Pablo A. Ferrari, Buenos Aires

Soliton decomposition of the Box Ball System in $\mathbb{Z}$

with

Chi Nguyen, Leonardo Rolla, Minmin Wang (arXiv:1806.02798)

Davide Gabrielli (In preparation)

Stochastic Networks, Edinburgh, June 2018

0	0	1	0	1	1	0	0	0	1	1	1	0	1	0	0	0	0	0	η
0	0	0	1	0	0	1	1	0	0	0	0	1	0	1	1	1	0	0	$T\eta$

Box at each integer $x \in \mathbb{Z}$.

Ball configuration $\eta \in \{0,1\}^{\mathbb{Z}}$

$$\eta(x) = 0 \rightarrow \text{empty box}, \qquad \eta(x) = 1 \rightarrow \text{ball at } x$$

Finite number of balls: Carrier visits boxes from left to right.

Carrier picks balls from occupied boxes

Carrier deposits one ball, if carried, at empty boxes.

0	0	1	0	1	1	0	0	0	1	1	1	0	1	0	0	0	0	0	η
0	0	1	0	1	2	1	0	0	1	2	3	2	3	2	1	0	0	0	carrier
0	0	0	1	0	0	1	1	0	0	0	0	1	0	1	1	1	0	0	$T\eta$

$T\eta$: configuration after the carrier visited all boxes.

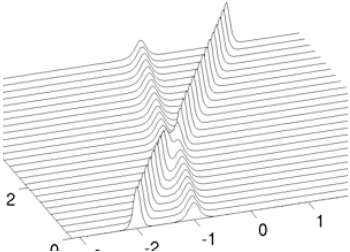
Ball-Box-System by Takahashi-Satsuma (1990)

Queue: 1's are arrivals and 0's are services. Carrier load = queue size.

Motivation: Korteweg & de Vries equation

$$\dot{u} = u''' + u\,u'$$

with $u(r,t) \in \mathbb{R}^+$, $r \in \mathbb{R}$, $t \in \mathbb{R}^+$. Interacting soliton solutions for KdV:



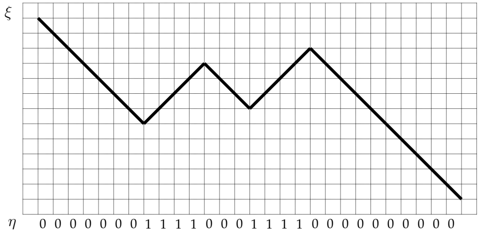
But there is no mathematical relation (yet) between KdV and BBS.

BBS with infinitely many balls

Walk representation ξ of ball configuration η:

$\xi \in \mathbb{Z}^{\mathbb{Z}}$ with $\xi(0) = 0$ and

$$\xi(x+1) - \xi(x) := 2\big(\eta(x+1) - 1\big)$$



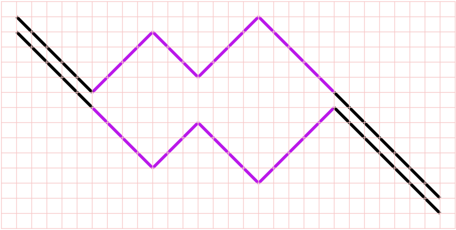
x is a (down) *record* for η if $\xi(x) < \xi(y)$,for all $y < x$.

Excursion:= configuration between two successive records.

BBS with infinitely many balls

$$T\eta(x) := (1 - \eta(x))\,\mathbf{1}\{x \text{ is not a record for } \eta\}$$

Non local! Coincides with previous definition for finite η.



Walk version of η and $T\eta$

Excursions are flipped down.

Configurations with *density* λ:

$$\mathcal{X}_\lambda := \Big\{\eta \in \{0,1\}^{\mathbb{Z}} : \lim_{n\to\pm\infty}\Big|\frac{1}{n}\sum_{x=0}^{n}\eta(x)\Big| = \lambda\Big\},$$

Lemma 1. *If* $\lambda \in (0, \frac{1}{2})$ *then*

$$\eta \in \mathcal{X}_\lambda \text{ implies } T\eta \in \mathcal{X}_\lambda.$$

Work on

$$\mathcal{X} := \cup_{\lambda\in(0,\frac{1}{2})}\mathcal{X}_\lambda.$$

A measure μ on $\mathcal{X}$ is invariant for T if $\eta \sim \mu$ implies $T\eta \sim \mu$.

