Intertwining operators between Dunkl operators of type B_n

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Dunkl theory

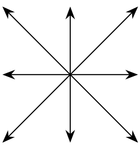
- … generalizes Euclidean harmonic analysis and important aspects of harmonic analysis on Riemannian symmetric spaces
- fundamental tool: Dunkl operators = differential reflection operators associated with root systems
- Special functions of several variables play an important role!
- Some applications/connections:
  - quantum integrable particle systems of Calogero-Moser type
  - representation theory of (double) affine Hecke algebras (Cherednik, 1990ies) $\rightsquigarrow$ Macdonald-Cherednik theory

- **"rational" DOs:** Dunkl (since 1989), Opdam, de Jeu,…
- **"trigonometric" DOs and hypergeometric functions:** Heckman, Opdam (since 1991), Cherednik, Schapira,…

Here: rational setting

Ingredients

- R: a (reduced) **root system** in $\mathbb{R}^n$, i.e.
  - $R \subset \mathbb{R}^n \setminus \{0\}$ finite
  - For each $\alpha \in R$, $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$
  - For each $\alpha \in R$, the reflection s_α in the hyperplane $\alpha^\perp$ leaves R invariant



- $W = \langle s_\alpha \, , \alpha \in R \rangle$ associated **reflection group (Weyl group)**
- $k : R \to \mathbb{C}$ a W-invariant **multiplicity function**

In this talk always: $k \geq 0$.

Examples:

- $R = A_{n-1} = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$
  $W = S_n$, acts on $\mathbb{R}^n$ by permutation of the coordinates
- $R = B_n = \{\pm e_i, \pm(e_i \pm e_j) : 1 \leq i < j \leq n\}$
  $W = S_n \ltimes \{\pm 1\}^n$, generated by permutations and sign changes
  Multiplicity: $k = (k_1, k_2)$; k_1 on $\pm e_i$, k_2 on $\pm e_i \pm e_j$

Dunkl operators associated with R and k:

$$T_\xi(k)f(x) = \partial_\xi f(x) + \frac{1}{2}\sum_{\alpha\in R} k(\alpha)\langle\alpha,\xi\rangle \frac{f(x)-f(s_\alpha x)}{\langle\alpha,x\rangle} \quad (x,\,\xi\in\mathbb{R}^n)$$

- Nice mapping properties (as usual partial derivatives).
- Case $k=0$: $T_\xi(0)=\partial_\xi$.
- **„Rank 1“ case:** $R=\{\pm 1\}\subset\mathbb{R}$, $W=\mathbb{Z}/2\mathbb{Z}$

$$T(k)f(x) = f'(x) + k\cdot\frac{f(x)-f(-x)}{x}$$

Theorem (Dunkl, '89) R, k fixed

The $T_\xi(k)$, $\xi\in\mathbb{R}^n$ commute.

Dunkl kernel and intertwining operator

For a spectral parameter $y \in \mathbb{C}^n$, consider the joint eigenvalue problem

$$(*) \quad \begin{cases} T_\xi(k)f = \langle \xi, y\rangle\, f \quad \forall\, \xi \in \mathbb{R}^n \\ f(0) = 1 \end{cases} \qquad (\langle .,. \rangle \text{ bilinear on } \mathbb{C}^n \times \mathbb{C}^n)$$

How to solve?

If $k = 0$, then $f(x) = e^{\langle x, y\rangle}$.

Nice method in the general case:

Dunkl's intertwining operator

There is a unique isomorphism V_k of the space $\mathbb{C}[\mathbb{R}^n]$ of polynomials on $\mathbb{R}^n$ which preserves the degree of homogeneity and satisfies

$$V_k(1) = 1; \;\; T_\xi(k)V_k = V_k\partial_\xi \;\forall\, \xi \in \mathbb{R}^n.$$

V_k extends to large classes of analytic functions (including exponentials); the intertwining properties remains.

