Higher dimensional shift spaces and k-graphs

ICMS 2019

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16 July, 2019

Shift spaces in one dimension

- Let $\mathcal{A}$ be a set, usually finite, of symbols, called the *alphabet*.
- Let $\mathcal{A}^*$ denote the Kleene Star of $\mathcal{A}$.
- The *full $\mathcal{A}$-shift* is the set $\mathcal{A}^{\mathbb{Z}}$ together with the shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by

$$(\sigma x)_n = x_{n+1} \quad n \in \mathbb{Z}.$$

- A *shift space* is a closed, shift invariant subset of $\mathcal{A}^{\mathbb{Z}}$ with product topology.
- Often a shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ can be described by giving a list $\mathcal{F} \subseteq \mathcal{A}^*$ of *forbidden words* that is

$$X = \mathsf{X}_{\mathcal{F}} := \{y \in \mathcal{A}^{\mathbb{Z}} : \text{ no subword of } y \text{ contains an element of } \mathcal{F}\}.$$

- The shift spaces (X_1, σ_1), (X_2, σ_2) are *conjugate* if there is a homeomorphism $h : X_1 \to X_2$ such that $h \circ \sigma_1 = \sigma_2 \circ h$.

Shifts of finite type

Definition 1

A shift space X over a finite alphabet $\mathcal{A}$ is of *finite type* if it may be described using a finite list of forbidden words.

Example 2

Let $E = (E^0, E^1, r, s)$ be a directed graph with E^0, E^1 finite. Let

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbb{Z}} : s(x_{i+1}) = r(x_i)\}$$

then X_E is a shift of finite type with $\mathcal{F} = \{ef : r(e) \neq s(f)\}$. The shift space X_E is called the *edge shift* associated to E.

Theorem 3 (Folklore)

Let X be a shift of finite type, then there is a directed graph E such that X is conjugate to X_E.

Proof uses overlapping words to build graph.

Higher dimensional shift spaces

- A higher dimensional shift is a closed subspace of $\mathcal{A}^{\mathbb{Z}^k}$ for some $k \geq 2$ which is invariant under the shift maps σ^b, $b \in \mathbb{Z}^k$ defined by

$$(\sigma^b x)_n = x_{n+b},\ n \in \mathbb{Z}^k.$$

- A major problem arises with specifying a higher dimensional shift using a collection of forbidden/allowed blocks. It is often impossible to decide if there are any configurations in $\mathcal{A}^{\mathbb{Z}^k}$ satisfying the conditions imposed.
- Various classes of higher dimensional shifts have been studied, each of which can directly be shown to be nonempty.
- One of the most successful classes are the **shifts of algebraic origin** introduced by Schmidt. The work of P-Raeburn and Weaver which we shall discuss, was inspired by his formalism.

More on higher dimensional shift spaces

- Let $S \subset \mathbb{Z}^k$ be a finite set, think of these as a set of shapes.
- Let $Q \subseteq \mathcal{A}^S$ a nonempty set, think of these as chosen *allowed* configurations to be found within the shapes of S.
- Let
$$\mathsf{X}_{S,Q} = \{x \in \mathcal{A}^{\mathbb{Z}^k} : x\,|_{S+n} \in Q \text{ for all } n \in \mathbb{Z}^k\}. \tag{1}$$
It is possible that $\mathsf{X}_{S,Q} = \emptyset$.

Definition 4

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^k}$ is a shift space such that $X = \mathsf{X}_{S,Q}$ for some S, Q then X is said to be a *shift of finite type*.

Note a subtle change from forbidden to allowed with the definition of finite type in one dimension.

