Lecture 5: discrete integrable systems

Mats Vermeeren*

19 November 2020

1 Introduction

In the previous lecture we studied the KdV equation

$$u_t = u_{xxx} + 6uu_x.$$

A closely related equation is obtained by introducing the potential variable v satisfying $u = v_x$:

$$v_t = v_{xxx} + 3v_x^2 \tag{1}$$

It is called the **potential KdV equation**. In this lecture we will have a look at a fully discrete analogue to this equation.[1] By **fully discrete** we mean that both space and time are discretised, so we replace $u : \mathbb{R}^2 \to \mathbb{R}$ (or v) by $U : \mathbb{Z}^2 \to \mathbb{R} : (m,n) \mapsto U_{m,n}$. The **lattice potential KdV (lpKdV) equation** is

$$(U_{m+1,n} - U_{m,n+1})(U_{m,n} - U_{n+1,m+1}) = q^2 - p^2. \tag{2}$$

In this lecture we will try to answer two questions:

- Is the lpKdV equation integrable?

  To answer this we first need to figure out what integrability means for difference equations. We will suggest a definition in Section 2 which is loosely analogous to the existence of commuting flows for integrable PDEs. We then look at some consequences of this definition: the existence of a discrete Lax pair in Section 3 and an algorithm to construct soliton solutions in Section 4.

- Why do we say that the strange-looking equation (2) is a discrete version of (1)?

  We will discuss a continuum limit procedure in Section 5. There are other answers to this question that we do not discuss here, but a hint at a different answer is given in Footnote 2.

*School of Mathematics, University of Leeds. m.vermeeren@leeds.ac.uk

[1] Both the KdV equation and the potential KdV equation have discrete counterparts. We will focus on the discrete potential KdV because it falls into a particularly interesting class of integrable difference equations.

Equation (2) is an example of a **quad equation**, which is a difference equation on a square stencil,

$$Q(U_{m,n}, U_{m+1,n}, U_{m,n+1}, U_{n+1,m+1}; p, q) = 0,$$

where Q is affine linear in each of its first four entries, and $p, q \in \mathbb{R}$ are parameters, which can be associated with each of the lattice directions in $\mathbb{Z}^2$. The linearity condition implies that whenever three of the entries U are given, we can solve for the fourth. Furthermore, we assume that a quad equation is symmetric under permutation of the lattice directions:

$$Q(U_{m,n}, U_{m,n+1}, U_{m+1,n}, U_{n+1,m+1}; p, q) = \pm Q(U_{m,n}, U_{m+1,n}, U_{m,n+1}, U_{n+1,m+1}; q, p).$$

In other words, as far as a quad equation is concerned, the only difference between the lattice directions is the value of the lattice parameter. In some contexts, it is helpful to go further and require quad equations to be invariant under the full symmetry group of the square D_4.

Notation In the literature there exist several notations to denote U evaluated at some lattice site. In some texts, the indices are used to denote lattice shifts with respect to some reference position $(m, n) \in \mathbb{Z}^2$, with "1" and "2" representing shifts in the horizontal and vertical direction respectively. In this notation Equation (2) reads

$$(U_1 - U_2)(U - U_{12}) = q^2 - p^2.$$

Often, "decorations" of U are used to denote lattice shifts, leading to an even more compact equation

$$(\widetilde{U} - \widehat{U})(U - \widehat{\widetilde{U}}) = q^2 - p^2.$$

2 Integrability as multidimensional consistency

An important aspect of integrability of the continuous KdV equation is that it is part of a hierarchy of commuting flows. A related notion exists for quad equations. To this end, let us consider an additional lattice direction and consider U as a function on $\mathbb{Z}^3$. We also introduce an additional parameter r, associated to the new lattice directions. We use $\widetilde{\ }$ and $\widehat{\ }$ to denote shifts in the original lattice direction and $\overline{\ }$ for shifts in the third lattice direction.

We can impose copies of the lpKdV equations on elementary squares in this lattice:

$$(\widetilde{U} - \widehat{U})(U - \widehat{\widetilde{U}}) = q^2 - p^2, \tag{3}$$

$$(\widehat{U} - \overline{U})(U - \widehat{\overline{U}}) = r^2 - q^2, \tag{4}$$

$$(\overline{U} - \widetilde{U})(U - \widetilde{\overline{U}}) = p^2 - r^2. \tag{5}$$

Due to the symmetry of the quad equation it does not matter which orientation we choose on the squares of the lattice.

