The groupoids of adaptable separated graphs and their type semigroups (I)

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Higher rank graphs: geometry, symmetry, dynamics
ICMS, July 16, 2019.

P. ARA, J. BOSA, E.P., A. SIMS,
The groupoids of adaptable separated graphs and their type semigroups
arXiv:1904.05197v2 (math.RA).

Outline

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Recently, great interest in the structure of the type semigroup $\mathrm{Typ}(\mathcal{G})$ of an ample groupoid $\mathcal{G}$

Used to determine the stably finite versus purely infinite dichotomy for $C_r^*(\mathcal{G})$.

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The Tarski notion of paradoxicality, transfered to the K-theoretic context, played a major role in recent approaches:

- In the case of actions of a discrete group G on the Cantor set X, Rørdam & Sierakowski introduced a semigroup $S(X,G)$ –an analog of Tarski's type semigroup– and tied the pure infiniteness of $C(X) \rtimes_r G$ to the existence of states on $S(X,G)$.
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- Rainone & Sims extended the idea, by defining a semigroup $S(\mathcal{G})$ associated to a étale groupoid $\mathcal{G}$ (see also Bönicke & Li's work).

Realization Problem for von Neumann regular rings (exchange rings) [Goodearl]:

> *Which kind of conical refinement monoids are realizable as $\mathcal{V}(R)$ for a suitable von Neumann regular ring (exchange ring)?*

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Advances: it is possible to construct such a ring for monoids associated to (directed) graphs E.

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CONNECTION: It is known $\mathcal{V}(L_K(E)) \cong M(E) \cong \mathrm{Typ}(\mathcal{G}_E)$.

A possibilty for extending the above result is to work with a monoid M such that there is an algebra A and a groupoid $\mathcal{G}_A$ with:

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What we will show in these two talks is that the previous schema works for conical, finitely generated refinement monoids.

Let me outline which is the strategy we follow for fill all the gaps.

(1) Basic tool: use separated graphs (E, C), because for any countable conical monoid M there exists (E, C) such that $M \cong M(E, C)$ [Ara-Goodearl].

(2) For each conical, finitely generated refinement monoid M, construct a finite I-system $\mathcal{J}$ so that $M \cong M(\mathcal{J})$ [Ara-P].

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(3) Given any finite I-system $\mathcal{J}$, construct an special separated graph (E, C) such that $M(E, C) \cong M(\mathcal{J})$ (this is an adaptable separated graph).

(4) Use the set of basic partial isometries of $L_K(E, C)$, and enlarge it for constructing an inverse semigroup $S(E, C)$.

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(5) Determine the topological space of tight filters $\hat{\mathcal{E}}_{\text{tight}}$ associated to the semilattice of idempotents of $S(E,C)$, and define a (partial) action $S(E,C) \curvearrowright \hat{\mathcal{E}}_{\text{tight}}$.

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IN THE SECOND TALK

(7) Construct the Steinberg algebra $A_K(\mathcal{G}(E,C))$, and show that it is isomorphic to the algebra $\mathcal{S}_K(E,C)$ defined for generators & relations of $S(E,C)$.

(8) Prove that $\mathrm{Typ}(\mathcal{G}(E,C)) \cong M(E,C)$.

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(9) Construct a von Neumann regular universal localization $Q_K(E,C)$ of $\mathcal{S}_K(E,C)$.

(10) Use a deconstruction (pullback) - reconstruction (pushout) procedure to show that $\mathcal{V}(Q_K(E,C)) \cong \mathrm{Typ}(\mathcal{G}(E,C))$.

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A commutative monoid M is *conical* if, for all x, y in M, $x + y = 0$ only when $x = y = 0$.

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M is a *refinement monoid* if, for all a, b, c, d in M such that $a + b = c + d$, there exist w, x, y, z in M such that $a = w + x$, $b = y + z$, $c = w + y$ and $d = x + z$.