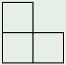
Example – Ledrappier shift

Example 5

The *Ledrappier shift* $X \subset \{0,1\}^{\mathbb{Z}^2}$ consists of those $x = (x_n)_{n \in \mathbb{Z}^2}$ with

$$x_n + x_{n+e_1} + x_{n+e_2} = 0 \text{ mod } 2$$

for all $n \in \mathbb{Z}^2$, where e_1, e_2 are the standard basis vectors for $\mathbb{Z}^2$. The Ledrappier shift, is a two-dimensional shift of finite type with

$S = \{0, e_1, e_2\} \subset \mathbb{Z}^2$ which we visualise as  and

$$Q = \left\{ \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline 0 & 0 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 0 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 0 & 1 \\ \hline \end{array} \right\} \subseteq \{0,1\}^S$$

Is there a graphical model? Let's take a small diversion first.

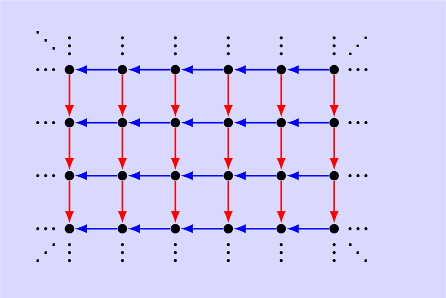
Example 6

Let

$$\Delta = \Delta_k = \{(m,n) : m,n \in \mathbb{Z}^k : m \leq n\}.$$

With structure maps $r(m,n) = m, s(m,n) = n$, so that $(\ell, n) = (\ell, m)(m,n)$ Δ_k becomes a categiory; and then a k-graph with functor $d(m,n) = n - m$.

The skeleton of Δ_2:



The edge shift associated to Λ part II

- The two-sided infinite path space of a k-graph Λ is

$$\Lambda^{\Delta} = \{x : \Delta \to \Lambda : x \text{ is a degree-preserving functor}\}.$$

- The space Λ^{Δ} may be given a locally compact (metric) topology. If Λ^n is finite for all $n \in \mathbb{N}^k$ then Λ^{Δ} is compact.
- For each $b \in \mathbb{Z}^k$ we define $\sigma_b : \Lambda^{\Delta} \to \Lambda^{\Delta}$ by

$$\sigma_b(x)(m, n) = x(m + b, n + b).$$

- The dynamical system $(\Lambda^{\Delta}, \sigma_b)$ is conjugate to a shift of finite type $(Y(\Lambda), \sigma^b)$ (where the allowed patterns come from the set of $(e_i + e_j)$-squares in Λ). In the case $k = 1$ reduces to the edge shift associated to a directed graph.
- Now let's return to our plan to model the Ledrappier shift graphically.

Basic Data

For our basic data we need.

- A *tile* T, which is a hereditary subset of $\mathbb{N}^2$ (in the sense that $(a,b) \leq (m,n) \in T \Rightarrow (a,b) \in T$).
- An integer $q \geq 2$ so that $\mathcal{A} = \mathbb{Z}/q\mathbb{Z}$ is the cyclic group of order q viewed as a ring.
- A weight function $w : T \to \mathcal{A}$.

Definition 7

The triple $BD = (T, q, w)$ is called the *basic data*.

Definition 8

Let $\Lambda(BD)^0 = \{v : T \to \mathcal{A} : \sum_{x \in T} v(x)w(x) = 0\}$

The set $\Lambda(BD)^0$ are the vertices in the 2-graph $(\Lambda(BD), d)$ whose construction we shall outline after this example.

Example 9 (The sock)

Let $T = \{0, e_1, e_2\} \subset \mathbb{N}^2$ be a tile, $q = 2$ so $\mathcal{A} = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, and weight function $w(x) = 1$ for all $x \in T$.

Then for the basic data $BD = (T, 2, w)$ the set $\Lambda(BD)^0$ contains four functions which we view as

$$\begin{array}{|c|c} \hline 0 & \\ \hline 0 & \multicolumn{1}{c|}{0} \\ \hline \end{array}, \quad \begin{array}{|c|c} \hline 1 & \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \quad \begin{array}{|c|c} \hline 1 & \\ \hline 1 & \multicolumn{1}{c|}{0} \\ \hline \end{array}, \quad \begin{array}{|c|c} \hline 0 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}.$$

- Let T be a tile, let c_1 be the maximum of the first coordinates in T and c_2 be the maximum of the second coordinates. In the example above $c_1 = c_2 = 1$.
- These configurations will become the vertices of a 2-graph. What about the edges?