Goals of lecture

1) *Soliton decomposition* of ball configurations.

2) Evolution is a *hierarchical translation* of soliton components.

3) Measures with independent soliton components are invariant for T.

4) Asymptotic speed of solitons.

5) Explicit soliton decomposition for iid Bernoulli, Ising models and other ball distributions.

Soliton: a solitary wave that propagates with little loss of energy and retains its shape and speed after colliding with another such wave

Conserved solitons. Motivation.

k-soliton: set of k successive ones followed by k zeroes.

Isolated k-solitons travel at speed k and conserve shape and distance:

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k-solitons and distances are conserved after interacting with m-solitons:

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Conserved solitons. Motivation.

k-soliton: set of k successive ones followed by k zeroes.

Isolated k-solitons travel at speed k and conserve the distances:

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.....111000...............111000..........
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k-solitons and distances are conserved after interacting with m-solitons:

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Conserved solitons. Motivation.

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Takahashi-Satsuma soliton identification

η with a finite number of balls.

Run: segments induced by broken lines in the walk representation.

Explore runs from left to right.

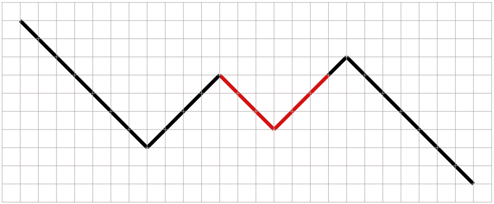
If run has length $k \leq m =$ lenght of the successive run, then

k boxes of short run and the first k boxes of long run is k-soliton.

Takahashi-Satsuma soliton identification

η with a finite number of balls.

Run: segments induced by broken lines in the walk representation.

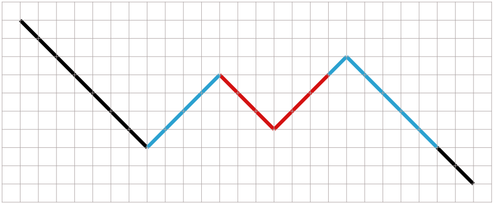


Explore runs from left to right.

If run has length $k \leq m =$ lenght of the successive run, then

k boxes of short run and the first k boxes of long run is k-soliton.

Ignore identified k-solitons and iterate:



head of k-soliton $\gamma :=$ position of ones $h(\gamma) = \{h_1, \dots, h_k\}$,

tail of k-soliton $\gamma :=$ zeroes $t(\gamma) = \{t_1, \dots, t_k\}$.

Infinite configurations:

Apply TS algorithm to each *excursion*, pretending it is isolated.

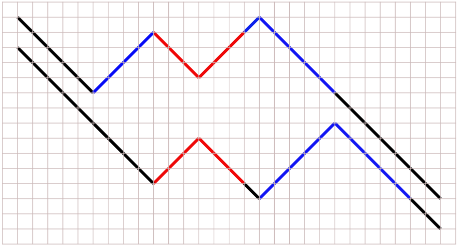
k-soliton conservation under T

Proposition. *For any $\eta \in \mathcal{X}$:*

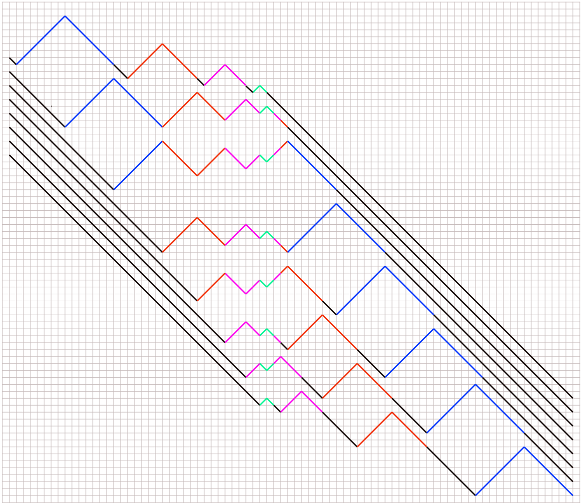
η has k-soliton γ with tail $t(\gamma) = a$

if and only if

$T\eta$ has k-soliton γ' with head $h(\gamma') = a$.



