LECTURE 1 – CLASSICAL INTEGRABLE SYSTEMS

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ABSTRACT. These notes are excerpts from my PhD thesis [11] and serve as complementary material to the first lecture [12] in the "Introduction to Integrability" online lecture series that was supported by the London Mathematical Society and hosted by ICMS Edinburgh.

1. Introduction

Integrable Systems is a broad area of research that joins seemingly unrelated problems of natural sciences amenable to exact mathematical treatment[1]. It serves as a busy crossroad of many subjects ranging from pure mathematics to experimental physics. As a result, the notion of 'integrability' is hard to pinpoint as, depending on context, it can refer to different phenomena, and "where you have two scientists you have (at least) three different definitions of integrability"[2]. Fortunately, the systems of our interest are integrable in the Liouville sense, which has a precise definition (see below). Loosely speaking, in such systems an abundance of conservation laws restricts the motion and allows the solutions to be exactly expressed with integrals, hence the name.

2. The golden age of integrable systems

Studying integrable systems is by no means a new activity as its origins can be traced back to the early days of modern science, when Newton solved the gravitational two-body problem and derived Kepler's laws of planetary motion (for more, see [24]). With hindsight, one might say that the solution of the Kepler problem was possible due to the existence of many conserved quantities, such as energy, angular momentum, and the Laplace-Runge-Lenz vector. In fact, the Kepler problem is a prime example of a (super)integrable system (also to be defined). As the mathematical foundations of Newtonian mechanics were established through work of Euler, Lagrange, and Hamilton, more and more examples of integrable/solvable mechanical problems were discovered. Just to name a few, these systems include the harmonic oscillator, the "spinning tops"/ rigid bodies [3] of Euler (1758), Lagrange (1788), and Kovalevskaya (1888), the geodesic motion on the ellipsoid solved by Jacobi (1839), and Neumann's oscillator model (1859). This golden age of integrable systems was ended abruptly in the late 1800s, when Poincaré, while trying to correct his flawed work on the three-body problem, realized that integrability is a fragile property, that even small perturbations can destroy [6]. This subsided scientific interest and the subject went into a dormant state for more than half a century.

3. Definition of Liouville integrability

In the Hamiltonian formulation of Classical Mechanics the state of a physical system, which has n degrees of freedom, is encoded by $2n$ real numbers. These numbers consist of (generalised) positions $q = (q_1, \dots, q_n)$ and (generalised) momenta $p = (p_1, \dots, p_n)$ and are collectively called canonical coordinates of the space of states, the phase space. The time

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[1]For those who are unfamiliar with Integrable Systems, we recommend reading the survey [22].

[2]A quote from another good read, the article *Integrability – and how to detect it* [15, pp. 31-94].

evolution of an initial state $(q_0, p_0) \in \mathbb{R}^{2n}$ is governed by Hamilton's equations of motion, a first-order system of ordinary differential equations that can be written as

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \dots, n,$$

where H is the Hamiltonian, i.e. the total energy of the system. In modern terminology, a Hamiltonian system is a triple (M, ω, H), where the phase space (M, ω) is a $2n$-dimensional symplectic manifold[3] and H is a sufficiently smooth real-valued function on M. An initial state $x_0 \in M$ evolves along integral curves of the Hamiltonian vector field X_H of H defined via $\omega(X_H, \cdot) = dH$. Darboux's theorem [2, 3.2.2 Theorem] guarantees the existence of canonical coordinates[4] (q, p) locally, in which by definition the symplectic form ω can be written as

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j,$$

and the equations of motion take the canonical form displayed above. The symplectic form ω gives rise to a Poisson structure on M, which is a handy device that takes two observables $f, g\colon M \to \mathbb{R}$ and turns them into a third one $\{f, g\}$, the Poisson bracket of f and g given by $\{f, g\} = \omega(X_f, X_g)$. In canonical coordinates, we have

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right).$$

It is bilinear, skew-symmetric, satisfies the Jacobi identity and the Leibniz rule. The equations of motion, for any $f\colon M \to \mathbb{R}$, can be rephrased using the Poisson bracket

$$\dot{f} = \{f, H\}.$$

Consequently, if $\{f, H\} = 0$, that is f Poisson commutes with the Hamiltonian H, then f is a constant of motion. In fact, this relation is symmetric, since $\{f, H\} = 0$ ensures that H is constant along the integral curves of the Hamiltonian vector field X_f.