Now consider an elementary cube in the lattice, and prescribe values of the field U in one half of it: fix U, $\widetilde{U}$, $\widehat{U}$ and $\overline{U}$. Then the equations (3)–(5) give unique values for $\widehat{\widetilde{U}}$, $\widehat{\overline{U}}$ and $\widetilde{\overline{U}}$. To calculate $\widehat{\widetilde{\overline{U}}}$, we can use any of the three shifted equations

$$(\widetilde{\overline{U}} - \widehat{\overline{U}})(\overline{U} - \widehat{\widetilde{\overline{U}}}) = q^2 - p^2, \tag{6}$$

$$(\widehat{\widetilde{U}} - \widetilde{\overline{U}})(\widetilde{U} - \widehat{\widetilde{\overline{U}}}) = r^2 - q^2, \tag{7}$$

$$(\widehat{\overline{U}} - \widehat{\widetilde{U}})(\widehat{U} - \widehat{\widetilde{\overline{U}}}) = p^2 - r^2. \tag{8}$$

Hence, a priory, we expect three different values for $\widehat{\widetilde{\overline{U}}}$. However, for the lpKdV equation this is not the case and $\widehat{\widetilde{\overline{U}}}$ is well-defined (single-valued). This surprising property is a discrete analogue of commuting PDEs. It can be considered as a definition of integrability for quad equations:

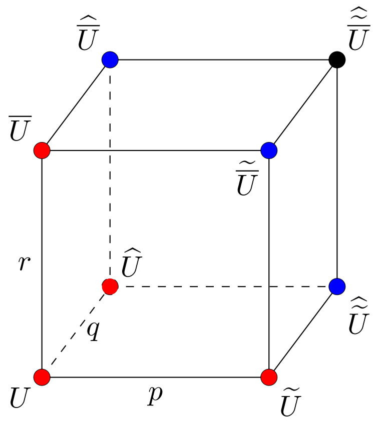

Figure 1: Evaluations of the discrete field U on an elementary cube in the lattice $\mathbb{Z}^3$.

Definition 1. We say that a quad equation $Q(U, \widetilde{U}, \widehat{U}, \widehat{\widetilde{U}}; p, q) = 0$ is **consistent around the cube** if $\widehat{\widetilde{\overline{U}}}$ is uniquely defined as a function of U, $\widetilde{U}$, $\widehat{U}$ and $\overline{U}$ by the system of equations

$$\begin{aligned}
&Q(U, \widetilde{U}, \widehat{U}, \widehat{\widetilde{U}}; p, q) = 0, \qquad && Q(\overline{U}, \widetilde{\overline{U}}, \widehat{\overline{U}}, \widehat{\widetilde{\overline{U}}}; p, q) = 0, \\
&Q(U, \widehat{U}, \overline{U}, \widehat{\overline{U}}; q, r) = 0, \qquad && Q(\widetilde{U}, \widehat{\widetilde{U}}, \widetilde{\overline{U}}, \widehat{\widetilde{\overline{U}}}; q, r) = 0, \\
&Q(U, \overline{U}, \widetilde{U}, \widetilde{\overline{U}}; r, p) = 0, \qquad && Q(\widehat{U}, \widehat{\overline{U}}, \widehat{\widetilde{U}}, \widehat{\widetilde{\overline{U}}}; r, p) = 0.
\end{aligned}$$

If a quad equation is consistent around the cube, it can be consistently imposed on all squares of a lattice $\mathbb{Z}^n$ of arbitrary dimension $n \geq 2$.

Exercise 1. Show the the lpKdV equation is consistent around the cube. You can do this by verifying that

$$\widehat{\widetilde{\overline{U}}} = -\frac{(q^2-r^2)\overline{U}\widehat{U} + (r^2-p^2)\widetilde{U}\overline{U} + (p^2-q^2)\widehat{U}\widetilde{U}}{(q^2-r^2)\widetilde{U} + (r^2-p^2)\widehat{U} + (p^2-q^2)\overline{U}} \tag{9}$$

no matter which of the equations (6)–(8) is used (along with two of the equations (3)–(5)).

Remark A curious observation regarding Equation (9) is that the formula for $\widehat{\widetilde{\overline{U}}}$ does not depend on U, only on $\widetilde{U}$, $\widehat{U}$ and $\overline{U}$. This is called the **tetrahedron property** because the four vertices of the cube that are involved in such a relation form a tetrahedron. A complete classification of D_4-invariant quad equations satisfying the tetrahedron property was obtained in Adler, Bobenko, and Suris [2003].

