \gamma)F|_{q_j=q_k+\epsilon} \qquad \text{for all } j<k. \tag{2.26}$$

Note that if F is invariant under the simple transposition σ_{jk} swapping j and k, then

$$\begin{aligned} \lim_{\epsilon\to 0^-} (\partial_j - \partial_k + \gamma)F|_{q_j=q_k+\epsilon} &= -\lim_{\epsilon\to 0^-} (\partial_k - \partial_j - \gamma)F|_{q_j=q_k+\epsilon} \\ &= -\lim_{\epsilon\to 0^-} (\partial_j - \partial_k - \gamma)F|_{q_k=q_j+\epsilon} \\ &= -\lim_{\epsilon\to 0^+} (\partial_j - \partial_k - \gamma)F|_{q_j=q_k+\epsilon} \end{aligned} \tag{2.27}$$

so $\lim_{\epsilon\to 0^-} (\partial_j - \partial_k + \gamma)F|_{q_j=q_k+\epsilon}$ vanishes. Hence, if F is S_n-fixed, the condition (2.26) is equivalent to the statement that all one-sided limits $\lim_{\epsilon\to 0^+} (\partial_j - \partial_k - \gamma)F|_{q_j=q_k+\epsilon} = 0$ vanish.

Since we are interested in S_n-fixed solutions, it suffices to specify F on $\mathbb{R}^n_+$ and hence we may simply solve the following problem for $F \in C(\overline{\mathbb{R}^n_+})$ with a C^1 extension (to some open set containing $\overline{\mathbb{R}^n_+}$):

$$(\Delta + E)F = 0, \tag{2.28}$$

$$\lim_{\epsilon\to 0^+} (\partial_j - \partial_{j+1} - \gamma)F|_{q_j=q_{j+1}+\epsilon} = 0 \qquad \text{for all } 1 \le j < n. \tag{2.29}$$

We then extend to the desired element of $CB^1(\mathbb{R}^n)^{S_n}$ by symmetry. If we want the solution also to be L-periodic in each variable, it sufficies to specify F on

$$\mathbb{R}^n_{+,L} = \{(q_1,\ldots,q_n) \in \mathbb{R}^n \,|\, L > q_1 > \ldots > q_n > 0\} \subset \mathbb{R}^n_+, \tag{2.30}$$

impose the periodicity condition

$$F(L, q_1, \ldots, q_{n-1}) = F(q_1, \ldots, q_{n-1}, 0) \tag{2.31}$$

and extend to a function on $\mathbb{R}^n_+$ by L-periodicity.

We now solve the eigenvalue problem of the free Hamiltonian. For "quasi-momenta" $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ consider $\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}} \in C^\infty(\mathbb{R}^n_{\mathrm{reg}})$ defined by

$$\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}}(\boldsymbol{q}) = \mathrm{e}^{\mathrm{i}\sum_{j=1}^n \lambda_j q_j}. \tag{2.32}$$

Then $e^{\mathrm{i}\boldsymbol{\lambda}}$ is an eigenfunction of $\mathcal{H}_0 = -\Delta$ with eigenvalue

$$E = \sum_{j=1}^n \lambda_j^2. \tag{2.33}$$

The CBA suggests a solution of the form

$$F_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) = \frac{1}{n!} \sum_{g\in S_n} a_\gamma(g) \mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}}(g^{-1}\cdot \boldsymbol{q}), \qquad \boldsymbol{q} \in \mathbb{R}^n_+, \tag{2.34}$$

with $a_0(g) = 1$ for all $g \in S_n$. Since the Laplacian commutes with the action of S_n, we have met condition (2.28). Furthermore, the exponential function $\mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}}$ on $\overline{\mathbb{R}^n_+}$ indeed has a C^1 (in fact, C^∞) extension to any open set containing $\overline{\mathbb{R}^n_+}$ (in fact, to $\mathbb{R}^n$). From now on we may assume that (2.34) is valid for $\boldsymbol{q} \in \overline{\mathbb{R}^n_+}$. It remains to check (2.29) and, if L-periodicity is desired, (2.31).