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A basic example of a refinement monoid is the monoid

$$M(E) = \left\langle a_v (v \in E^0) : a_v = \sum_{e \in s^{-1}(v)} a_{r(e)} \right\rangle$$

associated to a countable row-finite graph E.

If $x, y \in M$, $x \leq y$ if exists $z \in M$ such that $x + z = y$.

Definition

An element $p \in M$ is a *prime* if p is not invertible in M, and, whenever $p \leq a + b$ for $a, b \in M$, then either $p \leq a$ or $p \leq b$.

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A commutative monoid M is primely generated if every non-invertible $x \in M$ is a finite sum of prime elements of M.

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An element $x \in M$ is:

1. *regular* if $2x \leq x$.
2. *free* if $nx \leq mx$ implies $n \leq m$, for $n, m \in \mathbb{N}$.

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Let M be a finitely generated conical refinement monoid. We will express M in terms of "generalized graphs":

Definition (Ara-Goodearl)

A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v.

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The constructions we introduce revert to existing ones in case $C_v = \{s^{-1}(v)\}$ for each $v \in E^0$. We refer to a *non-separated graph* in that situation.

We assume throughout that (E, C) is *finitely separated*, i.e., $|X| < \infty$ for all $X \in C$.

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Let (E, C) be a separated graph. Its monoid is

$$M(E, C) = \left\langle a_v (v \in E^0) : a_v = \sum_{e \in X} a_{r(e)}, \forall X \in C_v, \forall v \in E^0 \right\rangle.$$

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The *Leavitt path algebra of the separated graph* (E, C) over a field K is the $*$-algebra $L_K(E, C)$ with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the following relations:

(V) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$,

(E) $s(e)e = er(e) = e$ for all $e \in E^1$,

(SCK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in X$, $X \in C$, and

(SCK2) $v = \sum_{e \in X} ee^*$ for every $X \in C_v$, $v \in E^0$.

Theorem (Ara-Goodearl)

For any separated graph (E, C)*,*

$$\mathcal{V}(L_K(E, C)) \cong M(E, C).$$

Outline

[Pierce]: Every conical primely generated antisymmetric monoid can be represented as a semilattice (a poset with an order-absorbing relation).

[Dobbertin]: Every primely generated conical regular refinement monoid can be represented as a semilattice of finitely generated abelian groups (a sort of partial order of finitely generated abelian groups).

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Let $I = (I, \leq)$ be a poset. An I-*system*

$$\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji}\,(i < j))$$

is given by the following data:

(a) A partition $I = I_{\text{free}} \sqcup I_{\text{reg}}$.

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We attach to each finite I-system $\mathcal{J}$ a conical, finitely generated refinement monoid $M(\mathcal{J})$.

$M(\mathcal{J})$ is the monoid generated by the M_is, with defining relations

$$x + y = x + \varphi_{ji}(y), \quad i < j, x \in M_j, y \in M_i.$$

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Theorem (Ara-P)

Let M be a conical, primely generated refinement monoid. Then, there exists an I-system $\mathcal{J}$ such that $M \cong M(\mathcal{J})$.

Corollary (Ara-P)

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1. $E^0 = \bigsqcup_{p \in I} E_p^0$.
2. All the connecting maps between E_ps go downwards on I.
3. The form of (E_p, C_p) allows to recover the monoid M_I when computing its associated monoid.

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To be precise, an *adaptable separated graph* is a finitely separated graph (E, C) s.t:

(1) $I = E^0/{\sim}$ is the antisymmetrization of E^0 with respect to the pre-order $v \geq w$ iff there is a path $v \to w$. Then I is finite and $I = I_{\text{free}} \sqcup I_{\text{reg}}$.

(2) $E^0 = \bigsqcup_{p \in I} E_p^0$, where E_p is a strongly connected row-finite graph if $p \in I_{\text{reg}}$ and $E_p^0 = \{v^p\}$ is a single vertex if $p \in I_{\text{free}}$.