Exercise 2. Repeat Exercise 1, but for the **lattice potential modified KdV equation**

$$p(U\widehat{U} - \widetilde{U}\widehat{\widetilde{U}}) - q(U\widetilde{U} - \widehat{U}\widehat{\widetilde{U}}) = 0. \tag{10}$$

You should find

$$\widehat{\widetilde{\overline{U}}} = -\frac{q^2(r\widetilde{U} - p\overline{U})\widehat{U} + r^2(p\widehat{U} - q\widetilde{U})\overline{U} + p^2(q\overline{U} - r\widehat{U})\widetilde{U}}{q^2(r\overline{U} - p\widetilde{U}) + r^2(p\widetilde{U} - q\widehat{U}) + p^2(q\widehat{U} - r\overline{U})}.$$

3 From CAC to a Lax pair

In the continuous case the commuting flows of higher (potential) KdV equations are symmetries of the (potential) KdV equation (1). In the same spirit, we can consider a shift in the third lattice direction as a transformation of one solution of the lpKdV equation into another. To emphasize this point of view, we introduce a new field $X : \mathbb{Z}^3 \to \mathbb{R}$ that is a shift of $U : \mathbb{Z}^3 \to \mathbb{R}$ in the third lattice direction, $X(m,n,k) = U(m,n,k+1)$, or simply $X = \overline{U}$. We then look for a difference equation satisfied by X with respect to the first two lattice variables m and n. From Equations (4) and (5) we find

$$\widehat{X} = \frac{UX + r^2 - q^2 - U\widehat{U}}{X - \widehat{U}},$$

$$\widetilde{X} = \frac{UX + r^2 - p^2 - U\widetilde{U}}{X - \widetilde{U}}.$$

We can linearise these equations by setting $X = \dfrac{F}{G}$, for some functions $F, G : \mathbb{Z}^3 \to \mathbb{R}$ to be determined. We then find

$$\begin{aligned}
\widehat{F} &= \gamma \left(UF + (r^2 - q^2 - U\widehat{U})G \right) \\
\widehat{G} &= \gamma \left(F - \widehat{U}G \right), \\
\widetilde{F} &= \gamma' \left(UF + (r^2 - p^2 - U\widehat{U})G \right) \\
\widetilde{G} &= \gamma' \left(F - \widetilde{U}G \right),
\end{aligned}$$

for some nonvanishing $\gamma, \gamma' : \mathbb{Z}^3 \to \mathbb{R}$. These equations can be written in matrix form as

$$\widehat{\begin{pmatrix} F \\ G \end{pmatrix}} = M \begin{pmatrix} F \\ G \end{pmatrix} \qquad \text{and} \qquad \widetilde{\begin{pmatrix} F \\ G \end{pmatrix}} = L \begin{pmatrix} F \\ G \end{pmatrix} \tag{11}$$

with

$$M = \gamma' \begin{pmatrix} U & r^2 - q^2 - U\widehat{U} \\ 1 & -\widehat{U} \end{pmatrix} \qquad \text{and} \qquad L = \gamma \begin{pmatrix} U & r^2 - p^2 - U\widetilde{U} \\ 1 & -\widetilde{U} \end{pmatrix}$$

Now, by virtue of the consistency around the cube of the lpKdV equation, the two ways of calculating $\widehat{\widetilde{X}} = \widehat{\widetilde{F/G}}$ must coincide for arbitrary F and G, hence

$$\widetilde{\widehat{\begin{pmatrix} F \\ G \end{pmatrix}}} = \gamma'' \widehat{\widetilde{\begin{pmatrix} F \\ G \end{pmatrix}}} \quad \Leftrightarrow \quad \widetilde{M \begin{pmatrix} F \\ G \end{pmatrix}} = \gamma'' \widehat{L \begin{pmatrix} F \\ G \end{pmatrix}} \quad \Leftrightarrow \quad \widetilde{M} L \begin{pmatrix} F \\ G \end{pmatrix} = \gamma'' \widehat{L} M \begin{pmatrix} F \\ G \end{pmatrix}$$

for some $\gamma'' : \mathbb{Z}^3 \to \mathbb{R}$. We can make sure that $\gamma'' = 1$ by choosing suitable γ and γ'. (Changing γ and γ' corresponds to multiplying F and G by the same function, which does not affect $X = \frac{F}{G}$.) Since the previous identity should hold for all F and G, it follows that

$$\widetilde{M} L = \widehat{L} M \tag{12}$$

We call this last equation the **discrete Lax equation** or **discrete zero-curvature condition** and the matrices L and M a **Lax pair**. This terminology is motivated by the fact that, just like a continuous Lax equation, Equation (12) expresses the compatibility of two linear equations (11).