2.2.1. *The special case* $n = 2$. The case $n = 2$ is instructive. Assume $q_1 \geq q_2$. We need to impose the condition

(2.35)
$$\lim_{\epsilon \to 0^+} (\partial_1 - \partial_2 - \gamma) F|_{q_1 = q_2 + \epsilon} = 0.$$

Denote the nontrivial element of S_2 by σ (the simple transposition swapping 1 and 2). The ansatz (2.34) specializes to

(2.36)
$$F_{\gamma;(\lambda_1,\lambda_2)}(q_1, q_2) = \frac{1}{2}\big(a_\gamma(\mathrm{id})\mathrm{e}^{\mathrm{i}(\lambda_1 q_1 + \lambda_2 q_2)} + a_\gamma(\sigma)\mathrm{e}^{\mathrm{i}(\lambda_2 q_1 + \lambda_1 q_2)}\big)$$

with $a_0(\mathrm{id}) = a_0(\sigma) = 1$. We impose the derivative jump condition (2.35). It yields

(2.37)
$$a_\gamma(\sigma) = \frac{\lambda_1 - \lambda_2 + \mathrm{i}\gamma}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma} a_\gamma(\mathrm{id}).$$

We may set

(2.38)
$$a_\gamma(\mathrm{id}) = 1, \qquad a_\gamma(\sigma) = \frac{\lambda_1 - \lambda_2 + \mathrm{i}\gamma}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}$$

provided $\lambda_1 \neq \lambda_2 + \mathrm{i}\gamma$; if $\lambda_1 = \lambda_2 + \mathrm{i}\gamma$ we may choose a different overall factorization (later we will see that this cannot occur for the repulsive quantum delta Bose gas on the circle). Note that $a_0(\mathrm{id}) = a_0(\sigma) = 1$ as desired. We have obtained

(2.39)
$$F_{\gamma;(\lambda_1,\lambda_2)}(q_1, q_2) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}(\lambda_1 q_1 + \lambda_2 q_2)} + \frac{\lambda_1 - \lambda_2 + \mathrm{i}\gamma}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}(\lambda_2 q_1 + \lambda_1 q_2)}\right), \qquad q_1 \geq q_2$$

Exercise 2.2. Show that the formula for $F_{\gamma;(\lambda_1,\lambda_2)}(q_1, q_2)$ for arbitrary $(q_1, q_2) \in \mathbb{R}^2$ is given by

(2.40)
$$\begin{aligned} F_{\gamma;(\lambda_1,\lambda_2)}(q_1, q_2) &= \frac{1}{2}\left(\frac{\lambda_1 - \lambda_2 - \mathrm{i}\gamma\,\mathrm{sgn}(q_1 - q_2)}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}(\lambda_1 q_1 + \lambda_2 q_2)} + \frac{\lambda_1 - \lambda_2 - \mathrm{i}\gamma\,\mathrm{sgn}(q_2 - q_1)}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}(\lambda_2 q_1 + \lambda_1 q_2)}\right) \\ &= \frac{1}{2}\left(\frac{\lambda_1 - \lambda_2 - \mathrm{i}\gamma\,\mathrm{sgn}(q_1 - q_2)}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}(\lambda_1 q_1 + \lambda_2 q_2)} - \frac{\lambda_2 - \lambda_1 - \mathrm{i}\gamma\,\mathrm{sgn}(q_1 - q_2)}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}(\lambda_2 q_1 + \lambda_1 q_2)}\right) \end{aligned}$$

where $\mathrm{sgn}(x) \in \{-1, 0, 1\}$ is the sign of the real number x. Show that $F_{\gamma;(\lambda,\lambda)}(q_1, q_2) = 0$ for all $(q_1, q_2) \in \mathbb{R}^2$ and argue that this means that the model displays fermionic behaviour. ⌀

From now on we assume that $\lambda_1 \neq \lambda_2$. To force L-periodicity of $F_{\gamma;(\lambda_1,\lambda_2)}$ we may assume furthermore that $L \geq q_1 \geq q_2 \geq 0$ and impose