Having conserved quantities can simplify things, since it restricts the motion to the intersection of their level surfaces, selected by the initial conditions. Thus one should aim at finding as many independent Poisson commuting functions as possible. By independence we mean that at generic points (on a dense open subset) of the phase space the functions have linearly independent derivatives. Of course, the non-degeneracy of the Poisson bracket limits the maximum number of independent functions in involution to n. If this maximum is reached, we found a Liouville integrable system.

Definition. A Hamiltonian system (M, ω, H), with n degrees of freedom, is called *Liouville integrable*, if there exists a family of independent functions $H_1, \dots, H_n$ in involution, i.e. $\{H_j, H_k\} = 0$ for all j, k, and H is a function of $H_1, \dots, H_n$.

The most prominent feature of Liouville integrable systems is the existence of action-angle variables. This is a system of canonical coordinates $I = (I_1, \dots, I_n)$, $\varphi = (\varphi_1, \dots, \varphi_n)$, in which the (transformed) Hamiltonians $H_1, \dots, H_n$ depend only on the action variables I, which are themselves first integrals, while the angle variables φ evolve linearly in time. An important result is the following

Liouville-Arnold theorem. [2, 5.2.24 Theorem] *Consider (M, ω, H) to be a Liouville integrable system with the Poisson commuting functions $H_1, \dots, H_n$. Fix the Hamiltonians at*

[3]A symplectic manifold (M, ω) is a manifold M equipped with a non-degenerate, closed 2-form ω.

[4]Notice the slight and customary abuse of notation as we use the symbols q_j, p_j for representing real numbers as well as coordinate functions on M. Hopefully, this does not cause any confusion.

some 'generic' values $c_1, \ldots, c_n$. Then the level set

$$M_c = \{x \in M \mid H_j(x) = c_j,\ j = 1, \ldots, n\}$$

is a smooth n-dimensional submanifold of M, which is invariant under the Hamiltonian flow of the system. Moreover, if M_c is compact and connected, then it is diffeomorphic to an n-torus $\mathbb{T}^n = \{(\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$, and the Hamiltonian flow is linear on M_c, i.e. the angle variables φ on M_c satisfy $\dot{\varphi}_j = \nu_j$, for some constants ν_j, $j = 1, \ldots, n$.

The action variables I are also encoded in the level set M_c. Roughly speaking, they determine the size of M_c, since I_j is obtained by integrating the canonical 1-form the phase space over the j-th cycle of the torus M_c.

Another relevant notion is superintegrability, which requires the existence of extra constants of motion.

Definition. A Liouville integrable system is called *superintegrable*, if in addition to the Hamiltonians $H_1, \ldots, H_n$ there exist independent first integrals $f_1, \ldots, f_k$ $(1 \leq k < n)$. If $k = n - 1$, then the system is *maximally superintegrable*.

Examples of maximally superintegrable systems include the Kepler problem, the harmonic oscillator with rational frequencies, and the rational Calogero-Moser system considered in Lecture 2. For more details on the theory of integrable systems, see [4].

Remark. It should be noted that, although there is no generally accepted notion of integrability at the quantum level, there are quantum mechanical systems that are called *integrable*.