The category $\Lambda(BD)$ – Part 1

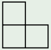
For $i \in \mathbb{N}^2$ and T a tile let $T + i = \{x + i : x \in T\}$ denote the translate of T by i. For $n \in \mathbb{N}^2$ let

$$T(n) = \bigcup_{i \leq n} T + i$$

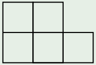
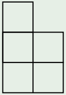
denote the collection of translates of T by an amount less than n.

Example 10 (Edges)

For $T = \{0, e_1, e_2\}$ as in Example 9 we visualise

$T =$ .

Then

$T(e_1) =$  and $T(e_2) =$ 

The category $\Lambda(BD)$ – Part 2

Definition 11

For $n \in \mathbb{N}^2$ let $\Lambda(BD)^n = \{\lambda : T(n) \to \mathcal{A} : \lambda|_{T+i} \in \Lambda^0$ for all $i \leq n\}$.
For $\lambda \in \Lambda(BD)^n$ let $r(\lambda) = \lambda|_T$ and $s(\lambda) = \lambda|_{T+n}$.

Example 12 (Red and blue edges for the sock)

Returning to Example 9 a typical element of $\Lambda(BD)^{e_1}$ is

$$\lambda = \begin{array}{ccc} 1 & 1 & \\ 1 & 0 & 1 \end{array} \text{ where } s(\lambda) = \begin{array}{cc} 1 & \\ 0 & 1 \end{array} \text{ and } r(\lambda) = \begin{array}{cc} 1 & \\ 1 & 0 \end{array}$$

and a typical element of $\Lambda(BD)^{e_2}$ has the form

$$\mu = \begin{array}{cc} 1 & \\ 0 & 1 \\ 1 & 1 \end{array} \text{ where } s(\mu) = \begin{array}{cc} 1 & \\ 0 & 1 \end{array} \text{ and } r(\mu) = \begin{array}{cc} 0 & \\ 1 & 1 \end{array}$$

The category $\Lambda(BD)$ – Part 3

Definition 13

The basic data $BD = (T, q, w)$ has *two invertible corners* if $w(c_1e_1)$ and $w(c_2e_2)$ are invertible in $\mathbb{Z}/q\mathbb{Z}$.

Lemma 14 (P-Raeburn-Weaver)

Let $BD = (T, q, w)$ be basic data with two invertible corners. Let $\lambda \in \Lambda(BD)^n$ and $\mu \in \Lambda(BD)^m$ be such that $s(\lambda) = r(\mu)$ then there exists unique $\lambda\mu \in \Lambda(BD)^{m+n}$ such that $\lambda\mu|_{T(n)} = \lambda$ and $\lambda\mu|_{(T+n)(m)} = \mu$

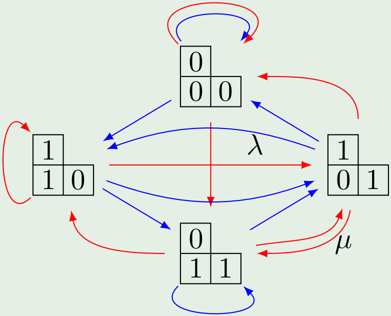
Let $\Lambda(BD)^* = \cup_{n\in\mathbb{N}^2}\Lambda(BD)^n$ and define $d : \Lambda(BD)^* \to \mathbb{N}^2$ by $d(\lambda) = n$ if and only if $\lambda \in \Lambda(BD)^n$.

Theorem 15 (P-Raeburn-Weaver)

Let $BD = (T, q, w)$ be basic data with two invertible corners, then $\Lambda(BD) = (\Lambda(BD)^0, \Lambda(BD)^*, r, s)$ is a 2-graph with no sinks and sources.

