Ample groupoid algebras

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Outline

- Groupoids
- Motivation: group algebras
- Ample groupoids and Steinberg algebras
- Groupoid C^*-algebras
- Some interesting results

A groupoid G

Definition

Let G be a set and $G^{(2)}$ be a subset of $G \times G$ such that there is a map $(\gamma, \alpha) \mapsto \gamma\alpha$ from $G^{(2)}$ to G.

Suppose there is an involution $\gamma \mapsto \gamma^{-1}$ on G.

Then we say G is groupoid if the following are satisfied:

- If (γ, α) and (α, β) are in $G^{(2)}$, then so are $(\gamma\alpha, \beta)$ and $(\gamma, \alpha\beta)$, and the equation $(\gamma\alpha)\beta = \gamma(\alpha\beta)$ is satisfied.
- We have $(\gamma^{-1}, \gamma) \in G^{(2)}$ for every $\gamma \in G$.
- If $(\gamma, \alpha) \in G^{(2)}$, then $(\gamma^{-1}\gamma)\alpha = \alpha$ and $\gamma(\alpha\alpha^{-1}) = \gamma$.

We call $G^{(2)}$ the set of composable pairs.

Terminology

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- Define maps s and $r : G \to G$ such that if $\gamma \in G$,
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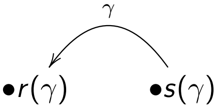
- Each $\gamma \in G$ can be viewed as a morphisms from $s(\gamma)$ to $r(\gamma)$.



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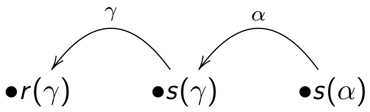

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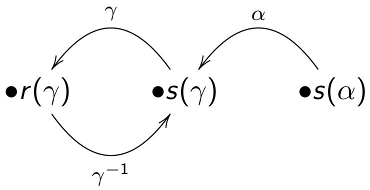

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Examples

Let G be a group. Then G is a groupoid with

- $G^{(2)} = G \times G$
- $s(\gamma) = \gamma^{-1}\gamma = e$
- $r(\gamma) = \gamma\gamma^{-1} = e$
- $G^{(0)} = \{e\}$

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Lemma

The following are equivalent:

1. *G is a group;*
2. *G is a groupoid and $G^{(0)}$ is a singleton;*
3. *G is a groupoid and $G^{(2)} = G \times G$.*

Examples

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- $G^{(2)} = \{((h_2, h_1 \cdot x), (h_1, x))\}$
- Composition: $(h_2, h_1 \cdot x)(h_1, x) = (h_2 h_1, x)$
- Inverse: $(h, x)^{-1} = (h^{-1}, h \cdot x)$
- $s((h, x)) = (h^{-1}, h \cdot x)(h, x) = (e, x)$
- $r((h, x)) = (h, x)(h^{-1}, h \cdot x) = (e, h \cdot x)$
- $G^{(0)} = \{e\} \times X \equiv X$

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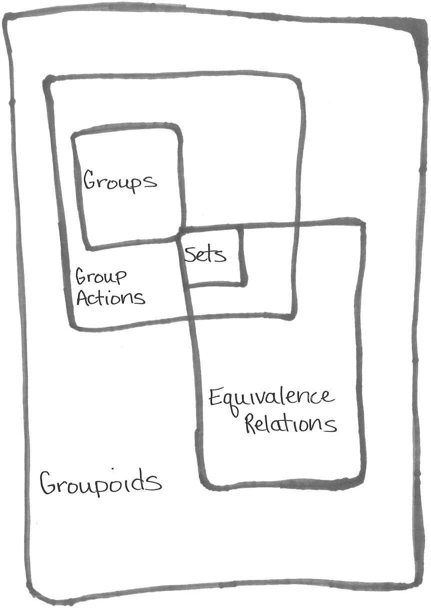
- Inverse: $(x,y)^{-1} = (y,x) \in Q$ (symmetric)
- For $(x,y) \in Q$,
  - $s((x,y)) = (y,x)(x,y) = (y,y) \in Q$
  - $r((x,y)) = (x,y)(y,x) = (x,x) \in Q$ (reflexive)
- $G^{(0)} = \Delta X \subseteq Q$

Lemma

A groupoid is principal if and only if it is (algebraically) isomorphic to an equivalence relation.