(2.41)
$$F_{\gamma;(\lambda_1,\lambda_2)}(L, q_1) = F_{\gamma;(\lambda_1,\lambda_2)}(q_1, 0)$$

which will "quantize" the parameters (λ_1, λ_2). We obtain

(2.42)
$$\frac{\lambda_1 - \lambda_2 + \mathrm{i}\gamma}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}\big(\mathrm{e}^{\mathrm{i}\lambda_2 L} - \mathrm{e}^{\mathrm{i}(\lambda_2 - \lambda_1)q_1}\big) = 1 - \mathrm{e}^{\mathrm{i}(\lambda_2 - \lambda_1)q_1}\mathrm{e}^{\mathrm{i}\lambda_1 L} \qquad \text{for all } q \in [0, L].$$

The constant function 1 and the function $q \mapsto \mathrm{e}^{\mathrm{i}(\lambda_2 - \lambda_1)q}$ are linearly independent, so that (2.41) is equivalent to the system

(2.43)
$$\mathrm{e}^{\mathrm{i}\lambda_1 L} = \frac{\lambda_1 - \lambda_2 + \mathrm{i}\gamma}{\lambda_1 - \lambda_2 - \mathrm{i}\gamma}, \qquad \mathrm{e}^{\mathrm{i}\lambda_2 L} = \frac{\lambda_2 - \lambda_1 + \mathrm{i}\gamma}{\lambda_2 - \lambda_1 - \mathrm{i}\gamma}.$$

Note that we have $\mathrm{e}^{\mathrm{i}(\lambda_1 + \lambda_2)L} = 1$: the "total momentum" $\lambda_1 + \lambda_2$ is an integer multiple of $\frac{2\pi}{L}$, just as in the non-interacting case (where each λ_j is an integer multiple of $\frac{2\pi}{L}$).

2.2.2. *General* n. We now present the result for general n.

Theorem 2.3 ([LL63, Section II]). *Let $n \in \mathbb{Z}_{>0}$, $\gamma \in \mathbb{C}$ and $\boldsymbol{\lambda} \in \mathbb{C}^n$ such that $\lambda_j - \lambda_k \neq \mathrm{i}\gamma$ for all $1 \leq j < k \leq n$. Fix $E = \sum_{j=1}^n \lambda_j^2$. Let $F_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) \in CB_1(\mathbb{R}^n)^{S_n}$ be uniquely determined by*

(2.44)
$$F_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) = \frac{1}{n!}\sum_{g \in S_n} \mathrm{sgn}(g)\left(\prod_{1 \leq k < \ell \leq n} \frac{\lambda_{g(k)} - \lambda_{g(\ell)} - \mathrm{i}\gamma}{\lambda_k - \lambda_\ell - \mathrm{i}\gamma}\right)\mathrm{e}^{\mathrm{i}\sum_{k=1}^n \lambda_{g(k)} q_k}, \qquad \boldsymbol{q} \in \mathbb{R}^n_+.$$

It is an eigenfunction of $\mathcal{H}_\gamma$ with eigenvalue E (i.e. it satisfies (2.25-2.26)). Furthermore, if $L > 0$ and assuming the λ_j are pairwise distinct, the L-periodicity of $F_{\gamma;\boldsymbol{\lambda}}$ (in each argument) is equivalent to the system of Bethe equations

$$\mathrm{e}^{\mathrm{i}\lambda_j L} = \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\lambda_j - \lambda_k + \mathrm{i}\gamma}{\lambda_j - \lambda_k - \mathrm{i}\gamma}, \qquad j \in \{1, 2, \ldots, n\}. \tag{2.45}$$

Proof. Owing to the remarks right before Section 2.2.1, we only need to verify (2.29) and (2.31). We follow the approach from [KBI93, Ch. 1], originally due to Gaudin [Ga14]. Observe that

$$(\lambda_{g(k)} - \lambda_{g(\ell)} - \mathrm{i}\gamma)\mathrm{e}^{\mathrm{i}\sum_{j=1}^n \lambda_{g(j)} q_j} = -\mathrm{i}(\partial_k - \partial_\ell + \gamma)\mathrm{e}^{\mathrm{i}\sum_{j=1}^n \lambda_{g(j)} q_j} \tag{2.46}$$

and consider the matrix $M_{\boldsymbol{\lambda}}(\boldsymbol{q}) = (\mathrm{e}^{\mathrm{i}\lambda_k q_\ell})_{1\leq k,\ell\leq n}$. Recalling the Leibniz definition of determinant, we have

$$\begin{aligned} F_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) &= \text{constant} \cdot \left(\prod_{1\leq k<\ell\leq n} (\partial_k - \partial_\ell + \gamma) \right) \sum_{g\in S_n} \mathrm{sgn}(g) \prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}\lambda_{g(j)} q_j} \\ &= \text{constant} \cdot \left(\prod_{1\leq k<\ell\leq n} (\partial_k - \partial_\ell + \gamma) \right) \det M_{\boldsymbol{\lambda}}(\boldsymbol{q}) \end{aligned} \tag{2.47}$$