k-soliton conservation under T

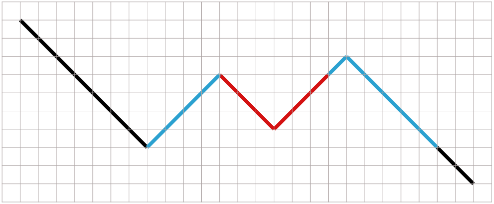


Slots

Given a configuration η with finite excursions we can identify its solitons.

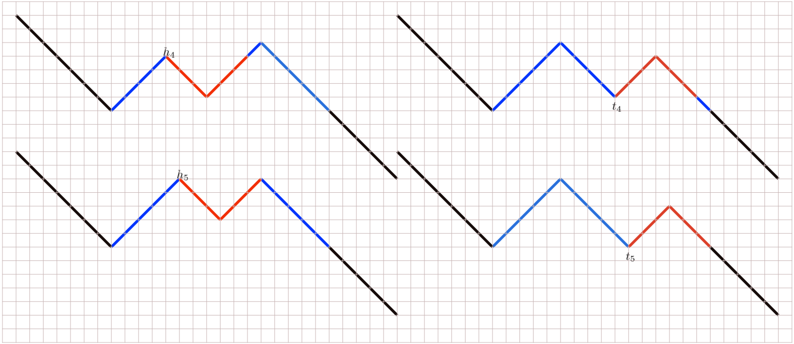
Will decompose η in soliton components.

Insert 3-soliton in 3-slot of 5-soliton:



Soliton components

3-slots: records and soliton boxes where a 3-soliton can be inserted.

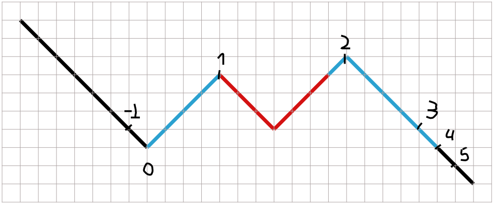


$$S_3\eta := \cup_{m>3} \cup_{\gamma \in \Gamma_m \eta} \{h_4(\gamma), \dots, h_m(\gamma), t_4(\gamma), \dots, t_m(\gamma)\} \cup R\eta$$

Soliton components

Enumerate 3-slots.

Insert 3-soliton in 3-slot number 1:



In this case denote:

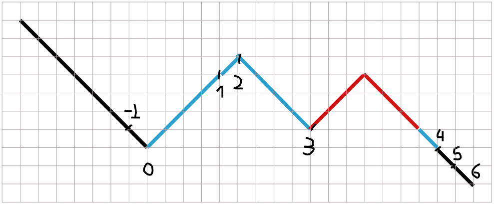
$$M_3\eta(1) = 1.$$

3-component at coordinate 1 has 1 soliton.
(The other coordinates in this picture have 0 solitons)

Soliton components

Soliton components.

Insert 3-soliton in 3-slot number 3:



$$M_3\eta(3) = 1.$$

Slots

ball configuration $\eta \mapsto$ *slot configuration* $S\eta$:

$S\eta : \mathbb{Z} \to \{0, 1, 2, \dots\} \cup \{\infty\}$

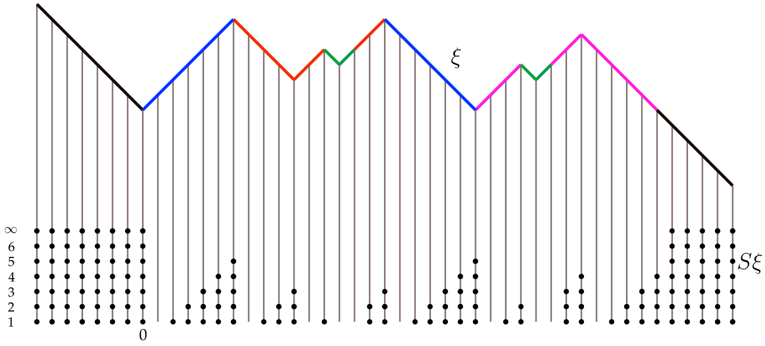
$$S\eta(x) = \begin{cases} i-1, & \text{if } x \in \{t_i(\gamma), h_i(\gamma)\}, \text{ for some } \gamma, k;\ i \in \{1, \dots, k\}, \\ \infty & \text{if } x \text{ is a record for } \eta \end{cases}$$

x is a k-*slot* for η if and only if $S\eta(x) \geq k$

$S_k\eta :=$ set of k-slots of η

$R\eta :=$ set of records of η

$$R\eta \subset S_{k+1}\eta \subset S_k\eta$$



Walk representation and slot configuration of a ball configuration η.

Records have infinite dots but we drawn only 7 in the picture.

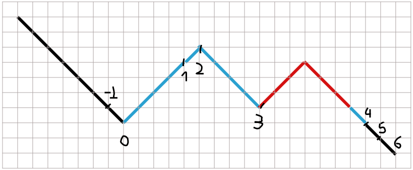
The numbered rows are the k-slots $S_k\eta$ for $k = 1, 2, \ldots, 7$.

Enumerate the k-slots:

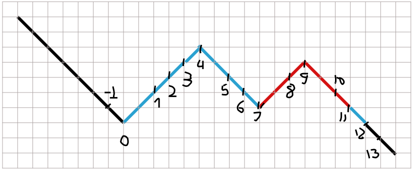
$s_k(\eta, 0) :=$ position of Record 0 for all $k \geq 1$

$s_k(\eta, i) :=$ position of the i-th k-slot counting from the Record 0

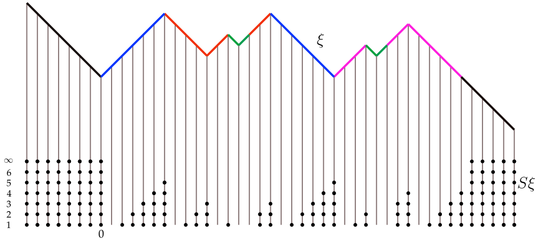
Enumerating 3-slots:

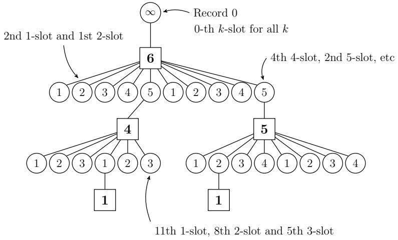


Enumerating 1-slots:



Soliton decomposition of ball configuration η





Soliton decomposition of ball configuration η

Let $\eta \in \mathcal{X}$.

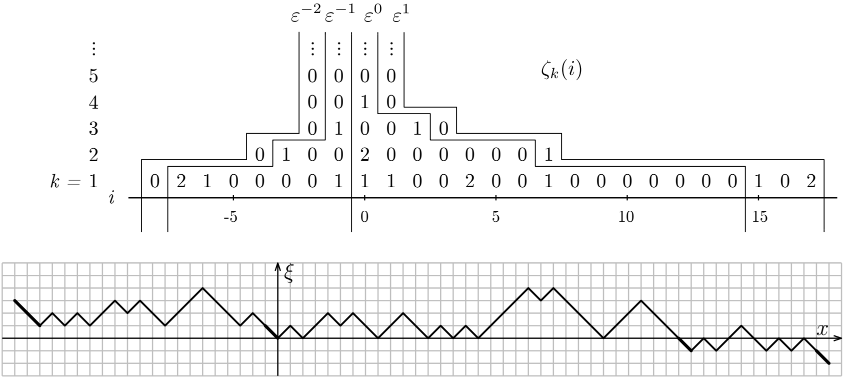
k-component $M_k\eta$:

$$M_k\eta(i) := \text{number of } k\text{-solitons appended to } i\text{-th } k\text{-slot}$$