Example 16 (The skeleton of the sock graph)

For the basic data $(T, 2, w)$ given in Example 9 we have $w \equiv 1$ and so T has invertible corners – the resulting 2-graph has the skeleton given below.



Where $\lambda = \begin{array}{|c|c|c} \hline 1 & 1 & \multicolumn{1}{c}{} \\ \hline 1 & 0 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$ and $\mu = \begin{array}{|c|c} \hline 1 & \multicolumn{1}{c}{} \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$

Connections with two dimensional shifts

Theorem 17 (P-Raeburn-Weaver 2009)

Suppose we have basic data $BD = (T, q, w)$ *with two invertible corners and let* $\Lambda = \Lambda(BD)$*. Then* $(\Lambda^\Delta, \sigma^b)$ *is conjugate to a shift of finite type.*

Definition 18

The basic data $BD = (T, q, w)$ has *three invertible corners* if $w(c_1 e_1)$, $w(c_2 e_2)$ and $w(0)$ are invertible in $\mathbb{Z}/q\mathbb{Z}$.

Theorem 19 (P-Raeburn-Weaver)

Suppose that $BD = (T, q, w)$ *is a set of basic data with three invertible corners then* $\Lambda(BD)$ *is strongly connected and aperiodic.*

In particular for the basic data $(T, 2, w)$ from Example 9, the 2-graph $\Lambda(BD)$ is strongly connected and aperiodic; moreover $(\Lambda(BD)^\Delta, \sigma^b)$ is conjugate to the Ledrappier shift.

Textile systems

Textile systems were originally championed by Nasu and Boyle. Here I have crafted their definitions to fit our conventions as we plan to construct a 2-graph from them.

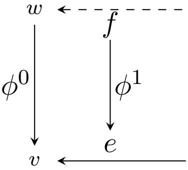
Definition 20

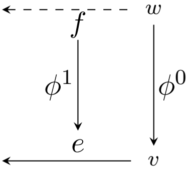
A *textile system* $T = (p, q : F \to E)$ consists of two directed graphs E, F and two graph morphisms $p, q : F \to E$ such that the map $F^1 \to E^1 \times F^0 \times F^0 \times E^1$ given by $f \mapsto (p(f), r(f), s(f), q(f))$ is injective. We shall frequently assume that p, q are surjective.

Are there many textile systems? You bet!

A graph morphism $\phi : F \to E$ has [unique] r-path (resp. s–path) lifting if for every $v \in \phi^0(F^0)$, $e \in \phi^1(F^1)$ with $r(e) = v$ (resp. $s(e) = v$) and $w \in F^0$ with $\phi^0(w) = v$ there is a [unique] $f \in \phi^1(F^1)$ with $r(f) = w$ (resp. $s(f) = w$) such that $\phi^1(f) = e$.

range





source

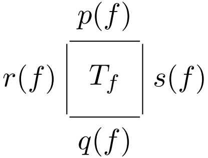
Lemma 21

Let E and F be directed graphs and $p, q : F \to E$ be graph morphisms. If p has unique r- or unique s-path lifting, then $(p, q : F \to E)$ is a textile system for any q. Similarly if q has unique r- or unique s-path lifting, then $(p, q : F \to E)$ is a textile system for any p.

Textile systems to Wang tiles

Let W be a non-empty, finite set of distinct, closed 1×1 squares (tiles) with coloured edges such that no horizontal edge has the same colour as a vertical edge: such a set W is called a collection of *Wang tiles* (introduced by Hao Wang in 1961). For each $\tau \in W$ we denote by $\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau)$ the colours of the right, top, left and bottom edges of τ. We write $C(W) = \{\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau) : \tau \in W\}$ for the set of colours occurring in W. Let $T = (p, q : F \to E)$ be a textile system, then for each $f \in F^1$ we may define a (unique) Wang tile T_f such that

