Ramanujan cubical complexes and non-residually finite CAT(0) groups in any dimension

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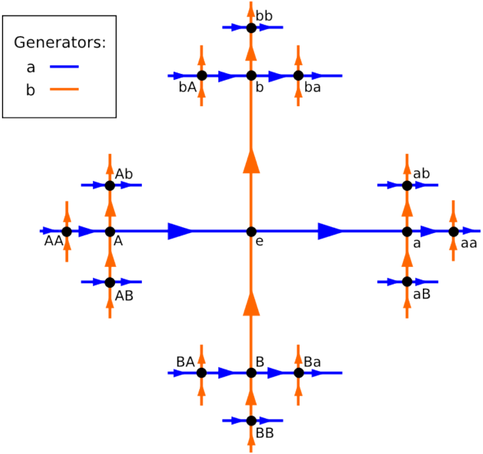
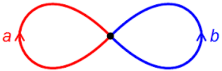
Outline

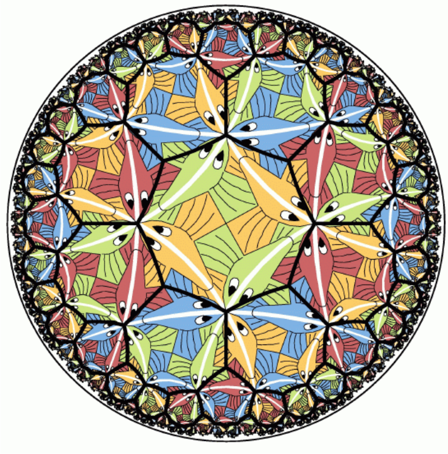
One-dimensional buildings: Cayley graphs of free groups





The four-valent tree is the *universal cover* of the wedge of two circles.

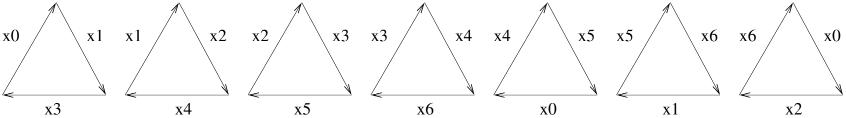
Example of an apartment: M.C.Escher - Circle Limit III



Polyhedra and links

Definition

A *polyhedron* is a two-dimensional complex which is obtained from several decorated p-gons by identification of corresponding sides.



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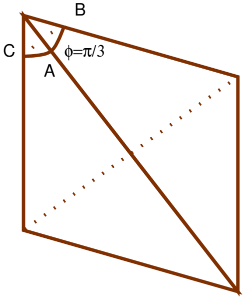
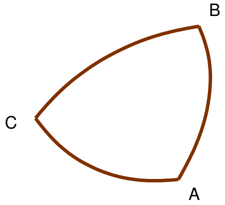
Take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

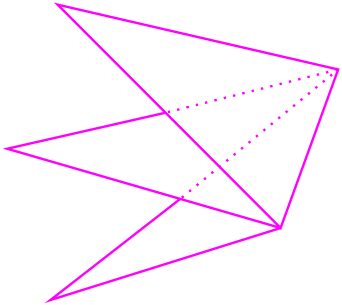




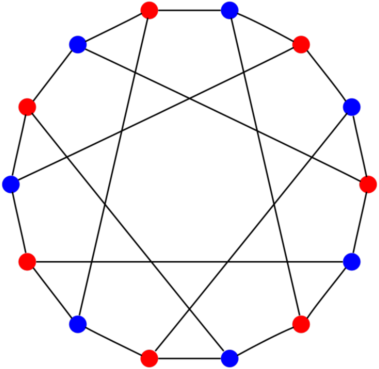
AB=BC=CA= $\pi/3$

Polyhedra and links

We consider *thick* polyhedra, which means that each edge is contained in at least three polygons.



Example of a link



This graph has *diameter* (the maximal distance between two vertices) three and *girth* (the length of the shortest cycle) six.