To be precise, an *adaptable separated graph* is a finitely separated graph (E, C) s.t:

(1) $I = E^0/{\sim}$ is the antisymmetrization of E^0 with respect to the pre-order $v \geq w$ iff there is a path $v \to w$. Then I is finite and $I = I_{\text{free}} \sqcup I_{\text{reg}}$.

(2) $E^0 = \bigsqcup_{p \in I} E^0_p$, where E_p is a strongly connected row-finite graph if $p \in I_{\text{reg}}$ and $E^0_p = \{v^p\}$ is a single vertex if $p \in I_{\text{free}}$.

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(3) For $v \in E^0_p$ with $p \in I_{\text{reg}}$, we have $|C_v| = 1$.

(4) For $p \in I_{\text{free}}$, we have that $s^{-1}(v^p) = \emptyset$ if and only if p is minimal in I. If p is not minimal there is a positive integer $k(p)$ such that $C_{v^p} = \{X^{(p)}_1, \dots, X^{(p)}_{k(p)}\}$, where each $X^{(p)}_i$ is of the form

$$X^{(p)}_i = \{\alpha(p,i), \beta(p,i,1), \beta(p,i,2), \dots, \beta(p,i,g(p,i))\},$$

for some $g(p,i) \geq 1$, where $\alpha(p,i)$ is a loop, i.e., $s(\alpha(p,i)) = r(\alpha(p,i)) = v^p$, and $r(\beta(p,i,t)) \in E^0_q$ for $q < p$.

(3) For $v \in E_p^0$ with $p \in I_{\mathrm{reg}}$, we have $|C_v| = 1$.

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Theorem

Let $\mathcal{J}$ be a finite I-system. Then there is an adaptable separated graph (E, C) such that

$$M(E, C) \cong M(\mathcal{J}).$$

Theorem

1. *If (E, C) is an adaptable separated graph, then $M(E, C)$ is a refinement monoid.*
2. *For any finitely generated conical refinement monoid M, there exists an adaptable separated graph (E, C) such that $M \cong M(E, C)$.*

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So, adaptable separated graphs provide a suitable combinatorial input for our construction.

Outline

Given an adaptable separated graph as before, we want to associate a groupoid to it.

We will follow Exel's construction of the tight groupoid of an inverse semigroup, so that we need first to get an inverse semigroup S based on the paths on E.

For this, we also consider Cuntz-Krieger relations (SCK1)-(SCK2) as before ...

... but we tame now our generating partial isometries in a different way, by introducing some auxiliary variables t_i^v, $i \in \mathbb{N}$, $v \in E^0$, and some commuting rules.

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... but we tame now our generating partial isometries in a different way, by introducing some auxiliary variables t_i^v, $i \in \mathbb{N}$, $v \in E^0$, and some commuting rules.

Definition

A semigroup T is an *inverse semigroup* if

- for every x in T, there exists a unique $x^* \in T$, such that $xx^*x = x$ and $x^*xx^* = x^*$,
- there exists a (necessarily unique) element $0 \in T$, called the zero element, such that $x0 = 0x = 0$, for all x in T.

If T is an inverse semigroup, then the set of idempotents of T, $\mathcal{E} = \mathcal{E}(T)$, is a semilattice with ordering $e \leq f$ if and only if $ef = e$, and $e \wedge f = ef$.

Notation

If $p \in I$ is **non-minimal** and **free**, we denote by σ^p the map $\mathbb{N} \to \mathbb{N}$ given by

$$\sigma^p(i) = i + k(p) - 1.$$

Moreover, if $1 \leq j \leq k(p)$, we denote by σ_j^p the unique bijective, non-decreasing map from $\{1, \dots, k(p)\} \setminus \{j\}$ onto $\{1, \dots, k(p) - 1\}$.