Exercise 3. Verify that the Lax equation (12) is equivalent to the lpKdV equation (3) if we set $\gamma = \gamma' = 1$ in the definition of L and M.

Exercise 4. Following the procedure above, construct a Lax pair for the lattice potential mKdV equation (10). You will need a different choice of γ and γ' for this example. Try $\gamma = \gamma' = \frac{1}{U}$.

Note that the discrete Lax equation (12) does not involve the variable X or shifts in the $\overline{}$-direction. The existence of a Lax pair is a feature of integrability that does not involve additional lattice dimensions. What we have sketched here is that consistency around the cube implies the existence of a Lax pair. This is one way in which the rather exotic definition of integrability as consistency around the cube has implications which perhaps are more easily recognizable as features of integrability.

4 Soliton solutions

Another parallel between integrable PDEs and their discrete counterparts is the existence of soliton solutions. In this section we show how the CAC property can be used to construct soliton solutions. Our approach will be similar to that of Section 3 in that we interpret the $\overline{}$ shift as a transformation between two solutions of the lattice KdV equation.

Some background and terminology If a set of equations $\mathcal{B}(X, V) = 0$ implies $\mathcal{A}(X) = 0$ upon elimination of V and $\mathcal{C}(V) = 0$ upon elimination of X, then $\mathcal{B}(X, V) = 0$ is called a **Bäcklund transformation** between $\mathcal{A}(X) = 0$ and $\mathcal{C}(V) = 0$. If $\mathcal{A} = \mathcal{C}$ we call it an **auto-Bäcklund transformation**.[2] In this section we will view the CAC-property as follows: the four equations on the vertical faces of the cube are an auto-Bäcklund transformation of the bottom/top equation. Then, starting with a very simple solution, we use this Bäcklund transformation to construct a soliton solution.

It turns out that the soliton solutions will have a simpler form if we transform the field $U_{m,n} \mapsto V_{m,n} = U_{m,n} + mp + nq$. In terms of V, the lpKdV equation reads

$$(V_{m+1,n} - V_{m,n+1} + q - p)(V_{m,n} - V_{n+1,m+1} + q + p) = q^2 - p^2 \tag{13}$$

Now it is easy to see that $V \equiv 0$ is a solution. We call this the **background solution**. The corresponding background solution of (2) is $U_{m,n} = -mp - nq$.

Now we repeat the approach of Section 3 with Equation (13) and write

$$X = \overline{V}$$

Calculations are simplified significantly by restricting our attention to the background solution $V = \widehat{V} = \widetilde{V} = \widehat{\widetilde{V}} = 0$. We find

$$(-X + r - q)(-\widehat{X} + r + q) = r^2 - q^2$$
$$(-X + r - p)(-\widetilde{X} + r + p) = r^2 - p^2$$

and hence

$$\widehat{X} = \frac{(q + r)X}{X + q - r}$$
$$\widetilde{X} = \frac{(p + r)X}{X + p - r}$$

[2] The notion of Bäcklund transformations also applies to PDEs. For example, the continuous KdV equation allows a family of Bäcklund transformations depending on a parameter. The Bäcklund transformations with distinct parameters p and q commute with each other, so subsequent transformations take the structure of a $\mathbb{Z}^2$ lattice, where horizontal shifts correspond to transformations with parameter p and vertical shifts to transformations with parameter q. This brings us into the realm of difference equations. In fact this approach can be used to derive the lpKdV equation from the continuous (potential) KdV equation (See [Hietarinta et al., 2016, Chapter 2]). We will relate these equations in a different fashion in Section 5

Considering $X = \frac{F}{G}$ as before, we find that

$$\widehat{\begin{pmatrix} F \\ G \end{pmatrix}} = M \begin{pmatrix} F \\ G \end{pmatrix} \qquad \text{and} \qquad \widetilde{\begin{pmatrix} F \\ G \end{pmatrix}} = L \begin{pmatrix} F \\ G \end{pmatrix}$$

with

$$M = \begin{pmatrix} q+r & 0 \\ 1 & q-r \end{pmatrix} \qquad \text{and} \qquad L = \begin{pmatrix} p+r & 0 \\ 1 & p-r \end{pmatrix}.$$