4. Solitary splendor: The renascence of integrability

About fifty years ago a revival has taken place in the field of Integrable Systems, when Zabusky and Kruskal [26] conducted a numerical study of the Korteweg-de Vries (KdV) equation[5], that is the nonlinear $(1+1)$-dimensional partial differential equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

and re-discovered its stable solitary wave solutions[6], whose interaction resembled that of colliding particles, hence they gave them the name *solitons*. Subsequently, Kruskal et al. [10] started a detailed investigation of the KdV equation and found an infinite number of conservation laws associated to it. More explicitly, they showed that the eigenvalues of the Schrödinger operator

$$L = \partial_x^2 + u$$

are invariant in time if the 'potential' u is a solution of the KdV equation. Moreover, they used the Inverse Scattering Method to reconstruct the potential from scattering data. Lax showed [17] that the KdV equation is equivalent to an equation involving a pair of operators, now called *Lax pair*, of the form

$$\dot{L} = [L, M],$$

where L is the Schrödinger operator above, M is a skew-symmetric operator and $[L, M] = LM - ML$. The commutator form of the Lax equation explains the isospectral nature of the operator L. The connection to integrable systems was made by Faddeev and Zakharov [27], who showed that the KdV equation can be viewed as a completely integrable Hamiltonian system with infinitely many degrees of freedom. These initial findings renewed interest in integrable systems and their applications. For example, Lax pairs associated to other integrable systems were found and used to generate conserved quantities.

[5] The motivation for Zabusky and Kruskal's work was to understand the recurrent behaviour in the Fermi-Pasta-Ulam-Tsingou problem [7], which turns into the KdV equation in the continuum limit.

[6] Korteweg and de Vries [14] devised their equation to reproduce such stable travelling waves, that were first observed by Russell [23] in the canals of Edinburgh.

The ideas and developments presented so far were all about the KdV equation. However, there are other physically relevant nonlinear PDEs with soliton solutions, which have been solved using the Inverse Scattering Method. For example, the sine-Gordon equation [1]

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0,$$

which can be interpreted as the equation that describes the twisting of a continuous chain of needles attached to an elastic band. It has different kinds of soliton solutions, called *kink*, *antikink*, and *breather*, that can interact with one another. It is a relativistic equation, since its solutions are invariant under the action of the Poincaré group of $(1+1)$-dimensional space-time.

The nonlinear Schrödinger equation [29] is another famous example. It reads

$$\mathrm{i}\psi_t + \frac{1}{2}\psi_{xx} - \kappa|\psi|^2\psi = 0,$$

where ψ is a complex-valued wave function and κ is constant. It is also an exactly solvable Hamiltonian system [28]. The equation is nonrelativistic (Galilei invariant).

Now let us list some applications of these soliton equations. The Korteweg-de Vries equation can be applied to describe shallow-water waves with weakly non-linear restoring forces and long internal waves in a density-stratified ocean. It is also useful in modelling ion acoustic waves in a plasma and acoustic waves on a crystal lattice. The kinks and breathers of the sine-Gordon equation can used as models of nonlinear excitations in complex systems in physics and even in cellular structures. The nonlinear Schrödinger equation is of central importance in fluid dynamics, plasma physics, and nonlinear optics as it appears in the Manakov system, a model of wave propagation in fibre optics.

Parallel to soliton theory, various exactly solvable quantum many-body systems appeared, that describe the interaction of quantum particles in one spatial dimension. These models proved to be a fruitful source of ideas and a great influence on the development of mathematical physics. Earlier important milestones include Bethe's solution of the one-dimensional Heisenberg model (Bethe Ansatz, 1931), Pauling's work on the 6-vertex model (1935), Onsager's solution of the planar Ising model (1944), and the delta Bose gas of Lieb-Liniger (1963). At the level of classical mechanics, a crucial step was Toda's discovery of a nonlinear, one-dimensional lattice model [25] with soliton solutions. The Toda lattice is an infinite chain of particles interacting via exponential nearest neighbour potential. The nonperiodic and periodic Toda chains are n particles with such interaction put on a line and a ring, and have the Hamiltonians

$$H_{\mathrm{np}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_{j+1}-q_j)}, \quad \text{and} \quad H_{\mathrm{per}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_{j+1}-q_j)} + g^2 e^{2(q_1-q_n)},$$

respectively. Hénon [13] found n conserved quantities for both of these systems, and Flashka [8, 9] and Manakov [18] found Lax pairs giving rise to these first integrals and proved them to be in involution. Therefore the Toda lattices are completely integrable. The scattering theory of the nonperiodic Toda lattice was examined by Moser [19]. Bogoyavlensky [5] generalised the Toda lattice to root systems of simple Lie algebras. Olshanetsky, Perelomov [20, 21] and Kostant [16] initiated group-theoretic treatments.