Now the condition (2.29) for fixed but arbitrary $j \in \{1, \ldots, n-1\}$ can be easily verified. Recalling that partial derivatives commute, first we write

$$F_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) = (\partial_j - \partial_{j+1} + \gamma)\widetilde{F}_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) \tag{2.48}$$

where

$$\begin{aligned} \widetilde{F}_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) =& \text{constant} \cdot \left(\prod_{1\leq k<j} (\partial_k - \partial_j + \gamma)(\partial_k - \partial_{j+1} + \gamma) \right) \left(\prod_{j+1<k\leq n} (\partial_j - \partial_k + \gamma)(\partial_{j+1} - \partial_k + \gamma) \right) \\ & \cdot \left(\prod_{\substack{1\leq k<\ell\leq n\\ k,\ell\notin\{j,j+1\}}} (\partial_k - \partial_\ell + \gamma) \right) \det M_{\boldsymbol{\lambda}}(\boldsymbol{q}). \end{aligned} \tag{2.49}$$

Note that the derivative jump condition (2.29) is equivalent to

$$\lim_{\epsilon\to 0^+} \left((\partial_j - \partial_{j+1})^2 - \gamma^2\right)\widetilde{F}_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q})|_{q_j = q_{j+1}+\epsilon} = 0. \tag{2.50}$$

The assignment (2.49) in fact yields a well-defined $\widetilde{F}_{\gamma;\boldsymbol{\lambda}} \in C^\infty(\mathbb{R}^n)$. By the antisymmetry property of the determinant, $\widetilde{F}_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q})$ changes by an overall sign if q_j and q_{j+1} are swapped; hence so does the expression $\left((\partial_j - \partial_{j+1})^2 - \gamma^2\right)\widetilde{F}_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q})$. Considering (2.50), we obtain (2.29).

To show (2.31), we use the formula (2.44). Let $\boldsymbol{q} \in \overline{\mathbb{R}^n_{+,L}}$ and set $C = (n!)^{-1}\prod_{k\leq\ell}(\lambda_k - \lambda_\ell - \mathrm{i}\gamma)^{-1} \neq 0$. For the left-hand side of (2.31) we observe that

$$F_{\gamma;\boldsymbol{\lambda}}(L, q_1, \ldots, q_{n-1}) = C \sum_{g\in S_n} \mathrm{sgn}(g) \left(\prod_{1\leq k<\ell\leq n} (\lambda_{g(k)} - \lambda_{g(\ell)} - \mathrm{i}\gamma) \right) \mathrm{e}^{\mathrm{i}\lambda_{g(1)} L} \mathrm{e}^{\mathrm{i}\sum_{k=2}^n \lambda_{g(k)} q_{k-1}}. \tag{2.51}$$

For the right-hand side, first note that for any group G, right-multiplication by $h \in G$ defines a bijection from G to itself. Hence for any $h \in S_n$ we have

$$F_{\gamma;\boldsymbol{\lambda}}(\boldsymbol{q}) = C \sum_{g\in S_n} \mathrm{sgn}(g)\mathrm{sgn}(h) \left(\prod_{1\leq k<\ell\leq n} (\lambda_{(gh)(k)} - \lambda_{(gh)(\ell)} - \mathrm{i}\gamma) \right) \mathrm{e}^{\mathrm{i}\sum_{k=1}^n \lambda_{(gh)(k)} q_k} \tag{2.52}$$

