Bisynchronous games

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Today's goals: review synchronous games and their properties, introduce bisynchronous describe some of their properties.

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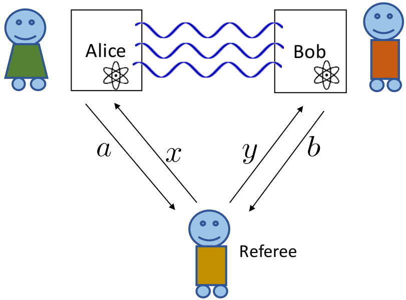

Figure: Alice, Bob and the Referee

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That is, p perfect iff $\big(\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0\big)$

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- C_{qc} which denotes the commuting model.

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- $C_{qs}(n, k) \subsetneq C_{qa}(n, k), \forall n \geq 5, k \geq 2,$ (Dykema, P, Prakash)
- $C_{qa}(n, k) = C_{qc}(n, k), \forall n, k \geq 2 \iff$ Connes' embedding problem has a positive answer(Junge etal, +Ozawa).

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Such densities are called *synchronous* and we use a superscript s to denote the sets of synchronous correlations.

- The graph colouring game is synchronous.
- Graph homomorphism games are synchronous.
- The CHSH game is *not* synchronous.
- Linear binary constraint games for $Ax = b$ are *not* synchronous, but
- they have synchronous versions with the same behaviour of perfect strategies.
- Graph isomorphism games are synchronous and bisynchronous.

Theorem (PSSTW)

$p(a, b|x, y) \in C_{qc}^s(n, k)$ iff there exists a C-algebra $\mathcal{A}$ generated by projections $\{E_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ satisfying $\sum_{a=1}^k E_{x,a} = I, \forall x$ and a trace $\tau : \mathcal{A} \to \mathbb{C}$ such that*

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Moreover, $p(a,b|x,y) \in C^s_q(n,k)$ iff $\mathcal{A}$ can be taken to be finite dimensional.

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Corollary (KPS; Dykema, P)

Connes' embedding conjecture has an affirmative answer if and only if $C_{qa}^s(n,k) = C_{qc}^s(n,k)$.

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Proposition (P-Rahaman)

If a bisynchronous game has a perfect qc-strategy, then $|I| = n \leq k = |O|$.

Quantum Permutations

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Leads to a theorem about when bisynchronous games have perfect strategies of the various types, in terms of $\mathcal{A}(\mathcal{G})$ and quantum permutations.

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The algebra $\mathcal{A}$ is called an *ancilla* for Φ. Unfortunately, no uniqueness theory for ancillas.

Given a density $p(a, b|x, y), 1 \leq a, b, x, y \leq n$ can define a map $\Phi_p : M_n \to M_n$ via

$$\Phi_p(|x\rangle\langle y|) = \sum_{a,b=1}^{n} p(a, b|x, y)|a\rangle\langle b|$$

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Let $p \in C_{qc}^{bs}$*. Then* Φ_p *is a factorizable UCPTP map via a* U *that is a quantum permutation.*

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Conversely, if $\Phi : M_n \to M_n$ is of the form $\Phi(X) = id_{M_n} \otimes \tau(U^*(X \otimes 1_{\mathcal{A}})U)$ with U a quantum permutation and we write $\Phi(|x\rangle\langle y|) = \sum_{a,b} q(a,b|x,y)|a\rangle\langle b|$, then $q \in C_{qc}^{bs}(n,n)$.

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Theorem (PR)

Let $p(a,b|x,y)$ be a bisynchronous density on n variables. Then $p \in C_{loc}^{bs}(n,n)$ iff $\Phi_p : M_n \to M_n$ is a mixed permutation.

The following result is related to the *orbital algebras* of Lupini, Mancinska and Roberson and the proof borrows heavily from their work on quantum graph automorphisms.

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Theorem (PR)

Let $p \in C_{qc}^{bs}(n,n)$ with $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$ and $U = (e_{x,a})$ a quantum permutation. For $A \in M_n$ T.F.A.E.

- $(A \otimes 1_{\mathcal{A}})U = U(A \otimes 1_{\mathcal{A}})$
- $\Phi_p(A) = A$,
- $A = (a_{i,j})$ *with* $a_{i,j} = a_{k,l}$ *whenever* $e_{i,j}e_{k,l} \neq 0$.

Corollary (PR)

Let $p \in C_{qc}^{bs}(n,n)$. *Then* $Fix(\Phi_P) := \{A : \Phi_p(A) = A\}$ *is an algebra in the usual product and is also closed under Schur product, i.e., if* $A = (a_{i,j}), B = (b_{i,j}) \in Fix(\Phi_P)$, *then* $AB \in Fix(\Phi_p)$ *and* $A \circ B := (a_{i,j}b_{i,j}) \in Fix(\Phi_p)$.

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Corollary (PR)

If $p \in C_{qc}^{bs}(n,n)$ is written as $p(a,b|x,y) = \tau(e_{x,a}e_{y,b}) = \gamma(f_{x,a}f_{y,b})$ with $(\mathcal{A},\tau)$, $(\mathcal{B},\gamma)$ C-algebras with traces and $U = (e_{x,a}), V = (f_{x,a})$ both quantum permutations. Then for $A \in M_n$,*

$$(A \otimes 1_{\mathcal{A}})U = U(A \otimes 1_{\mathcal{A}}) \iff (A \otimes 1_{\mathcal{B}})V = V(A \otimes 1_{\mathcal{B}}).$$

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Let G and H be graphs on n vertices with adjacency matrices A_G and A_H and let $t \in \{loc, q, qa, qc\}$. TFAE:

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3. *there is a map* $\Phi : M_n \to M_n$ *that is t-factorizable via a quantum permutation with the property that* $\Phi(A_G) = A_H$ *and* $\Phi^*(A_H) = A_G$*, where* Φ^* *is the adjoint of* Φ*.*

Open Questions

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- Is $C_q^{bs}(n,n)^- = C_{qa}^{bs}(n,n)$ (the bisynchronous approximation problem)?
- Is CEP true $\overset{?}{\iff}$ $C_{qa}^{bs}(n,n) = C_{qc}^{bs}(n,n), \forall n$?
- Is every nearly quantum permutation near to a quantum permutation?
- If $\mathcal{G}$ is bisynchronuos with n inputs and n outputs and $\mathcal{A}(\mathcal{G}) \neq (0)$ then does $\mathcal{G}$ have a perfect qc-strategy?(True for graph isomorphism games)

Thanks!

References

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