Polyhedra and links

Theorem (Ballmann, Brin 1994)

Let X be a compact two-dimensional thick polyhedron. If all links are graphs of diameter m and girth $2m$, then the universal cover of the polyhedron is a two-dimensional building.

A polygonal presentation is a set of words satisfying certain combinatorial properties (AV, 2000).

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Theorem (AV,2002)

A polyhedron with given links can be constructed explicitly using a polygonal presentation. Any connected bipartite graph can be realized as a link of every vertex a 2-dimensional polyhedron with $2k$-gonal faces.

A Result of Jacobi

In p is an odd prime, the number of

$$(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$$

such that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$$

is

$$8(p+1).$$

Suppose $p \equiv 1 \pmod 4$. Then exactly one a_j is odd and the number of representations with a_0 odd, $a_0 > 0$, is

$$p+1.$$

Consequence:
Let S_p be the set of integer quaternions

$$a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}(\mathbb{Z})$$

with $\mathbf{i}^2$=$\mathbf{j}^2$=$\mathbf{k}^2$=$\mathbf{ijk}$=-1 and $a_0 > 0$, a_0 odd, $|a|^2 = p$. Then

$$S_p = p+1.$$

An Arithmetic Construction

If $p \equiv 1 (\text{mod } 4)$ is prime, then $x^2 \equiv -1 (\text{mod p})$ has a solution in $\mathbb{Z}$, so, by Hensel's Lemma, $x^2 = -1$ has a solution i_p in $\mathbb{Q}_p$.

Define

$$\psi_p : \mathbb{H}(\mathbb{Z}) \mapsto PGL_2(\mathbb{Q}_p)$$

by

$$\psi_p(a) = \begin{pmatrix} a_0 + a_1 i_p & a_2 + a_3 i_p \\ -a_2 + a_3 i_p & a_0 - a_1 i_p \end{pmatrix}$$

Theorem (Lubotzky, Phillips, Sarnak; Margulis1988)

$\psi_p(S_p)$ contains $p+1$ elements and generates a free group Γ_p of rank $(p+1)/2$. Γ_p acts freely and transitively on the vertices of the $(p+1)$-regular tree T_{p+1}. Ramanujan graphs are Cayley graphs of $PGL_2(\mathbb{Z}/q\mathbb{Z})$ with respect to generators $\psi_p(S_p)$ for $p \neq q$.

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- Infinite series of lattices acting on products of trees of different valencies: Burger-Mozes (2000), Rattaggi (2004)
- Arithmetic lattices acting on products of trees of the same valency: joint work with Jakob Stix (2013) (and later developments)
- The same valency is needed to get Ramanujan complexes.

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- From classical Hamiltonian quaternions, we've moved to more general quaternionic algebras, namely, the quaternionic multiplication is the following:
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Example

For $p = 3$ we get four squares with labels

$$a_2 b_1 a_2 b_2^{-1}, a_1 b_2^{-1} a_2^{-1} b_2^{-1}, a_1 b_1 a_1 b_2, a_1 b_1^{-1} a_2 b_1^{-1}, \textit{where}$$

$$a_1 = t + \mathbf{j} + \mathbf{k}, a_2 = t + \mathbf{j} - \mathbf{k}, b_1 = t + \mathbf{j}, b_2 = t + \mathbf{k}.$$

Cubes and Products of Trees

The four squares define a group G which belongs to a family constructed by J.Stix and AV

$$G = \langle a_1, a_2, b_1, b_2 \mid a_2 b_1 a_2 b_2^{-1}, a_1 b_2^{-1} a_2^{-1} b_2^{-1}, a_1 b_1 a_1 b_2, a_1 b_1^{-1} a_2 b_1^{-1} \rangle.$$

Let $S = \{a_1, a_2, b_1, b_2\}$. Then $Cay(G, S)$ is a one-skeleton of a thick Euclidean building (product of two trees) with the following properties:

- lots of (equilateral) squares (chambers)

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- G is an arithmetic lattice in $PGL(2, \mathbb{F}_3((t))) \times PGL(2, \mathbb{F}_3((t)))$

Arithmetic lattices acting simply transitively on products of trees

Theorem (Jakob Stix,AV)

Let q be an odd prime power, then one can explicitly construct quaternionic groups acting simply transitively on product of trees of valency (q+1), and the number of non-commensurable classes of such groups is at least $(q+1)/2$.