Now fix h to be the n-cycle $(12\cdots n)$; in particular $h(k)=k+1$ if $k<n$, $h(n)=1$ and $\mathrm{sgn}(h)=(-1)^{n-1}$. As a consequence,

$$\begin{aligned}\mathrm{sgn}(h)\left(\prod_{1\le k<\ell\le n}(\lambda_{(gh)(k)}-\lambda_{(gh)(\ell)}-\mathrm{i}\gamma)\right)&=\\ &=(-1)^{n-1}\left(\prod_{1\le k<\ell<n}(\lambda_{g(k+1)}-\lambda_{g(\ell+1)}-\mathrm{i}\gamma)\right)\left(\prod_{k=1}^{n-1}(\lambda_{g(k+1)}-\lambda_{g(1)}-\mathrm{i}\gamma)\right)\\ &=\left(\prod_{1<k<\ell\le n}(\lambda_{g(k)}-\lambda_{g(\ell)}-\mathrm{i}\gamma)\right)\left(\prod_{\ell=2}^{n}(\lambda_{g(1)}-\lambda_{g(\ell)}+\mathrm{i}\gamma)\right)\end{aligned}\tag{2.53}$$

and

$$\mathrm{e}^{\mathrm{i}\sum_{k=1}^{n}\lambda_{(gh)(k)}q_k}=\mathrm{e}^{\mathrm{i}\lambda_{g(1)}q_n}\mathrm{e}^{\mathrm{i}\sum_{k=2}^{n}\lambda_{g(k)}q_{k-1}}\tag{2.54}$$

Now we set $q_n=0$ and compare the result with (2.51); it yields

$$\sum_{g\in S_n}\mathrm{sgn}(g)\left(\prod_{1<k<\ell\le n}(\lambda_{g(k)}-\lambda_{g(\ell)}-\mathrm{i}\gamma)\right)c_{\boldsymbol\lambda}(g)\mathrm{e}^{\mathrm{i}\sum_{k=2}^{n}\lambda_{g(k)}q_{k-1}}=0\tag{2.55}$$

where

$$c_{\boldsymbol\lambda}(g)=\left(\prod_{\ell=2}^{n}(\lambda_{g(1)}-\lambda_{g(\ell)}-\mathrm{i}\gamma)\right)\mathrm{e}^{\mathrm{i}\lambda_{g(1)}L}-\left(\prod_{\ell=2}^{n}(\lambda_{g(1)}-\lambda_{g(\ell)}+\mathrm{i}\gamma)\right).\tag{2.56}$$

Any $g\in S_n$ is determined by $\{g(2),\ldots,g(n)\}\subset\{1,2,\ldots,n\}$; in particular, since the λ_j are all distinct, the maps $(q_1,\ldots,q_{n-1})\mapsto\mathrm{e}^{\mathrm{i}\sum_{k=2}^{n}\lambda_{g(k)}q_{k-1}}$ are all distinct and hence linearly independent. We deduce that the periodicity condition (2.31) is equivalent to the statement that $c_{\boldsymbol\lambda}(g)=0$ for all $g\in S_n$. Writing $g(1)=j$, we have $\{g(2),\ldots,g(n)\}=\{1,\ldots,n\}\backslash\{j\}$ and we see that $c_{\boldsymbol\lambda}(g)=0$ for all $g\in S_n$ is equivalent to the system (2.45). □

Exercise 2.3. Show that $F_{\gamma;\boldsymbol\lambda}$ defined by (2.44) satisfies $F_{0;\boldsymbol\lambda}=\Sigma_n\mathrm{e}^{\mathrm{i}\boldsymbol\lambda}$ and directly check that this indeed satisfies $\mathcal{H}_0F_{0;\boldsymbol\lambda}=EF_{0;\boldsymbol\lambda}$ with $E=\sum_j\lambda_j^2$. ⌀

Consider the solution set $\mathrm{BA}^{(n)}_{\gamma,L}:=\{\boldsymbol\lambda\in\mathbb{C}^n\,|\,\lambda_j\neq\lambda_k,\ (2.45)\text{ is satisfied}\}$.

Exercise 2.4. Prove that the set $\mathrm{BA}^{(n)}_{\gamma,L}$ is stable under the action of S_n and under shifts by the vector $\frac{2\pi}{L}(1,1,\ldots,1)$. Also prove that if $\boldsymbol\lambda\in\mathrm{BA}^{(n)}_{\gamma,L}$ then $\lambda_1+\lambda_2+\ldots+\lambda_n$ is an integer multiple of $\frac{2\pi}{L}$ (i.e. the total momentum is quantized). ⌀

The key result about the structure of the set $\mathrm{BA}^{(n)}_{\gamma,L}$ is as follows.