Definition

Given an adaptable separated graph (E, C), denote by $S(E, C)$ the $*$-semigroup (with 0) generated by

$$E^0 \cup E^1 \cup \{(t_i^v)^{\pm} \mid i \in \mathbb{N}, v \in E^0\}$$

with the defining relations given below, except B1(ii)(d) and B2(1)(ii).

RELATIONS

BLOCK 1:

1. (i) For all $v, w \in E^0$, we have $v \cdot w = \delta_{v,w} v$ and $v = v^*$.
2. (ii) For all $e \in E^1$, we have:
   - (a) $e = s(e)e = er(e)$
   - (b) $e^* e = r(e)$
   - (c) $e^* f = \delta_{e,f} r(e)$ if $e, f \in X \subseteq C_{s(e)}$.
   - (d) $v = \sum_{e \in X} ee^*$, for $X \in C_v$, $v \in E^0$.

BLOCK 2:

(1) For each **free** prime $p \in I$ and $i = 1, \dots, k(p)$, we have:

- (i) $\alpha(p,i)^*\alpha(p,i) = v^p$
- (ii) $\alpha(p,i)\alpha(p,i)^* = v^p - \sum_{t=1}^{g(p,i)} \beta(p,i,t)\beta(p,i,t)^*$
- (iii) For $i \neq j$, $\alpha(p,i)\alpha(p,j) = \alpha(p,j)\alpha(p,i)$, and $\alpha(p,i)\alpha(p,j)^* = \alpha(p,j)^*\alpha(p,i)$.
- (iv) $\beta(p,i,s)^*\beta(p,j,t) = 0$ if either $i \neq j$, or $i = j$ and $s \neq t$.
- (v) $\alpha(p,i)^*\beta(p,i,t) = 0 = \beta(p,i,t)^*\alpha(p,i)$ for all $1 \leq i \leq k(p)$ and all $1 \leq t \leq g(p,i)$.

(2) For the $\{t_i^v\}$, we impose the following relations:

- (i) For each $v \in E^0$, $\{(t_i^v)^{\pm} : i \in \mathbb{N}\}$ is a family of mutually commuting elements such that

$$vt_i^v = t_i^v = t_i^v v, \qquad t_i^v (t_i^v)^{-1} = v = (t_i^v)^{-1} t_i^v, \qquad (t_i^v)^* = (t_i^v)^{-1}.$$

- (ii) If $p \in I$ is **regular**, $e \in E^1$ is such that $s(e) \in E_p^0$ and $i \in \mathbb{N}$,

$$t_i^{s(e)} e = e t_i^{r(e)}.$$

- (iii) If $p \in I$ is **free**, $i \in \mathbb{N}$, $1 \leq j \leq k(p)$ and $1 \leq s \leq g(p,j)$,

$$(t_i^{v^p})^{\pm} \beta(p,j,s) = \beta(p,j,s)(t_{\sigma^p(i)}^{r(\beta(p,j,s))})^{\pm},$$

(iv) If $p \in I$ is **free**, $i \neq j$, and $1 \leq s \leq g(p,j)$,

$$\alpha(p,i)\beta(p,j,s) = \beta(p,j,s)t_{\sigma_j^p(i)}^{r(\beta(p,j,s))}$$

and

$$\alpha(p,i)^*\beta(p,j,s) = \beta(p,j,s)(t_{\sigma_j^p(i)}^{r(\beta(p,j,s))})^{-1}.$$

(iiv) If $p \in I$ is **free**, $t_i^{v^p}\alpha(p,j) = \alpha(p,j)t_i^{v^p}$ and $t_i^{v^p}\alpha(p,j)^* = \alpha(p,j)^*t_i^{v^p}$ for all $i \in \mathbb{N}$ and $j \in \{1,\dots,k(p)\}$.

We provide a different description of $S(E,C)$. This will be given via the paths that one can intuitively associate to any adaptable separated graph.