Arithmetic lattices acting simply transitively on products of trees

Let q be a prime power. Let

$$\delta \in \mathbb{F}_{q^2}^{\times}$$

be a generator of the multiplicative group of the field with q^2 elements. If $i, j \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ are

$$i \not\equiv j \pmod{q-1},$$

then $1 + \delta^{j-i} \neq 0$, since otherwise

$$1 = (-1)^{q+1} = \delta^{(j-i)(q+1)} \neq 1,$$

a contradiction. Then there is a unique $x_{i,j} \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ with

$$\delta^{x_{i,j}} = 1 + \delta^{j-i}.$$

With these $x_{i,j}$ we set $y_{i,j} := x_{i,j} + i - j$, so that

$$\delta^{y_{i,j}} = \delta^{x_{i,j}+i-j} = (1 + \delta^{j-i}) \cdot \delta^{i-j} = 1 + \delta^{i-j}.$$

We set

$$l(i,j) := i - x_{i,j}(q-1),$$
$$k(i,j) := j - y_{i,j}(q-1).$$

Let $M \subseteq \mathbb{Z}/(q^2-1)\mathbb{Z}$ be a union of cosets stable under multiplication by q, and by addition of $q-1$.

Theorem (RSV 2018)

Each group $\Gamma_{M,\delta}$ acts simply transitively on a product of $d = |M|$ trees.

$$\Gamma_{M,\delta} = \left\langle a_i \textit{ for all } i \in M \;\middle|\; \begin{array}{c} a_{i+(q^2-1)/2} a_i = 1 \textit{ for all } i \in M, \\ a_i a_j = a_{k(i,j)} a_{l(i,j)} \textit{ for all } i,j \in M \textit{ with } i \not\equiv j \pmod{q-1} \end{array} \right\rangle$$

if q is odd, and if q is even:

$$\Gamma_{M,\delta} = \left\langle a_i \textit{ for all } i \in M \;\middle|\; \begin{array}{c} a_i^2 = 1 \textit{ for all } i \in M, \\ a_i a_j = a_{k(i,j)} a_{l(i,j)} \textit{ for all } i,j \in M \textit{ with } i \not\equiv j \pmod{q-1} \end{array} \right\rangle.$$

3D example

$$\Gamma = \left\langle \begin{array}{c} a_1, a_5, a_9, a_{13}, a_{17}, a_{21}, \\ b_2, b_6, b_{10}, b_{14}, b_{18}, b_{22}, \\ c_3, c_7, c_{11}, c_{15}, c_{19}, c_{23} \end{array} \middle| \begin{array}{c} a_i a_{i+12} = b_i b_{i+12} = c_i c_{i+12} = 1 \text{ for all } i \,, \\ a_1 b_2 a_{17} b_{22},\ a_1 b_6 a_9 b_{10},\ a_1 b_{10} a_9 b_6, \\ a_1 b_{14} a_{21} b_{14},\ a_1 b_{18} a_5 b_{18},\ a_1 b_{22} a_{17} b_2, \\ a_5 b_2 a_{21} b_6,\ a_5 b_6 a_{21} b_2,\ a_5 b_{22} a_9 b_{22}, \\ a_1 c_3 a_{17} c_3,\ a_1 c_7 a_{13} c_{19},\ a_1 c_{11} a_9 c_{11}, \\ a_1 c_{15} a_1 c_{23},\ a_5 c_3 a_5 c_{19},\ a_5 c_7 a_{21} c_7, \\ a_5 c_{11} a_{17} c_{23},\ a_9 c_3 a_{21} c_{15},\ a_9 c_7 a_9 c_{23}, \\ b_2 c_3 b_{18} c_{23},\ b_2 c_7 b_{10} c_{11},\ b_2 c_{11} b_{10} c_7, \\ b_2 c_{15} b_{22} c_{15},\ b_2 c_{19} b_6 c_{19},\ b_2 c_{23} b_{18} c_3, \\ b_6 c_3 b_{22} c_7,\ b_6 c_7 b_{22} c_3,\ b_6 c_{23} b_{10} c_{23}. \end{array} \right\rangle .$$