Theorem 2.4. *Let $L>0$, $n\in\mathbb{Z}_{>0}$ and $\gamma>0$. The set $\mathrm{BA}^{(n)}_{\gamma,L}$ is contained in $\mathbb{R}^n$ and in one-to-one correspondence with $\mathbb{Z}^n$.*

Proof. We refer to [KBI93, Thms. 1 and 2] for the detailed proofs. The first statement follows from a careful comparison of the absolute values of factors of the form $\mathrm{e}^{\mathrm{i}\lambda L}$ and $\frac{\lambda+\mathrm{i}\gamma}{\lambda-\mathrm{i}\gamma}$. For the second statement, we first re-write the equations in logarithmic form:

$$L\lambda_j+\sum_{k=1}^{n}2\tan^{-1}\left(\frac{\lambda_j-\lambda_k}{\gamma}\right)=2\pi N_j,\qquad j\in\{1,2,\ldots,n\}\tag{2.57}$$

where the N_j are integers if n is odd and half-integers if n is even. For fixed $\{N_1,\ldots,N_n\}$, one observes that (2.57) is the extremum condition of the *Yang-Yang action* [YY69]

$$S_{YY}=\frac{L}{2}\sum_{j=1}^{n}\lambda_j^2-2\pi\sum_{j=1}^{n}N_j\lambda_j+\frac12\sum_{1\le j\neq k\le n}\int_0^{\lambda_j-\lambda_k}\tan^{-1}\left(\frac{\mu}{\gamma}\right)\mathrm{d}\mu,\tag{2.58}$$

which has a positive definite matrix of second derivatives (so that S_{YY} has a unique extremum). □

Exercise 2.5. Show that the extremum condition of (2.58) is indeed given by (2.57). ⌀

Remark 2.5. Note that the reality of the λ_j implies that we cannot have $\lambda_j-\lambda_k=\mathrm{i}\gamma$ for any $j,k\in\{1,\ldots,n\}$ so that the condition in Theorem 2.3 is satisfied in the periodic case. As a consequence, the Bethe wavefunction is well-defined for all $\boldsymbol\lambda\in\mathrm{BA}^{(n)}_{\gamma,L}$. ⌀

2.2.3. *Integrability and Dunkl-type operators.* This system is integrable in the following sense. The Hamiltonian $\mathcal{H}_\gamma = \mathcal{H}_\gamma^{(2)}$ can be embedded into a family of n independent commuting operators

$$\mathcal{H}_\gamma^{(1)}, \ldots, \mathcal{H}_\gamma^{(n)} \tag{2.59}$$

which have the $F_{\gamma;\boldsymbol{\lambda}}$ as joint eigenfunctions; in fact, they can be conveniently defined on the span of the Bethe wavefunctions via

$$\mathcal{H}_\gamma^{(r)} F_{\gamma;\boldsymbol{\lambda}} = p_r(\boldsymbol{\lambda}) F_{\gamma;\boldsymbol{\lambda}}, \tag{2.60}$$

where $p_r(\boldsymbol{\lambda})$ is the r-th power sum polynomial:

$$p_r(\boldsymbol{\lambda}) = \sum_{j=1}^n \lambda_j^r. \tag{2.61}$$

Note that $\{p_r(\boldsymbol{\lambda}) \,|\, r = 1, \ldots, n\}$ forms a generating set of the commutative algebra of all polynomials symmetric in the indeterminates $\lambda_1, \ldots, \lambda_n$, i.e. an arbitrary symmetric polynomial expression in the λ_j can be written uniquely as a polynomial in the $p_j(\boldsymbol{\lambda})$ with complex coefficients:

$$\mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n} = \mathbb{C}[p_1(\boldsymbol{\lambda}), \ldots, p_n(\boldsymbol{\lambda})]. \tag{2.62}$$

Exercise 2.6. Let $n = 2$. Write $p_4(\boldsymbol{\lambda})$ as a polynomial in $p_1(\boldsymbol{\lambda})$ and $p_2(\boldsymbol{\lambda})$. $\varnothing$