In the example

$$M_6\eta(0) = 1, \quad M_4\eta(3) = 1,$$
$$M_5\eta(2) = 1, \quad M_1\eta(9) = 1, \quad M_1\eta(18) = 1.$$

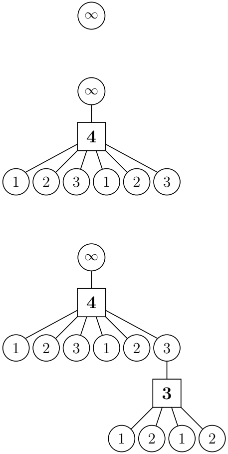
Reconstruction of ξ from ζ.

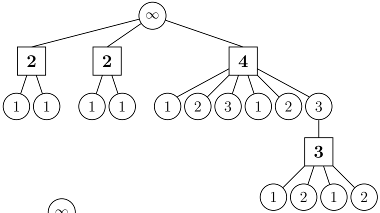


In the lower part we show Records -2 to 2 in boldface and the excursions between them. Above we show the parts of the field ζ that used in the reconstruction of $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^{0}, \varepsilon^{1}$.

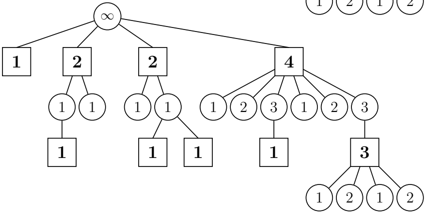
Reconstruction of excursion 0 of previous page.

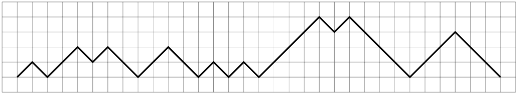
Reconstruction of ε^0

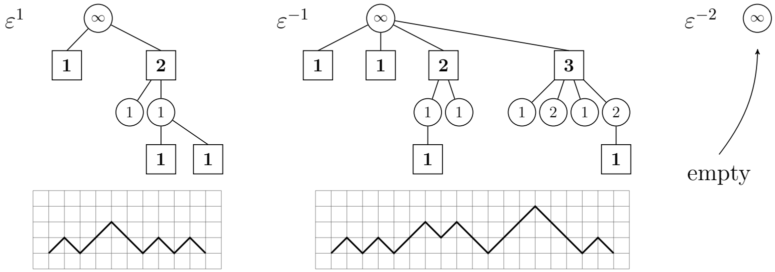




ε^0







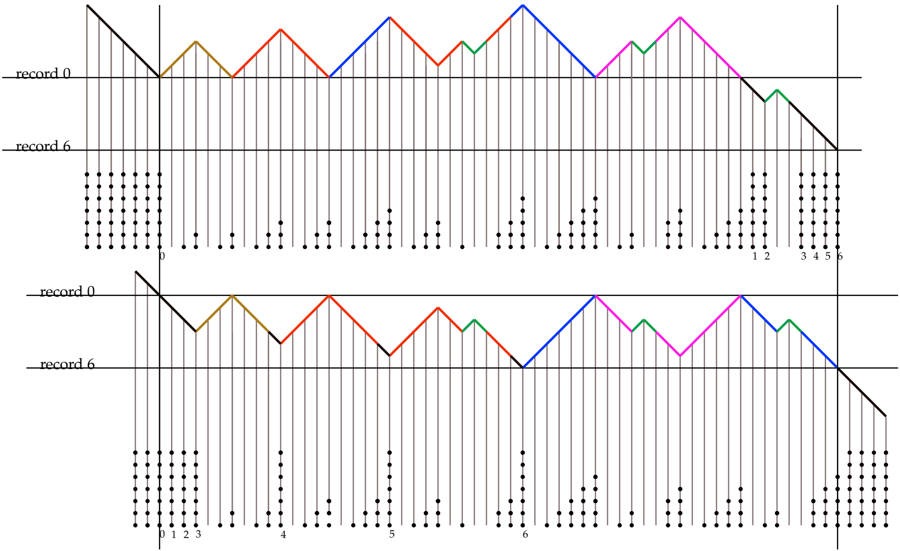
Reconstruction algorithm for the other excursions.