Dunkl kernel (associated with R, k)

$$E_k(x,y) := V_k\big(e^{\langle\,\cdot\,,y\rangle}\big)(x), \quad x \in \mathbb{R}^n,\ y \in \mathbb{C}^n.$$

Consequence:
$f(x) = E_k(x,y)$ is the unique real analytic solution of the EVP $(*)$

Basic properties of E_k:

- $T_\xi(k)E_k(\,.\,,y) = \langle \xi, y\rangle E_k(\,.,y)$.
- $E_k(0,y) = 1$
- E_k extends analytically to $\mathbb{C}^n \times \mathbb{C}^n$.
- $E_k(x,y) = E_k(y,x)$
- $E_k(\lambda x, y) = E_k(x, \lambda y),\ E_k(wx, wy) = E_k(x,y)\ \forall \lambda \in \mathbb{C}, w \in W$

Explicit expressions for E_k and V_k are a topic of intensive current research!

Rank 1 (Dunkl '91):

- $V_k f(x) = c \cdot \int_{-1}^{1} f(tx)(1-t)^{k-1}(1+t)^k dt$
- $E_k(x,y) = j_{k-1/2}(ixy) + \frac{ixz}{2k+1} j_{k+1/2}(ixy)$ with

  $j_\alpha(z) = {}_0F_1\big(\alpha+1; -z^2/4\big)$ (normalized Bessel function)

Type A: Dunkl '95; Amri '14; Sawyer '17 (recursive formula); further partial results by Xu as well as de Bie/Lian (both arXiv '20)

Dihedral groups: Amri/Demni '17: Xu '19; de Bie/Lian arXiv'20

Harmonic analysis: the Dunkl transform

Fact: $|E_k(x, iy)| \leq 1$ for all $x, y \in \mathbb{R}^n$ **(back to this later!)**

$$\omega_k(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{k(\alpha)} \quad (W\text{-invariant weight})$$

For $f \in L^1(\mathbb{R}^n, \omega_k)$,

$$\widehat{f}^{\,k}(\xi) := c_k \int_{\mathbb{R}^n} f(x) E_k(x, -i\xi) \omega_k(x) dx$$

Many results for the Euclidean Fourier transform carry over (Fourier inversion, Plancherel theorem, Paley-Wiener theorem...)

Bessel functions

Bessel function associated with R and k:

$$J_k(x,y) := \frac{1}{|W|}\sum_{w\in W} E_k(wx,y) \quad (W\text{-invariant in } x,y)$$

rank 1 case: $J_k(x,y) = j_{k-1/2}(ixy), \quad y\in\mathbb{C}.$

For $k = \frac{n-1}{2}\,(n\in\mathbb{N})$, these are the smooth and even eigenfunctions of the $SO(n)$-radial part of the Laplacian on $\mathbb{R}^n$,

$$\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}.$$

Further Example : $R = B_n\,,\ k = (k_1, k_2),\ k_2 > 0$

$$J_k^B(x,y) = \sum_{\lambda \geq 0} \frac{1}{[\mu]_\lambda^\alpha \, 4^{|\lambda|} |\lambda|!} \cdot \frac{C_\lambda^\alpha(x^2) C_\lambda^\alpha(y^2)}{C_\lambda^\alpha(\mathbf{1})}, \quad x, y \in \mathbb{C}^n$$

with $\alpha = \frac{1}{k_2}\,,\ \mu = k_1 + k_2(n-1) + \frac{1}{2}$ (Baker/Forrester '97)

- the sum is over all partitions of length $\leq n$
- $[\mu]_\lambda^\alpha$: a generalized Pochhammer symbol
- C_λ^α : **Jack polynomials** of index α in n variables. (Important in algebraic combinatorics, theory of symmetric functions)
- the C_λ^α are symmetric, homogeneous of degree $|\lambda|$ and orthogonal on the torus $\mathbb{T}^n$ w.r.t. the weight $\prod_{i<j} |z_i - z_j|^{2/\alpha}$
- The C_λ^α generalize the powers x^m in one variable; for $\alpha = 1$: Schur polynomials

Excursion: Geometric cases

Setting: G/K a Riemannian symmetric space of the non-compact type, i.e. G non-compact Lie group of „Harish-Chandra class", $K \leq G$ maximal compact subgroup.