Roughly, a finite path is as follows: consider a sequence of elements $p_1 > p_2 > \ldots > p_n$ of the poset I, and for each i a path γ_i in E_{p_i}; we form a finite path by connecting the γ_i together via the connectors β. Diagrammatically, we may write

$$p_1 \curvearrowright^{\beta_{1,2}} p_2 \curvearrowright^{\beta_{2,3}} \ldots \curvearrowright^{\beta_{n-1,n}} p_n .$$

We now define the monomials as the possible multiplicative expressions one can form using generators (excluding connectors) corresponding to a given prime. They will be denoted by $\mathbf{m}(p)$ for $p \in I$. Namely,

(1) if p is a **free** prime, we define

$$\mathbf{m}(p) = (t_{i_1}^{v^p})^{d_1} \dots (t_{i_r}^{v^p})^{d_r} \prod_{j=1}^{k(p)} \alpha(p,j)^{k_j} (\alpha(p,j)^*)^{l_j}$$

for $d_1, \dots, d_r \in \mathbb{Z} \setminus \{0\}, r \geq 0, k_j, l_j \geq 0$

(2) if p is a **regular** prime, we define

$$\mathbf{m}(p) = (t^v_{i_1})^{d_1} \dots (t^v_{i_r})^{d_r} \gamma \nu^*,$$

where γ, ν are paths of finite length in E_p satisfying $s(\gamma) = v$, $v \in E^0_p$, and $r(\gamma) = r(\nu)$.

Definition

S is the union of $\{0\}$ and the set of all triples $(\gamma, \mathbf{m}(p), \eta)$, where γ, η are finite paths, $\mathbf{m}(p)$ is a monomial at some prime $p \in I$, and $r(\gamma) = s(\mathbf{m}(p))$, $r(\eta) = r(\mathbf{m}(p))$.

So, S consists of combinations $\gamma\mathbf{m}(p)\eta^*$ of a finite path, a monomial and the star of a finite path.

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So, S consists of combinations $\gamma\mathbf{m}(p)\eta^*$ of a finite path, a monomial and the star of a finite path.

Proposition

Let (E, C) be an adaptable separated graph. Then, there is a natural $*$-isomorphism $S(E, C) \cong S$.

The idempotents in S are of the form $\gamma \mathrm{m}(p)\gamma^*$; moreover, the idempotents commute. So

Proposition

Let (E, C) be an adaptable separated graph. Then, $S(E, C)$ is an inverse semigroup.

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Let (E, C) be an adaptable separated graph. Then, $S(E, C)$ is an inverse semigroup.

An inverse semigroup is E^*-unitary if given an idempotent e and an element s, if $e = se$ then s is idempotent.

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This guarantees that the groupoid we will construct is Hausdorff.

Outline

Definition

A *groupoid* $\mathcal{G}$ is an small category in which every homomorphism is an isomorphism. We will denote by $\mathcal{G}^{(0)}$ its set of units, and by $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ the range and source maps $r(\gamma) = \gamma\gamma^*$ and $s(\gamma) = \gamma^*\gamma$.

Definition

A *topological groupoid* is a groupoid endowed with a topology under which multiplication and inversion are continuous maps; in particular, r and s are continuous maps.

Definition

A topological groupoid $\mathcal{G}$ is said to be *étale* if r (and so s) is a local homeomorphism from $\mathcal{G}$ to $\mathcal{G}^{(0)}$.

If $\mathcal{G}$ is étale, then $\mathcal{G}^{(0)}$ is open. We will always asume that our groupoids are étale, locally compact, and $\mathcal{G}^{(0)}$ is Hausdorff in the relative topology.

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A locally compact étale groupoid $\mathcal{G}$ is said to be *ample* if $\mathcal{G}^{(0)}$ is totally disconnected.

This is equivalent to assume that the topology of $\mathcal{G}$ has a basis of open compact bisections. Here, a bisection is a subset $U \subseteq \mathcal{G}$ such that r and s are injective on U.