Powers of these matrices take a particularly simple form:

$$M^n = \begin{pmatrix} (q+r)^n & 0 \\ \frac{(q+r)^n-(q-r)^n}{2r} & (q-r)^n \end{pmatrix} \qquad \text{and} \qquad L^m = \begin{pmatrix} (p+r)^m & 0 \\ \frac{(p+r)^m-(p-r)^m}{2r} & (p-r)^m \end{pmatrix}.$$

Hence

$$\begin{aligned} \begin{pmatrix} F_{m,n} \\ G_{m,n} \end{pmatrix} &= M^n L^m \begin{pmatrix} F_{0,0} \\ G_{0,0} \end{pmatrix} \\ &= \begin{pmatrix} (p+r)^m(q+r)^n & 0 \\ \frac{(p+r)^m(q+r)^n-(p-r)^m(q-r)^n}{2r} & (p-r)^m(q-r)^n \end{pmatrix} \begin{pmatrix} F_{0,0} \\ G_{0,0} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} X_{m,n} = \frac{F_{m,n}}{G_{m,n}} &= \frac{(p+r)^m(q+r)^n F_{0,0}}{\frac{(p+r)^m(q+r)^n-(p-r)^m(q-r)^n}{2r} F_{0,0} + (p-r)^m(q-r)^n G_{0,0}} \\ &= 2r \frac{\left(\frac{p+r}{p-r}\right)^m \left(\frac{q+r}{q-r}\right)^n X_{0,0}}{\left(\left(\frac{p+r}{p-r}\right)^m \left(\frac{q+r}{q-r}\right)^n - 1\right) X_{0,0} + 2r}. \end{aligned}$$

We find the soliton solution

$$X_{m,n} = 2r \frac{c\rho_{m,n}}{c\rho_{m,n}+1} \tag{14}$$

with

$$\rho_{m,n} = \left(\frac{p+r}{p-r}\right)^m \left(\frac{q+r}{q-r}\right)^n \qquad \text{and} \qquad c = \frac{X_{0,0}}{2r - X_{0,0}}.$$

Exercise 5. Sketch a plot of the soliton solution (14) with parameters $r = 0.5$, $c = 1$ for the equation with lattice parameters $p = 5$ and $q = 2$. Compare your sketch with Figure 2

Hint: Observe that X tends to 0 or $2r$ unless $\rho \approx \frac{1}{c}$, which happens close to the line

$$\log\left(\frac{p+r}{p-r}\right) m + \log\left(\frac{q+r}{q-r}\right) n + \log(c) = 0$$

in the (m,n)-plane.

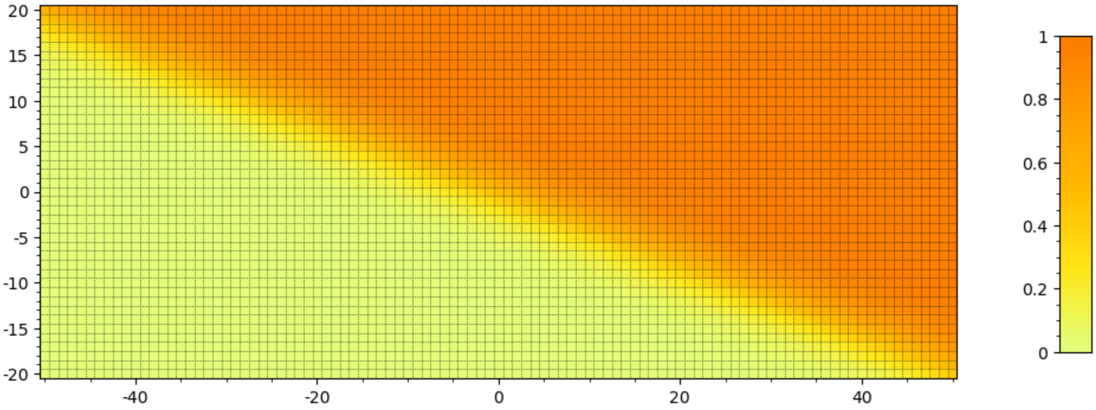

Figure 2: A 1-soliton solution with lattice parameters $p = 5$, $q = 2$ and solution parameters $r = 0.5$, $c = 1$. For the sake of visualisation it is plotted as a function of $\mathbb{R}^2$ rather than $\mathbb{Z}^2$. The lattice points are the intersections of the mesh lines.