Groups
Sets
Group Actions
Equivalence Relations
Groupoids

Directed graph conventions CAUTION

- Let E be a row-finite directed graph with no sources.
- We say a vertex is
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  $F(E) := \{e_1 \ldots e_n : e_i \in E^1 \text{ such that } s(e_i) = r(e_{i+1})\} \cup E^0$

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- Our path conventions help.

- Length is a functor $d : F(E) \to \mathbb{N}$ such that if $d(\lambda) = m + n$, then there exist unique μ and ν in $F(E)$ with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Higher-rank graphs

- A higher-rank graph or *k-graph* is a category Λ with objects Λ^0 (called vertices) such that there is a functor $d : \Lambda \to \mathbb{N}^k$ that satisfies the unique factorisation property:

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- Path categories of directed graphs $\iff$ 1-graphs.
- Infinite paths in Λ
  - When $k = 1$, $\Lambda^\infty = E^\infty$
  - Recall the k-graph $\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$
  - $\Lambda^\infty := \{x : \Omega_k \to \Lambda : x \text{ is a degree-preserving functor}\}$
  - For $x \in \Lambda^\infty$, write $r(x)$ for $r(x(0))$.
  - We can compose a finite path $\lambda \in \Lambda$ and an infinite path x with $s(\lambda) = r(x)$ to get $\lambda x \in \Lambda^\infty$.

The boundary path groupoid

- Let Λ be a row-finite k-graph with no sources.

$$G_\Lambda := \{(\mu x, d(\mu) - d(\nu), \nu x) : \mu, \nu \in \Lambda, x \in \Lambda^\infty, s(\mu) = s(\nu) = r(x)\}$$

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- $(w, n, y) \in G_\Lambda$ has range $(w, 0, w)$ and source $(y, 0, y)$

- $G_\Lambda^{(0)} = \{(x, 0, x) : x \in \Lambda^\infty\}$ which we identify with Λ^∞

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- For $h \in H$, let $1_h : H \to R$ denote the function

$$1_h(x) = \begin{cases} 1 & \text{if x=h} \\ 0 & \text{otherwise.} \end{cases}$$

- $RH := \text{span}\{1_h : h \in H\}$
  - Addition and scalar multiplication of functions are defined pointwise.
  - Multiplication of functions $f, g \in RH$ is given by convolution: $fg(h) = \sum_{h_1 h_2 = h} f(h_1)g(h_2)$.
  - Multiplication of generators is given by $1_{h_1} 1_{h_2} = 1_{h_1 h_2}$.

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- To generalise observe: The set of singletons is a basis for the discrete topology on G where each element of the basis is a compact open set such that s and r restrict to injective maps.

Ample Groupoids

Let G be a topological groupoid: G is equipped with a topology such that composition and inversion are continuous.

Definitions

1. We say an open set $B \subseteq G$ is an open bisection if $r_{|B}$ and $s_{|B}$ are homeomorphisms onto an open subset of $G^{(0)}$.
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- A topological group is discrete if and only if it is an ample groupoid.
- If B and D are compact open bisections then so is $BD = \{bd : b \in B, d \in D \text{ and } s(b) = r(d)\}$.
- The collection of all copact open bisections forms an inversre semigroup.

Ample groupoids

Examples

- discrete space
- Cantor set
- discrete group
- discrete groupoid
- the boundary path groupoid G_Λ
- ...and many more

Steinberg Algebra

- Let G be an ample groupoid and R be a commutative unital ring.
- For $B \subseteq G$, define $1_B : G \to R$ such that $1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$
- The Steinberg Algebra:

  $A_R(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$

  - Addition and scalar multiplication of functions are defined pointwise.
  - Multiplication is given by convolution $fg(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$
  - Multiplication of generators is given by $1_B 1_D = 1_{BD}$.
- Introduced by Steinberg in 2010 to model discrete inverse semigroup algebras.