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Definition

Let T be an inverse semigroup, and let $\mathcal{E}$ be its semilattice of idempotents. A filter in $\mathcal{E}$ is a nonempty subset $\eta \subseteq \mathcal{E}$ such that:

1. $0 \notin \eta$,
2. closed under $\wedge$,
3. $f \geq e \in \eta$ implies $f \in \eta$.

We denote the set of filters by $\widehat{\mathcal{E}}_0$.This is a locally compact totally disconnected Hausdorff space when equipped with the cylinder topology:

Given finite subsets $X, Y \subseteq \mathcal{E}$, consider the set

$$U(X,Y) = \{\eta \in \hat{\mathcal{E}}_0 : X \subseteq \eta,\ Y \subseteq \mathcal{E} \setminus \eta\}.$$

Then each $U(X,Y)$ is an open set and the collection of all such is easily seen to form a basis for the topology of $\hat{\mathcal{E}}_0$.

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A filter η is an *ultrafilter* if it is not properly contained in another filter. We denote $\widehat{\mathcal{E}}_\infty \subseteq \widehat{\mathcal{E}}_0$ the space of ultrafilters.

Definition

The set $\widehat{\mathcal{E}}_{\text{tight}}$ of *tight filters* is the closure of $\widehat{\mathcal{E}}_\infty$ into $\widehat{\mathcal{E}}_0$.

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Definition

We define a standard partial action of T on $\widehat{\mathcal{E}}_0$ as follows:

1. For each $e \in \mathcal{E}$, $D^\beta_e = \{\eta \in \widehat{\mathcal{E}}_0 : e \in \eta\}$,
2. Given $s \in T$,

$$\begin{aligned} \beta_s : D^\beta_{s^*s} &\longrightarrow D^\beta_{ss^*} \\ \eta &\longrightarrow \beta_s(\eta) = \{f \in \mathcal{E} : f \geq ses^* \text{ for every } e \in \eta\} \end{aligned}$$

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β restricts to an action of T on ultrafilters and on tight filters.

Consider the transformation groupoid $T \times \widehat{\mathcal{E}}_{\text{tight}}$.

The elements are the pairs (s, ω) such that $\omega \in \text{Dom}(s) = D^{\beta}_{s^*s}$.

We fix the germ relation: $(s, \omega) \sim (t, \eta)$ if $\omega = \eta$ and there exists an idempotent $e \leq t, s$ with $\omega \in \text{Dom}(e)$ such that $se = te$.

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Definition (Tight groupoid of the inverse semigroup T)

Define $\mathcal{G}_{\text{tight}}(T) = T \times \widehat{\mathcal{E}}_{\text{tight}} / \sim$, with:

1. $d([s,x]) = x$ and $r([s,x]) = \beta_s(x)$,
2. $[s,z] \cdot [t,x] = [st,x]$ if and only if $z = \beta_t(x)$,
3. $[s,x]^{-1} = [s^*, \beta_s(x)]$,
4. $\mathcal{G}^{(0)}_{\text{tight}}(T) = \{[e,x] : e \in \mathcal{E}\} \cong \widehat{\mathcal{E}}_{\text{tight}}$

$\mathcal{G}_{\text{tight}}(T)$ is ample, but in general is neither Hausdorff nor amenable.

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$\mathcal{G}_{\text{tight}}(T)$ is ample, but in general is neither Hausdorff nor amenable.

Given $s \in T$, $U \subseteq D_{s^*s}$ open subset, the set

$$\Theta(s,U) = \{[s,\xi] : \xi \in U\}$$

is an open compact bisection.

In fact, the set $\{\Theta(s,U) : s \in T, U \subseteq D_{s^*s}\}$ is a basis of the topology of $\mathcal{G}_{\text{tight}}(T)$.

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In fact, the set $\{\Theta(s, U) : s \in T, U \subseteq D_{s^*s}\}$ is a basis of the topology of $\mathcal{G}_{\text{tight}}(T)$.