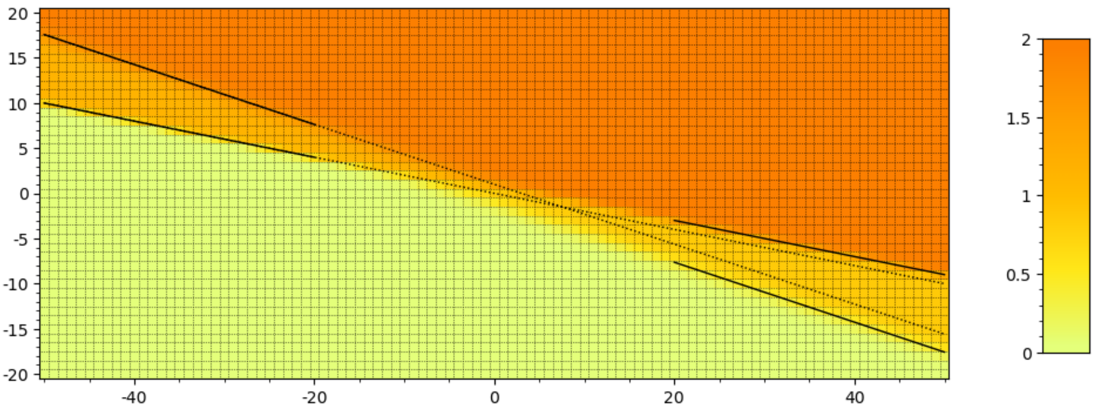

Figure 3: A 2-soliton solution can be understood as an approximate superposition of two 1-soliton solutions, with a phase shift.

If we restrict the 1-soliton solution to a horizontal line in the lattice, we get smoothed step function which is approximately zero on the far left and approximately 1 on the far right. If we do the same for a space-time plot of a soliton for the continuous KdV equation (i.e. look at the wave profile at a fixed time[3]), we get a bump function which tends to zero on both sides. However, a soliton of the continuous *potential* KdV equation is also a smoothed bump function, since its x-derivative produces a KdV soliton. A precise connection between soliton solutions of the lattice potential KdV equation and the continuous potential KdV equation will be made in Exercise 7.

We have obtained the 1-soliton solution as a Bäcklund transformation of the background solution. Similarly (but with much more cumbersome calculations) a 2-solition solution can be obtained as a Bäcklund transformation of the 1-soliton solution. In the language of multidimensional consistency, this would be the double shift $\overline{\overline{V}}$. As in the continuous case, a 2-soliton solution is approximately a superposition of two 1-soliton solutions, but with a phase shift, as you can see in Figure 3.

Whenever we have a multidimensionally consistent quad equation, we can use this approach to construct solutions. However, there is no guarantee that every additional Bäcklund transformation yields a truly new solution. It could be a trivial transformation of the solution we started with.

5 Continuum limits

It is tempting to think of the $\mathbb{Z}^2$-lattice as a discretization of space-time and of the parameters p and q as the mesh size. Then to compute a continuum limit we would take

$$\frac{\widetilde{U} - U}{p} \to u_x, \qquad \frac{\widehat{U} - U}{q} \to u_t \tag{15}$$

as $p, q \to 0$. In this limit, Equation (2) becomes

$$(pu_x - qu_t)(-pu_x - pu_t) = q^2 - p^2 \quad \Leftrightarrow \quad p^2(u_x^2 - 1) = q^2(u_t^2 - 1).$$

which does not look like the KdV equation at all. The reason for this failure lies in the symmetries.

The lpKdV equation and other quad equations are by definition symmetric under interchange of the lattice directions (along with their parameters). This is very unlike integrable PDEs such as the KdV equation, where space and time play a very different role. Hence we need to replace the continuum limit (15) with something that suitably breaks the symmetry to create distinct space and time variables.