Examples:

- $GL_n(\mathbb{F})/U_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$;
- $SO_0(n,1)/SO(n)$ (real hyperbolic spaces),
- $SO_0(p,q)/SO(p) \times SO(q)$ (noncompact Grassmann manifolds)

Consider (G, K)**:**

- Cartan decomposition of the Lie algebra of G: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
- $\mathfrak{p}$: Euclidean space with the Killing form as scalar product
- K acts on $\mathfrak{p}$ via Ad (orthogonal transformations)
- Consider $\mathfrak{p}$ as a (flat) symmetric space with this action:

$$\mathfrak{p} \cong (K \ltimes \mathfrak{p})/K \quad (K \ltimes \mathfrak{p}\text{: Cartan motion group})$$

Basis functions for „radial“ harmonic analysis on $\mathfrak{p}$:

The **spherical functions of** $(\mathfrak{p}, K)$, i.e. the smooth, K-invariant functions on $\mathfrak{p}$ which are eigenfunctions of all K-invariant constant coefficient differential operators on $\mathfrak{p}$.

By their K-invariance, the spherical functions can be considered as functions on a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.

Example: $G/K = SO_0(n,1)/SO(n)$

$\mathfrak{p} \cong \mathbb{R}^n$, $\mathfrak{a} \cong \mathbb{R}$; the spherical functions are the 1-variable Bessel functions $x \mapsto j_{k-1/2}(ixy)$, $y \in \mathbb{C}, k = \frac{n-1}{2}$.

Important observation by Heckman:

The spherical functions of $(\mathfrak{p}, K)$, considered as functions on $\mathfrak{a} \cong \mathbb{R}^n$, are given by

$$\varphi(x) = J_k(x,y),\ y \in \mathbb{C}^n$$

J_k : Dunkl-type Bessel function. R, k: associated with G/K;
the $k(\alpha)$ are half-integer dimension numbers ($\frac{1}{2}\times$ root multiplicities)

Examples

(1) $G/K = GL_n(\mathbb{F})/U_n(\mathbb{F})$

$\mathfrak{p} = \{X \in M_n(\mathbb{F}) : X = X^*\}$ (Hermitian matrices)

$K = U_n(\mathbb{F})$ acts by conjugation

$\mathfrak{a} = \{\mathrm{diag}(x_1, \dots, x_n) : x_i \in \mathbb{R}\} \cong \mathbb{R}^n$

$\rightsquigarrow$ **Bessel functions of type** A_{n-1}, $k = \frac{d}{2}$, $d = \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\}$

(2) $G/K = SO_0(p, q)/SO(p) \times SO(q)$, similar over $\mathbb{C}, \mathbb{H}$; $p \geq q \geq 2$

$\rightsquigarrow$ **Bessel functions of type** B_q

$k = (k_1, k_2) = \left(\frac{d}{2}(p - q + 1) - \frac{1}{2}, \frac{d}{2}\right)$

Harish-Chandra integral

In the geometric cases described before,

$$J_k(x,y) = \int_K e^{\langle k.x,y\rangle} dk \qquad \langle .,. \rangle : \text{ Killing form on } \mathfrak{p}$$

$$\underset{\text{Kostant}}{=} \int_{co(W.x)} e^{\langle \xi, y\rangle} d\mu_x(\xi)$$

$co(W.x) \subseteq \mathfrak{a} \cong \mathbb{R}^n$: convex hull of the Weyl group orbit of x

μ_x a probability measure

Now back to the general Dunkl setting!

There is an abstract generalization of the Harish-Chandra integral formula, not only for the Bessel function, but also for the Dunkl kernel:

Positivity of V_k and abstract Harish-Chandra formula

Theorem (R. '99) R, $k \geq 0$ arbitrary

(1) The intertwiner V_k is positive on $\mathbb{C}[\mathbb{R}^n]$, i.e. $p \geq 0 \implies V_k p \geq 0$.