Let us identify what happens in the case of T being $S(E, C)$.

Let γ be a finite path. A **semifinite path** μ starting at γ is one of the following:

(1) If $r(\gamma) = v^p$, with p a free prime, then

$$\mu = \gamma \prod_{j=1}^{k(p)} \alpha(p,j)^{k_j},$$

where $0 \leq k_j \leq \infty$ for all $j \in \{1, \dots, k(p)\}$. We say that μ is an **infinite path** if $k_j = \infty$ for all $j \in \{1, \dots, k(p)\}$.

(2) If $r(\gamma) = v$ with $v \in E_p^0$ and p a regular prime, then

$$\mu = \gamma\lambda,$$

where λ is either a finite or an infinite path in the graph E_p. We say that μ is an **infinite path** if λ is an infinite path in E_p.

Theorem

Let $\mathcal{S}$ be the collection of all semifinite paths. Then

1. *There is a bijective correspondence $\varphi \colon \mathcal{S} \to \hat{\mathcal{E}}_0$.*
2. *φ restricts to a bijection between the set of infinite paths and the set $\hat{\mathcal{E}}_\infty$ of ultrafilters.*
3. *The space $\hat{\mathcal{E}}_\infty$ of ultrafilters is closed in the space $\hat{\mathcal{E}}_0$ of filters. Consequently, $\hat{\mathcal{E}}_\infty = \hat{\mathcal{E}}_{tight}$.*

Definition

We denote by $\mathcal{P}$ the set of semifinite paths of the form $\mu = \gamma\lambda$, where γ is a finite path, and λ is a path of finite length in the component of a regular prime, or $\lambda = \prod_{j=1}^{k(p)} \alpha(p,j)^{k_j}$ for $k_j \in \mathbb{Z}^+$, $1 \leq j \leq k(p)$ for a free prime p.

Notice that every $e \in \mathcal{E}$ is of the form $e(\mu)$ for a unique $\mu \in \mathcal{P}$. Accordingly, elements of $\mathcal{P}$ will be called $\mathcal{E}$-paths.

For $\mu \in \mathcal{P}$, write

$$\mathcal{Z}(\mu) = \{\eta \in \hat{\mathcal{E}}_\infty \mid \mu\mu^* \in \eta\}.$$

Depending on the situation, $\mathcal{Z}(\mu)$ might also be denoted by the idempotent it determines, i.e., $\mathcal{Z}(e(\mu))$. Notice that $\mathcal{Z}(\mu) = \mathcal{U}(\{\mu\mu^*\}, \emptyset) \cap \hat{\mathcal{E}}_\infty$.

Corollary

The space $\hat{\mathcal{E}}_\infty$ of ultrafilters admits a basis of compact open subsets, namely the family $\{\mathcal{Z}(\mu)\}_{\mu\in\mathcal{P}}$. Moreover, every compact open subset of $\hat{\mathcal{E}}_\infty$ is a finite disjoint union of sets of the form $\mathcal{Z}(\mu)$, for $\mu \in \mathcal{P}$.

Corollary

The set $\{\Theta(\mu\mu^*, \mathcal{Z}(\mu)) : \mu \in \mathcal{P}\}$ is a basis of open compact bisections of the tight groupoid $\mathcal{G}(E,C) := \mathcal{G}_{\textit{tight}}(S(E,C))$.

Since $S(E,C)$ is E^*-unitary, we conclude by results of [Exel]

Proposition

If (E,C) is a finitely separated graph, then $\mathcal{G}(E,C)$ is a Hausdorff groupoid.

By using a “graph-goupoid” type picture of $\mathcal{G}(E, C)$, we are able to prove

Proposition

Let (E, C) be an adaptable separated graph. The groupoid $\mathcal{G}(E, C)$ is amenable.

The groupoids of adaptable separated graphs and their type semigroups (I)

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ICMS, July 16, 2019.