5. Exercises

Exercise 1. Introduce the notation $x = (q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ and

$$J = \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix}$$

with $\mathbf{1}_n$ and $\mathbf{0}_n$ standing for the $n \times n$ unit and zero matrices, respectively.

a) Show that Hamilton's equations can be written as

$$\dot{x} = J\,\nabla H(x)$$

where $(\nabla H)_j = \partial H/\partial x_j$, $j = 1, 2, \ldots, 2n$ is the gradient of the Hamiltonian H.

b) Show that the matrix J has the following properties:

$$J^2 = \mathbf{1}_{2n}, \quad J^{-1} = J^\top = -J, \quad \det(J) = +1.$$

Exercise 2. Consider the Kepler problem defined on $M = \{(\vec{q}, \vec{p}) : \vec{q} \in \mathbb{R}^3 \setminus \{\vec{0}\},\ \vec{p} \in \mathbb{R}^3\}$ by the Hamiltonian

$$H = \frac{|\vec{p}|^2}{2m} - \frac{k}{|\vec{q}|} = \frac{p_1^2 + p_2^2 + p_3^2}{2m} - \frac{k}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$$

with $k > 0$ being a fixed positive constant.

a) Show using Hamilton's equations that the angular momentum vector

$$\vec{L} = \vec{q} \times \vec{p}$$

is a conserved quantity, that is $\frac{dL_j}{dt} = 0$, $j = 1, 2, 3$.

b) Show using Hamilton's equations that the Laplace-Runge-Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{q}}{|\vec{q}|}$$

is a conserved quantity, that is $\frac{dA_j}{dt} = 0$, $j = 1, 2, 3$.

Exercise 3. Show that the Poisson bracket $\{.,.\}$ has the following properties:

a) antisymmetry: $\{g, f\} = -\{f, g\}$ for all $f, g \in C^\infty(M)$

b) bilinear: $\{c_1 f_1 + c_2 f_2, g\} = c_1\{f_1, g\} + c_2\{f_2, g\}$,
$\{f, d_1 g_1 + d_2 g_2\} = d_1\{f, g_1\} + d_2\{f, g_2\}$
for all $c_1, c_2, d_1, d_2 \in \mathbb{R}$ and $f, f_1, f_2, g, g_1, g_2 \in C^\infty(M)$

c) Jacobi identity: $\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0$
for all $f_1, f_2, f_3 \in C^\infty(M)$

d) Leibniz rule: $\{f_1 f_2, g\} = \{f_1, g\}f_2 + f_1\{f_2, g\}$
$\{f, g_1 g_2\} = \{f, g_1\}g_2 + g_1\{f, g_2\}$
for all $f, f_1, f_2, g, g_1, g_2 \in C^\infty(M)$.

Exercise 4. Show that the quantities $H, |\vec{L}|^2, L_3$ (appearing in Exercise 2) form a Poisson commuting family of functions, that is

$$\{H, |\vec{L}|^2\} = 0, \quad \{|\vec{L}|^2, L_3\} = 0, \quad \{L_3, H\} = 0.$$

Exercise 5. Show that for $n \times n$ matrix L with entries depending on a single variable t, we have

$$\frac{d\mathrm{tr}(L^k)}{dt} = k\mathrm{tr}\left(L^{k-1}\frac{dL}{dt}\right) \quad \text{for all } k = 1, 2, 3, \ldots$$

Use this result to show that if $\dot{L} = [L, M] = LM - ML$, then $\frac{d\mathrm{tr}(L^k)}{dt} = 0$ $(k = 1, 2, 3, \ldots)$.

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