(2) For each $x \in \mathbb{R}^n$ there exists a (unique) probability measure μ_x^k on $co(W.x)$ such that

$$E_k(x, y) = \int_{co(W.x)} e^{\langle \xi, y \rangle} d\mu_x^k(\xi) \quad \forall y \in \mathbb{C}^n.$$

Some consequences:

- $E_k(x, y) > 0$ for all $x, y \in \mathbb{R}^n$ $\rightsquigarrow$ probabilistic applications
- Good bounds on E_k, e.g. $|E_k(x, iy)| \leq 1$ for all $x, y \in \mathbb{R}^n$
- positivity of generalized translations (in the Dunkl sense) of radial functions $\rightsquigarrow$ Useful for harmonic analysis

Intertwiner between $T_\xi(k)$ and $T_\xi(k')$

For multiplicities $k, k' \geq 0$ on the same root system R, the operator $V_{k',k} := V_{k'} \circ V_k^{-1}$ intertwines the Dunkl operators w.r.t k and k':

$$T_\xi(k')V_{k',k} = V_{k',k}T_\xi(k)$$

Old conjecture (P): If $k' \geq k$, i.e. $k'(\alpha) \geq k(\alpha)\ \forall \alpha$, then $V_{k',k}$ is positive.

Equivalent: If $k' \geq k$, then for each $x \in \mathbb{R}^n$ there exists a (unique) compactly supported probability measure $\mu_x^{k',k}$ on $\mathbb{R}^n$ such that

$$(*) \quad E_{k'}(x,y) = \int_{\mathbb{R}^n} E_k(\xi,y)d\mu_x^{k',k}(\xi) \quad \forall y \in \mathbb{C}^n \quad \textbf{(Sonine formula)}$$

Then also

$$J_{k'}(x,y) = \int_{\mathbb{R}^n} J_k(\xi,y)d\nu_x^{k',k}(\xi)$$

with a unique W-invariant probability measure $\nu_x^{k',k}$.

(P) is true in rank 1 (Y. Xu '03): $k' > k \implies$

$$V_{k',k}f(x) = c_{k',k}\int_{-1}^{1} f(xt)|t|^{2k}(1+t)(1-t^2)^{k'-k-1}dt.$$

In this case, the Sonine formula for J_k is just the classical Sonine formula for one-variable Bessel functions (N. Sonine, 1849–1915):

$$j_\beta(x) = c_{\alpha,\beta}\int_0^1 j_\alpha(xt)t^{2\alpha+1}(1-t^2)^{\beta-\alpha-1}dt \quad \forall\, \beta > \alpha > -1.$$

But (P) is not true in general!

We have positive examples, but also counterexamples for $R = B_n$.

Results on conjecture (P) for $R = B_n$ (with M. Voit, 2020)

- $B_n = \{\pm e_i, \pm e_i \pm e_j, 1 \leq i < j \leq n\} \subset \mathbb{R}^n$
- $k = (k_1, k_2)$ with $k_1 \geq 0$ (on $\pm e_i$), $k_2 > 0$ (on $\pm e_i \pm e_j$)
- Consider $k' = (k_1 + h, k_2)$ with $h \geq 0$.

Theorem 1 (Necessary condition)

If $V_{k',k} = V_{k'} V_k^{-1}$ is positive, then **either** $h > k_2(n-1)$, **or** h belongs to the discrete set $\{0, k_2, \ldots, k_2(n-1)\} - \mathbb{N}_0$

Proof:

(1) V_k is a topological isomorphism of $\mathcal{E}(\mathbb{R}^n) \implies$ for fixed $x \in \mathbb{R}^n$,

$$\varphi \mapsto V_{k',k}\,\varphi(x),\ \mathcal{E}(\mathbb{R}^n) \to \mathbb{C}$$

defines a compactly supported distribution on $\mathbb{R}^n$.

(2) With $\varphi(x) = J_k^B(x,y) \implies J_{k'}^B(1,y) = \langle u_{k,h}\,, J_k^B(\,.\,,y)\rangle\ \forall y$, with a unique B_n-invariant distribution $u_{k,h} \in \mathcal{E}'(\mathbb{R}^n)$.

If $V_{k',k}$ is positive, then $u_{k,h}$ must be a positive measure.

(3) Series expansion of J_k^B in terms of Jack polynomials shows:
For $h > k_2(n-1)$, $u_{k,h}$ is a positive measure with a compactly supported probability density:

$$\langle u_{k,h}, \varphi \rangle = \int_{\mathbb{R}^n} \varphi(x) f_{k,h}(x) dx,$$

$$f_{k,h}(x) = c_{k,h} \cdot \prod_{j=1}^{n} x_j^{2k_1}(1 - x_j^2)^{h-k_2(n-1)-1} \prod_{i<j} |x_i^2 - x_j^2|^{2k_2} \cdot 1_{]-1,1[^n}(x)$$

As a function of h, $f_{k,h}(x)$ extends analytically to $\{\mathrm{Re}\, h > 0\}$.