Adjacency operators for graphs and Ramanujan graphs

Let X be a connected graph with uniformly bounded valencies. We consider X as a 1-dimensional cubical complex and write X_0 for the set of vertices of X. We write $P \sim Q$ if two vertices $P, Q \in V(X)$ are adjacent, and we denote by $\mu(P, Q)$ the number of edges that connect P with Q.

Definition

The ***adjacency operator*** *A_X acting on the space of L^2-functions $f : X_0 \to \mathbb{C}$ is defined as*

$$A_X(f)(P) = \sum_{Q \sim P} \mu(P, Q) f(Q),$$

where we sum over adjacent vertices with the multiplicity of the number of edges linking them.

The adjacency operator commutes with the induced right action of the group of graph automorphisms on $L^2(X_0)$.

Adjacency operators for graphs and Ramanujan graphs

Let X be a finite graph of constant valency $q + 1$. The **trivial eigenvalues** of A_X acting on $L^2(X_0)$ are $\lambda = \pm(q + 1)$. These are obtained by the constant non-zero function for $\lambda = q + 1$, and by the 'alternating function' with $f(P) = -f(Q) \neq 0$ for all $P \sim Q$ for $\lambda = -(q + 1)$. The latter only exists if X has a bipartite structure.

Adjacency operators for graphs and Ramanujan graphs

Alon and Boppana prove that asymptotically in families of finite $(q+1)$-regular graphs X_n with diameter tending to ∞ the largest absolute value of a non-trivial eigenvalue $\lambda(X_n)$ of the adjacency operator A_{X_n} has limes inferior

$$\varliminf_{n\to\infty} \lambda(X_n) \geqslant 2\sqrt{q}.$$

This estimate motivates the definition as follows.

Definition

A finite $(q+1)$-regular graph X is defined to be a ***Ramanujan graph*** *if all non-trivial eigenvalues λ of the adjacency operator A_X have absolute value $\lambda \leqslant 2\sqrt{q}$.*

Higher-dimensional Ramanujan cube complexes

We write $P \sim_v Q$ if two vertices in the product of d trees are adjacent in v-direction, $v \in \{1, ..., d\}$.

Definition

*We define an **adjacency operator** A_v **in** v**-direction** on $L^2(G/K)$ by*

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Definition

*Let $X \to \Delta^d$ be a finite cubical complex of dimension d that has constant valency $q_v + 1$ in all directions. Then X is a **cubical Ramanujan complex**, if for each $v \in \{1, \ldots, d\}$, the eigenvalues λ of A_v are trivial, i.e., $\lambda = \pm(q_v + 1)$, or non-trivial and then bounded by*

$$\lambda \leqslant 2\sqrt{q_v}.$$

Higher-dimensional Ramanujan cube complexes

Theorem

Let $\Gamma \subseteq \Gamma_{M,\delta}$ be a congruence subgroup. Then the quotient X_Γ of a product of d trees by Γ is a cubical Ramanujan complex.

We conjecture that infinitely many of the Ramanujan complexes of the Theorem above are higher-dimensional coboundary expanders of bounded degree in the sense of Gromov.

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- Why difficult? Each k-D cube group gives k 2D groups, which need to be compatible, and remain compatible after doubling.

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- Further applications of harmonic maps to study of buildings and higher-dimensional complexes;
- New applications of polygonal presentations to algebraic geometry: Beauville surfaces and fake quadrics.