Remark 2.6. The algebra $\mathbb{C}[\lambda_1, \ldots, \lambda_n]^{S_n}$ has many more useful bases, in particular the elementary symmetric polynomials and the complete homogeneous symmetric polynomials. Power-sum polynomials are useful from the physical perspective since $p_1(\boldsymbol{\lambda})$ can be interpreted as total momentum and $p_2(\boldsymbol{\lambda})$ as total energy. $\varnothing$

It is possible to express operators corresponding to these constants of motion in terms of well-defined commuting operators acting on elements of $C^\infty(\mathbb{R}^n_{\rm reg})$. For $j = 1, \ldots, n$ consider the Dunkl-type operator [Op95, MW96]

$$\partial_{j,\gamma} := \partial_j - \gamma \sum_{k<j} \theta(q_j - q_k)\sigma_{jk} + \gamma \sum_{k>j} \theta(q_k - q_j)\sigma_{jk} \tag{2.63}$$

where σ_{jk} swaps the coordinates q_j and q_k and θ is the step function defined in Exercise 2.1. Furthermore, $\partial_{j,\gamma} = \partial_j$ if we restrict to functions on $\mathbb{C}^\infty(\mathbb{R}^n_+)$ and we have $\mathcal{H}_\gamma = -\sum_{j=1}^n \partial_{j,\gamma}^2$ as operators on $C^\infty(\mathbb{R}^n_{\rm reg})$.

Exercise 2.7. Show for $n = 2$ that

$$\partial_{1,\gamma} := \partial_1 + \gamma\theta(q_2 - q_1)\sigma_{12}, \qquad \partial_{2,\gamma} := \partial_2 - \gamma\theta(q_2 - q_1)\sigma_{12} \tag{2.64}$$

commute. Also show that $\mathcal{H}_\gamma = -(\partial_{1,\gamma}^2 + \partial_{2,\gamma}^2)$ as operators on $C^\infty(\mathbb{R}^n_{\rm reg})$. $\varnothing$

Then it follows, see e.g. [Vl11, Prop. 3.5.7], that for all $f \in CB^\infty(\mathbb{R}^n)$ the system

$$\partial_{j,\gamma} f = \mathrm{i}\lambda_j f \qquad \text{for all } 1 \le j \le n \tag{2.65}$$

implies

$$f \text{ satisfies } -\Delta f = Ef \text{ with } E = \sum_j \lambda_j^2 \text{ and the jump conditions (2.22) on } L_{jk}. \tag{2.66}$$

This can be used to show that the Bethe wavefunctions $F_{\gamma;\boldsymbol{\lambda}}$ are joint eigenfunctions of the $\partial_{j,\gamma}$, which was first observed in [Gu87]. Assuming the span of the Bethe wavefunctions is dense in $L^2([0,L]^n)^{S_n}$ (a property known as *completeness*), we deduce that symmetric expressions in the Dunkl-type operators are the integrals of motion of the quantum delta Bose gas.

2.2.4. *Orthogonality, completeness and norm formulae.* Orthogonality and completeness of the Bethe wavefunctions of the repulsive quantum delta Bose gas on a circle were proven by Dorlas in [Do93] using completeness of the plane waves, a continuity argument at $\gamma = 0$ and algebraic Bethe ansatz techniques. The result by Dorlas entails that the set

$$\left\{ F_{\gamma,\boldsymbol{\lambda}} \,\middle|\, \boldsymbol{\lambda} \in \mathrm{BA}_{\gamma,L}^{(n)} \right\} \tag{2.67}$$

is an orthogonal basis for a dense subspace of $L^2([0,L]^n)^{S_n}$.

Norms of quantum-mechanical wavefunctions are important for the computation of probabilities; namely if F is an arbitrary wavefunction then $\widetilde{F} := F/\|F\|$, where $\|F\|$ is the Hilbert space norm of F satisfies $\|\widetilde{F}\| = 1$ so that $\boldsymbol{q} \mapsto |\widetilde{F}(\boldsymbol{q})|^2$ can be interpreted as a genuine probability density. The norms of the Bethe wavefunctions of the repulsive quantum delta Bose gas on a circle were conjectured by Gaudin [Ga72, Ga14] to be given by determinantal formula, which was proven by Korepin [Ko82] using algebraic Bethe ansatz techniques.

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