We will label the continuous independent variables $t_1, t_2, \ldots$, where $t_1 = x$ is the space variable and $t_2, t_3, \ldots$ are the time variables of the PDEs in an integrable hierarchy. To

[3]The horizontal and vertical directions in the lattice should not be identified with space and time, as we will see in Section 5. However, for this qualitative analysis, any line that is not parallel to the soliton will do.

obtain a sensible continuum limit we identify lattice shifts with **Miwa shifts** of the continuous variable $v : \mathbb{R}^k \to \mathbb{R}$: if $V = v(t_1, t_2, \ldots, t_k)$, then

$$\widetilde{V} = v(t_1 + 2\lambda, t_2 + \lambda^2, \ldots, t_k + \tfrac{2}{k}\lambda^k),$$
$$\widehat{V} = v(t_1 + 2\mu, t_2 + \mu^2, \ldots, t_k + \tfrac{2}{k}\mu^k),$$

where $\lambda = p^{-1}$ and $\mu = q^{-1}$ are the inverse lattice parameters. Like quad equations, Miwa shifts are symmetric under exchange of the lattice directions and their parameters, but contrary to (15) the continuous variables each enter in a different way.

We will be able to identify $t_1 = x$, use t_3 as time variable for the potential KdV equation and $t_5, t_7, \ldots$ as time variables for the higher equations of the hierarchy. It turns out that even-numbered times will only yield the trivial equations $\partial_{t_{2j}} v = 0$, so we will omit them in the rest of the discussion.

Using Miwa shifts, we write the lpKdV equation in terms of the interpolating continuous field v:

$$\begin{aligned}&\big(v(t_1 + 2\lambda, t_3 + \tfrac{2}{3}\lambda^3, \ldots) - v(t_1 + 2\mu, t_3 + \tfrac{2}{3}\mu^3, \ldots) + \mu^{-1} - \lambda^{-1}\big)\\ &\qquad \cdot \big(v(t_1, t_3, \ldots) - v(t_1 + 2\lambda + 2\mu, t_3 + \tfrac{2}{3}\lambda^3 + \tfrac{2}{3}\mu^3, \ldots) + \mu^{-1} + \lambda^{-1}\big) = \mu^{-2} - \lambda^{-2}\end{aligned}$$

Expanding the left hand side as a double power series in μ and λ, and dividing both sides by $(\lambda + \mu)(\lambda - \mu)$ we observe that all terms of negative order in μ or λ cancel. We are left with

$$-\frac{4}{3}\partial_{t_1}^3 v + \frac{4}{3}\partial_{t_3} v - 4(\partial_{t_1} v)^2 + \mathcal{O}(\lambda + \mu) = 0.$$

Hence as the leading order equation we recover the **potential KdV** equation:

$$\partial_{t_3} v = \partial_{t_1}^3 v + 3(\partial_{t_1} v)^2.$$

Finally we can justify calling Equation (2) the "lattice potential KdV equation".

The rest of the potential KdV hierarchy can be obtained from the higher order terms of the series expansion. They can be computed one after the other, because in the Miwa shifts the higher-numbered times occur only at higher powers of μ and λ.

Why Miwa shifts? The calculations of this continuum limit are done in terms of a single equation, but still multi-dimensional consistency plays an important role. A key property of the Miwa shifts is that distinct parameters produce linearly independent shifts in $\mathbb{R}^k$. (To see this, note that shifts in different coordinates t_i are each given by a different power of the parameter.) Because of this, a k-dimensional lattice will be embedded into $\mathbb{R}^k$, while the multidimensional consistency property corresponds to the commutativity of the continuous flows. Taking a formal limit $k \to \infty$, we see that one single quad equation with the CAC property encodes a full hierarchy of commuting PDEs.

This transformation was introduced for a similar purpose to ours by Miwa [1982]. Interestingly, this predates the notion of multi-dimensional consistency, so his motivation must have been different.

Many other integrable quad equations allow a continuum limit of this nature. Often, a suitable transformation needs to be performed before a meaningful continuum limit can be taken. In the case of lpKdV this transformation consisted of the fairly simple change of (2) to (13) and inverting the lattice parameters. In some cases this is more challenging and it is not known if a suitable transformation exists for every integrable quad equation. Nevertheless, the many cases where quad equations have been shown to produce integrable hierarchies in a continuum limit are evidence of a deep connection between multi-dimensionally consistent quad equations and integrable PDEs.

Exercise 6. Fill in the details of the calculation sketched above.