Examples of Steinberg Algebras

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- The algebra of $n \times n$ matrices with entries in R
  - Suppose X contains n elements
  - Let $G := X \times X$, the trivial equivalence relation
  - G is ample with respect to the discrete topology
  - $\{1_{(x_i, x_j)}\}$ is a set of matrix units that generate $A_R(G)$

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- Discrete inverse semigroup algebras

G_Λ as an ample Hausdorff groupoid

- Suppose Λ is a row-finite directed graph with no sources.
- Recall
  $G_\Lambda := \{(\mu x, d(\mu) - d(\nu), \nu x) : \mu, \nu \in \Lambda, x \in \Lambda^\infty, s(\mu) = s(\nu) = r(x)\}$.
- For each $\mu \in \Lambda$ define $Z(\mu) := \{\mu x \mid x \in \Lambda^\infty, s(\mu) = r(x)\} \subseteq X$

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- The sets $Z(\mu)$ are a basis of compact open sets for a Hausdorff topology on $G_\Lambda^{(0)} \sim \Lambda^\infty$.

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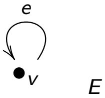
- The sets $Z(\mu)$ are a basis of compact open sets for a Hausdorff topology on $G_\Lambda^{(0)} \sim \Lambda^\infty$.
- For finite paths μ, ν with $s(\mu) = s(\nu)$, define
$Z(\mu, \nu) := \{(\mu x, |\mu| - |\nu|, \nu x) : x \in X, s(\mu) = r(x)\}$, and
- The collection of $Z(\mu, \nu)$ form a basis of compact open bisections for a Hausdorff topology on G_Λ.
- Since we view $G^{(0)} \subseteq G$, we identify (for example) $Z(v)$ and $Z(v, v)$.

E

E

- Let $x = eee....$ Then $E^\infty = \{x\}$



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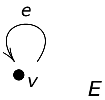

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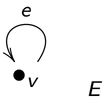

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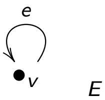
- It is isomorphic to the group $\mathbb{Z}$.



- Let $x = eee....$ Then $E^\infty = \{x\} \equiv \{(x,0,x)\}$
- The groupoid $G_E = \{(x,n,x) : n \in \mathbb{Z}\}$
- It is isomorphic to the group $\mathbb{Z}$.
- The topology is discrete:

  If $n > 0$, then $Z(\underbrace{e...e}_{n}, v) = \{(x,n,x)\}$.

  If $n < 0$, then $Z(v, \underbrace{e...e}_{-n}) = \{(x,n,x)\}$.

  If $n = 0$, then $Z(v,v) = \{(x,0,x)\}$.

Groupoid C^*-algebras

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- From now on assume G is amenable.

Higher-rank graph algebras are groupoid C^*-algebras

- $C^*(\Lambda) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$
- The collections $\{1_{Z(v)}\}$ and $\{1_{Z(s(\lambda),\lambda)}\}$ form a Cuntz-Krieger Λ-family in $C^*(G)$.
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  - Surjective ✓
  - Injective by gauge invariant uniqueness theorem.
- Restricts to a isomorphism of the Kumjian-Pask algebra $\text{KP}_{\mathbb{C}}(\Lambda)$ onto the Steinberg algebra $A_{\mathbb{C}}(G_\Lambda)$.

Connections between the algebra and the C^*-algebra

Theorem (Brown-C-Farthing-Sims 2014)

Let G be a Hausdorff ample groupoid. Then the following are equivalent:

1. *G is minimal and effective.*
2. *A_K is simple for any field K.*
3. *$C^*(G)$ is simple.*

Connections between the algebra and the C^*-algebra

Theorem (Renault, C-Edie-Michell-an Huef-Sims)

Suppose G is strongly effective, Hausdorff ample groupoid. The following are lattice isomorphic:

1. *The open invariant subsets of $G^{(0)}$.*
2. *The ideals in $A_K(G)$ for any field K.*
3. *The ideals in $C^*(G)$.*

Connections between the algebra and the C^*-algebra

Theorem (Et al)

Suppose G is effective. The following are equivalent:

1. *There is a diagonal preserving C^*-isomorphism from $C^*(G_1)$ onto $C^*(G_2)$.*
2. *There is a diagonal preserving ring isomorphism from $A_{\mathbb{C}}(G_1)$ onto $A_{\mathbb{C}}(G_2)$.*
3. $G_1 \cong G_2$.

The End