Component evolution are hierarchical translations

$$r^t(\eta,0) := \text{ position of Record 0 in } T^t\xi$$

$$\hat{T}^t\xi := \theta^{r^t(\eta,0)}T^t\eta \quad \text{process as seen from Record 0}$$

$$s_k^t(\eta,0) := \text{ position of the tagged 0-th } k\text{-slot in } \hat{T}^t\eta$$


record 0
record 6
0
1 2
3 4 5 6
record 0
record 6
0 1 2 3
4
5
6

Component evolution

Flow of m-solitons through Record 0:

$$J_m^t(\eta) := \#\{m\text{-solitons to the left of record 0 in } \eta \text{ and to the right of record 0 in } \hat{T}\eta\}$$

Label of zero k-slot in $M_k \hat{T}^t \eta$:

$$o_k^t(\eta, 0) := \sum_{m>k} 2(m-k) J_m^t(\eta) \tag{1}$$

Position of zero k-slot in $M_k \hat{T}^t \eta$:

$$s_k^t(\eta, 0) := s_k\Big(\hat{T}^t \eta, o_k^t(\eta, 0)\Big) \tag{2}$$

$o_k^t(\eta, 0)$ and $s_k^t(\eta, 0)$ are functions of $(M_m \eta : m > k)$.

Component evolution. Solitons!

Theorem 2.

k-soliton component of $\hat{T}^t\eta$ is a shift of the k-soliton component of η:

$$M_k\hat{T}^t\eta = \theta^{o_k^t(\eta,0)+kt}M_k\eta$$

$$o_k^t(\eta,0) := \sum_{m>k} 2(m-k)J_m^t(\eta)$$

$J_m^t(\eta) :=$ Flow of m-solitons thru 0-record

Solitons!

Theorem [FNRW] We have proven

$$M\hat{T}\eta = \Theta M\eta$$

Dynamics of components is a hierarchical translation and components conserve distances and shapes (solitons).

$$\Theta(\zeta_k : k \geq 1) := (\theta^{s_k^t(\eta,0)+kt}\zeta : k \geq 1)$$

$M : \{0,1\}^{\mathbb{Z}} \to (\mathbb{N}^{\mathbb{Z}})^N$ given by $M\eta = (M_k\eta : k \geq 1)$

Independent-component invariant measures

Theorem [FNRW]

$\zeta_k \in \mathbb{N}^{\mathbb{Z}}$ independent k-soliton configurations with shift-stationary law.

Let $\bar\zeta = (\zeta_k : k \ge 1)$ and assume

$M^{-1}\bar\zeta \in \mathcal{X}^o$ *a.s.*

Let $\mu :=$ law of $\eta := M^{-1}\bar\zeta$, ball configuration with components ζ_k.

Then μ is $\hat T$-invariant: $\mu\hat T = \mu$.

Independent-component invariant measures

Proof. Call $\eta := M^{-1}\bar{\zeta}$.

Want to show that slot decomposition of $\hat{T}\eta$ has same law as $\bar{\zeta}$:

$$E\Big(\prod_{k=1}^{n} F_k(M_k\hat{T}^t\eta)\Big) = \prod_{k=1}^{n} EF_k(\zeta_k), \qquad n \geq 2,$$

F_k are local functions.

$$\begin{aligned} & E\big(F_k(M_k\hat{T}^t\eta) \,|\, \mathcal{F}(M_m\hat{T}^t\eta : m > k)\big) \\ & \quad = E\big(F_k(\theta^{s_k^t(\eta,0)+kt} M_k\eta)|\mathcal{F}(\zeta_m : m > k)\big) \qquad \text{(Propo 2)} \\ & \quad = E\big(F_k(\theta^{s_k^t(\eta,0)+kt} \zeta_k)|\mathcal{F}(\zeta_m : m > k)\big) \qquad \text{(definition)} \\ & \quad = EF_k(\zeta_k) \end{aligned}$$

because ζ_k is independent of $s_k^t(\eta, 0) \in \mathcal{F}(\zeta_m : m > k)$. □

Asymptotic speed of k-solitons (in records per unit time)

For a k-soliton γ define $y_k^t := \#$ (records between record-0 at time t and γ^t) .