Exercise 7. One can also take the continuum limit of a particular solution. Do this for the 1-soliton solution (14):

- Let $m, n \to \infty$ and $\lambda, \mu \to 0$ such that

$$m = \frac{3t}{2\lambda(\lambda^2 - \mu^2)} + \frac{x}{4\lambda} + O(1) \qquad \text{and} \qquad n = \frac{3t}{2\mu(\mu^2 - \lambda^2)} + \frac{x}{4\mu} + O(1).$$

  Then

$$2m\lambda + 2n\mu \to x \qquad \text{and} \qquad \tfrac{2}{3}m\lambda^3 + \tfrac{2}{3}n\mu^3 \to t.$$

- Verify that

$$\log(\rho_{m,n}) = m \log\left(\frac{1 + r\lambda}{1 - r\lambda}\right) + n \log\left(\frac{1 + r\mu}{1 - r\mu}\right) \to rx + r^3 t,$$

  hence

$$X_{m,n} = 2r\frac{c\rho_{m,n}}{c\rho_{m,n} + 1} \to v(x,t) = 2r\frac{\exp(rx + r^3 t + a)}{\exp(rx + r^3 t + a) + 1},$$

  where $a = \log c$. This is a soliton solution to the potential KdV equation.

- Check that $\frac{\partial}{\partial x} v(x,t)$ is the 1-soliton solution of the KdV equation.

Further reading

An excellent source on discrete integrable sytems is the textbook Hietarinta et al. [2016]. It discusses many more aspects of discrete integrable systems than we did here, such as Hirota's bilinear method, discrete Painlevé equations, and variational principles. Sections 1-4 of these notes are based in part on Chapter 3 of this book.

The notion of consistency around the cube, though implicit in some earlier works, appeared around the turn of the century. See for example Doliwa and Santini [1997] and Nijhoff and Walker [2001]. The connection to Lax pairs was established in Bobenko and Suris [2002] and Nijhoff [2002]. A classification of integrable quad equations in Adler et al. [2003].

Even before the era of multi-dimensional consistency, the lattice potential KdV equation was known as an integrable lattice equation; an interesting review article from this time is Nijhoff and Capel [1995].

The construction of solution solutions by using the CAC property as a Bäcklund transformation appear among others in Atkinson et al. [2008]. With methods different from the one presented above (multi-)soliton solutions for several quad equations were obtained in Nijhoff et al. [2009], Hietarinta and Zhang [2009]

Continuum limits of integrable difference equations were studied already in the 1980s. The Miwa shifts have their origin in Miwa [1982] and the continuum limit presented above for the lpKdV equation is equivalent to the one found in Wiersma and Capel [1987]. A discussion that is (for obvious reasons) closer to the presentation here can be found in Vermeeren [2019].

References

V. E. Adler, A. I. Bobenko, and Y. B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Communications in Mathematical Physics*, 233(3):513–543, 2003.

J. Atkinson, J. Hietarinta, and F. Nijhoff. Soliton solutions for Q3. *Journal of Physics A: Mathematical and Theoretical*, 41(14):142001, 2008.

A. I. Bobenko and Y. B. Suris. Integrable systems on quad-graphs. *International Mathematics Research Notices*, 2002(11):573–611, 2002.

A. Doliwa and P. M. Santini. Multidimensional quadrilateral lattices are integrable. *Physics Letters A*, 233(4-6):365–372, 1997.

J. Hietarinta and D.-j. Zhang. Soliton solutions for ABS lattice equations: II. Casoratians and bilinearization. *Journal of Physics A: Mathematical and Theoretical*, 42(40): 404006, 2009.

J. Hietarinta, N. Joshi, and F. W. Nijhoff. *Discrete systems and integrability.* Cambridge university press, 2016.

T. Miwa. On Hirota's difference equations. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 58(1):9–12, 1982.

F. Nijhoff and H. Capel. The discrete Korteweg-de Vries equation. *Acta Applicandae Mathematica*, 39(1-3):133–158, 1995.

F. Nijhoff, J. Atkinson, and J. Hietarinta. Soliton solutions for ABS lattice equations: I. Cauchy matrix approach. *Journal of Physics A: Mathematical and Theoretical*, 42 (40):404005, 2009.

F. W. Nijhoff. Lax pair for the Adler (lattice Krichever–Novikov) system. *Physics Letters A*, 297(1-2):49–58, 2002.

F. W. Nijhoff and A. J. Walker. The discrete and continuous Painlevé VI hierarchy and the Garnier systems. *Glasgow Mathematical Journal*, 43(A):109–123, 2001.

M. Vermeeren. Continuum limits of pluri-Lagrangian systems. *Journal of Integrable Systems*, 4(1):xyy020, 2019.

G. L. Wiersma and H. Capel. Lattice equations, hierarchies and Hamiltonian structures. *Physica A: Statistical Mechanics and its Applications*, 142(1-3):199–244, 1987.