Arguments of A. Sokal '11 show:
If the distribution $u_{k,h}$ with $h > 0$ is a measure, then $f_{k,h}$ must be locally integrable on $\mathbb{R}^n$.

$\Longrightarrow$ **either** $h > k_2(n-1)$, **or** $c_{k,h} = 0$ $\rightsquigarrow$ discrete values of h.

Corollary

If $J^B_{(k_1+h,k_2)}$ has a positive Sonine integral representation w.r.t. $J^B_{(k_1,k_2)}$, then either $h > k_2(n-1)$, or $h \in \{0, k_2, \dots k_2(n-1)\} - \mathbb{N}_0$.

The same holds for the Dunkl kernel.

A further consequence:

Multivariate Jacobi polynomials (Heckman-Opdam polynomials of type BC) allow limit transitions to Bessel functions of type B. Theorem 1 implies that there occur negative connection coefficients between Jacobi polynomials from multiplicity k to $k' \geq k$, which do not show up in the 1-variable case.

Positive results for B_n

Again $k = (k_1, k_2), k_1 \geq 0, k_2 > 0$; $k' = (k_1 + h, k_2), \; h \geq 0.$

Conjecture: $V_{k',k}$ is positive iff h belongs to the **„generalized Wallach set“**

$$\{0, k_2, \ldots, k_2(n-1)\} \cup \,]k_2(n-1), \infty[\,.$$

For certain half-integer values of k_2, this set is well-known in the analysis on symmetric cones (e.g. the cone of positive definite matrices).
It characterizes those Riesz distributions which are positive measures (Gindikin, '75).

Theorem 2 (k_2 „geometric“, h large)

Denote by $\widetilde{V}_{k',k}$ the restriction of $V_{k',k}$ to B_n-invariant functions.

If $k_2 \in \{\frac{1}{2}, 1, 2\}$ and $h > k_2(n-1)$, then $\widetilde{V}_{k',k}$ is positive.

Proof: Based on explicit Sonine integrals for J_k^B which are derived from known Sonine formulas for Bessel functions on symmetric cones.

Theorem 3 (arbitrary k_2, discrete values of h)

If $k_2 > 0$ is arbitrary and $h \in \{0, k_2, 2k_2, \ldots\}$, then $\widetilde{V}_{k',k}$ is also positive.

Proof: based on multivariate extensions of the following properties of the classical Laguerre polynomials:

(a) $\lim_{n\to\infty} \widetilde{L}_n^\alpha\left(\frac{x}{n}\right) = j_\alpha(2\sqrt{x})$; $\quad \widetilde{L}_n^\alpha = \frac{L_n^\alpha}{L_n^\alpha(0)}$

(b) If $\beta > \alpha$, then $\widetilde{L}_n^\beta(x) = \sum_{k=0}^{n} c_{n,k} \widetilde{L}_k^\alpha(x)$ with $c_{n,k} \geq 0$, $\sum_{k=0}^n c_{n,k} = 1$.

Idea: Take the limit (a) in formula (b) $\Longrightarrow$ there exists a probability measure μ on $[0,1]$ such that

$$j_\beta(2\sqrt{x}) = \int_0^1 j_\alpha(2\sqrt{x\xi}\,)d\mu(\xi) \quad \forall\, x \geq 0.$$

For **multivariate** Laguerre polynomials L_λ^α (indexed by partitions λ), the connection coefficients as in (b) are ≥ 0 if $\beta - \alpha \in \{0, k_2, 2k_2, \ldots\}$.

But: There exist $\beta > \alpha$, where negative connection coefficients occur!

References:

- M. Rösler, M. Voit, Sonine formulas and intertwining operators in Dunkl theory. To appear in IMRN; https://doi.org/10.1093/imrn/rnz313; arXiv:1902.02821.
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Thank you for your attention!