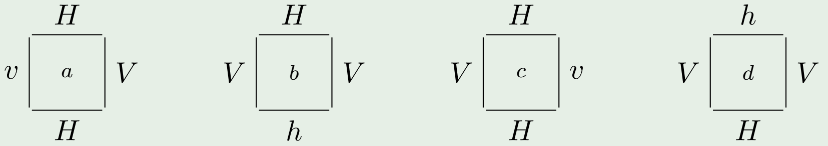
$$\mathsf{t}(T_f) = p(f),\ \mathsf{b}(T_f) = q(f),$$
$$\mathsf{l}(T_f) = r(f),\ \mathsf{r}(T_f) = s(f).$$



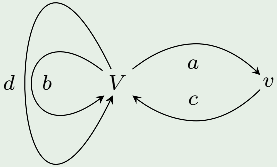
Let $W_T = \{T_f : f \in F^1\}$ and then $C(W_T) = E^1 \sqcup F^0$.
Can also go from Wang tiles to Textile systems, but it is more complicated.

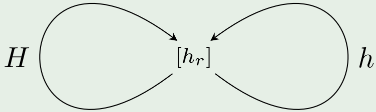
Example 22 (Domino tiles)

The set $W = \{a, b, c, d\}$ is a collection Wang tiles, called the domino tiles:



These tiles come from textile system $T_W : (p_W, q_W : F_W \to E_W)$:

$F_W :=$ 

$E_W :=$ 

Where $p^1_W(a) = p^1_W(b) = p^1_W(c) = H$, $p^1_W(d) = h$ and
$q^1_W(a) = q^1_W(c) = q^1_W(d) = H$ and $q^1_W(b) = h$.

Tiles should lead to tiling of the plane, right?

Theorem 23 (Pask, Sims, Tang)

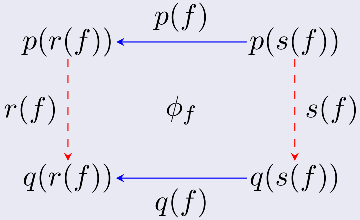
Let $T = (p, q : F \to E)$ *be a textile system then the Wang tiles* W_T *tile the plane if* p, q *have* r, s*-path lifting.*

- The domino textile system given in Example 22 gives a tiling of the plane but does not satisfy the above theorem, so the above result is not an if and only if.
- Where are the 2-graphs? Here we go....

2-coloured graphs from textile systems

Definition 24

Let $T = (p, q : F \to E)$ be a textile system. Define the 2-coloured graph G_T as follows: Let $(G_T)^0 = E^0$, $(G_T)^1 = E^1 \cup F^0$. For $e \in E^1$ define $s(e) = s_E(e)$, $r(e) = r_E(e)$ and $c(e) = c_1$, and for $w \in F^0$ define $s(w) = p(w)$, $r(w) = q(w)$ and $c(w) = c_2$.
For $f \in F^1$ let ϕ_f be the 2-coloured square in G_T:



Denote the collection of 2-coloured squares in G_T by $\mathcal{C}_T$.

2-graphs

To get a 2-graph from the two-coloured graph G_T, we may apply the results of Hazelwood, Raeburn, Sims and Webster to the squares $\mathcal{C}_T$. To cut a long story short, it suffices to find conditions on p, q which imply that $\mathcal{C}_T$ is a complete collection of squares. From what we have seen before the following should not be a surprise.

Theorem 25 (Pask, Sims, Tang, Webster, Kang)

Let $T = (p, q : F \to E)$ be a textile system, then $\mathcal{G}_T$ is a 2-graph if and only if p has unique r-path lifting and q has unique s-path lifting.

Since there are lots of textile systems with these properties, we may use them to build 2-graphs. However not every textile system gives a 2-graph.

Example 26

The category $\mathcal{G}_{T_W}$ from Example 22 is not a 2-graph as p_W does not have unique r-path lifting.

THANK YOU