Theorem 3. *There exists $v = (v_k)_{k\geq 1}$ such that,*

$$\lim_{t\to\infty} \frac{y_k^t}{t} = v_k, \quad \mu\text{-a.s.} \tag{3}$$

If $\sum_k k^2 \rho_k < \infty$, the vector $(v_k)_{k\geq 1}$ is the unique finite solution of

$$v_k = k + \sum_{m>k} 2(m-k)(v_m - v_k)\rho_m, \quad k \geq 1. \tag{4}$$

Asymptotic speed of the position of Record 0 is given by

$$\lim_{t\to\infty} \frac{r^t(\eta, 0)}{t} = -\sum_{m\geq 1} 2m\rho_m v_m, \quad \mu\text{-a.s.} \tag{5}$$

and the asymptotic speed of γ^t is

$$\lim_{t\to\infty} \frac{\text{Position of } \gamma^t}{t} = v_k + \sum_{m\geq 1} 2m\rho_m (v_k - v_m), \quad \mu\text{-a.s.} \tag{6}$$

k-components of iid Bernoulli are independent iid geometrics.

With Davide Gabrielli

Let $\lambda \in (0, \frac{1}{2})$.

Let $q_1 := \lambda(1-\lambda)$ and for $k \geq 2$,

$$q_k := \frac{q_1^k}{\prod_{j=1}^{k-1}(1-q_j)^{2(k-j)}}\,.$$

Theorem 4 (F, Gabrielli). *If* $(\eta(x) : x \in \mathbb{Z})$ *iid Bernoulli*(λ)*, then*

$(M_k\eta(s) : s \in \mathbb{Z})$ *iid Geometric*$(1-q_k)$ *and*

$(M_k\eta : k \geq 1)$ *are independent.*

Other measures with independent geometric k-components.

Let $\alpha_k \geq 0$ such that $\sum_{k\geq 0} \alpha_k < \infty$.

Let ε be an excursion between Record 0 and Record 1 and

$n_k(\varepsilon) :=$ number of k-solitons of ε.

$$\text{weight} \quad w_\alpha(\varepsilon) := \prod_{k=1}^{\infty} \alpha_k^{n_k(\varepsilon)} \tag{7}$$

induces a measure

$$\nu_\alpha(\varepsilon) = \frac{w(\varepsilon)}{Z_\alpha} \tag{8}$$

Concatenate independent excursions to obtain a measure $\hat{\mu}_\alpha$ on $\widehat{\mathcal{X}}$.

$\mu :=$ Anti-Palm(μ) has independent components geometric with parameters q_i

Theorem 5. *Let* $\nu_\alpha = \bigotimes_{i\in\mathbb{Z}} \nu_\alpha^i$ *(independent excursions induced by weights* w*)*

If $(\eta(x) : x \in \mathbb{Z}) \sim \nu_\alpha$*, then*

$(M_k\eta(s) : s \in \mathbb{Z})$ *iid Geometric*$(1 - q_k)$ *and*

$(M_k\eta : k \geq 1)$ *are independent.*

Special cases

- $\alpha_i = [\lambda(1-\lambda)]^i$, $\lambda < \frac{1}{2}$. Product measure with density λ
- $\alpha_i = e^{2J}e^{ih}$, $h < 0$. Ising measure with pair interaction J and external field h.

When the k-components of invariant measures are independent?

Let $\hat{\mu}$ on $\mathcal{X}^o$ be invariant for $\hat{T}$ and record-mixing(?). Then,

$$\hat{\mu}M = \bigotimes_{k\geq 1} \hat{\mu}M_k.$$

That is, if η has law $\hat{\mu}$,

$(M_k\eta : k \geq 1)$ is a family of independent configurations.

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